• All problems are worth approximately the same weight, even if some may be harder than others.

• Part A must be submitted online through Sakai, as a scan or as good pictures. This is because the grader for this part works remotely.

• Part B must be submitted in paper (handwritten or typed), in class, on the due date, in the first 10 minutes of lecture. Use a staple if you have more than one sheet.

• There’s no need to copy down the problem statement; simply indicate which problem is being solved.

• Circle your final answer when appropriate, but make sure to include the work to get there.

• It is OK to work with others, as long as you acknowledge them and write down your solutions in your own words. It is not OK to copy a solution from whatever source.

Part A

1. You are given the following matrices:

\[
A = \begin{bmatrix}
3 & 0 & -2 & 0 \\
1 & 1 & 1 & 1 \\
-4 & -2 & 0 & 1 \\
1 & -1 & -1 & 1 \\
0 & 8 & 0 & 1
\end{bmatrix}, \quad
B = \begin{bmatrix}
6 & 1 \\
1 & -1 \\
2 & 2 \\
3 & 2 \\
8 & 7
\end{bmatrix}
\]

Compute the only one of the following matrices that is actually well-defined:

\[
AB, \quad A^T B, \quad AB^T, \quad A^T B^T
\]

2. Find the inverse of each of the following matrices (note that they are elementary matrices):

a) \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{3} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

b) \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

3. Answer the following items.

a) Verify that the matrices

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
3 & -1 & 2 \\
6 & 1 & 2
\end{bmatrix}, \quad
B = \frac{1}{35} \begin{bmatrix}
-4 & -1 & 7 \\
6 & -16 & 7 \\
9 & 11 & -7
\end{bmatrix}
\]

are inverses of each other. Note that the scalar 1/35 can be brought all the way to the front of a matrix multiplication and only applied to the end result, for convenience.

b) Find a matrix \(X\) such that \(AX = C\), where \(C\) is given below. Keep fractions as fractions; don’t convert to decimals.

\[
C = \begin{bmatrix}
3 & 0 \\
0 & -1 \\
1 & 1
\end{bmatrix}
\]

*Hint: Is there a way to solve for \(X\) in the equation \(AX = C\)?*
4. Use the matrix inversion algorithm to find the inverse of $A$ (or show that it is not invertible):

$$A = \begin{bmatrix} -1 & -2 & -3 & -4 \\ 0 & 2 & 3 & 0 \\ 0 & 2 & 0 & -3 \\ 1 & -2 & 0 & 7 \end{bmatrix}$$

5. Find the RREF form $R$ of $A$ and an invertible matrix $P$ such that $R = PA$:

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 13 & 0 \\ 0 & 0 & 0 & 1 & -5 & -1 \\ 0 & 50 & 150 & 50 & -200 & -250 \\ 0 & 0 & 0 & 1 & -5 & -3 \end{bmatrix}$$

6. Write the following system in the form $Ax = b$ and use matrix inversion to solve it.

$$\begin{align*}
    x_1 + x_2 + x_3 &= 6 \\
    3x_1 + 2x_3 &= -1 \\
    2x_1 + x_2 + 2x_3 &= 1
\end{align*}$$

7. Find the general solution of this system using an LU decomposition of the coefficients matrix:

$$\begin{align*}
    x_1 + 2x_2 + 2x_3 + 2x_4 + x_5 &= 8 \\
    x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 5 \\
    2x_1 + 4x_2 + 2x_3 + x_4 &= 12 \\
    -3x_1 - 2x_2 - 3x_4 - 5x_5 &= -8
\end{align*}$$

8. Are the following matrices inverses of each other?

$$A = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ 5 & -4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

9. Suppose that a matrix $A$ has a total of 12 entries and a matrix $B$ has a total of 18 entries. Suppose that the product $AB$ is defined. What are the possible dimensions of $AB$? Hint: List all possible dimensions that $A$ and $B$ can have. Rule out a few based on the fact that $AB$ is defined.

10. We have learned that the motivation for the notion of matrix inverses is that, whenever we have an equation like $AB = AC$, we can cancel out the $A$, leaving $B = C$, as long as $A$ has an inverse. It turns out that $A$ having an inverse is important for this: consider

$$A = \begin{bmatrix} 2 & -3 & 6 \\ -4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

Verify that $AB = AC$ (but note that $B \neq C$). This is happening because $A$ is not invertible.

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**Part B**

11. Suppose that $A, B$ are matrices such that both the sum $A + B$ and the product $AB$ are defined. Does this imply that they must be square matrices? If yes, explain why. If no, give a counterexample.

12. When $u, v$ are vectors of the same dimension $m$, the matrix product $uv^T$ is defined and yields a square $m \times m$ matrix. It turns out that matrices of this form all have rank 1 (when $u, v \neq 0$). In this question, we verify this fact for a particular choice of $u, v$. **Come up with two 4D vectors $u, v \neq 0$ of your liking and use Gaussian elimination to compute the rank of $A = uv^T$.** But to make things interesting, please choose your vectors to be different from each other, with each having at most one 0 entry, all four entries distinct, and at least one positive and one negative entry.

13. The trace of a square matrix $A$, denoted $\text{Tr}[A]$, is by definition the sum of the elements along its main diagonal:

$$\text{Tr}[A] = a_{11} + a_{22} + \ldots + a_{mm} \quad \text{(where $A$ is an $m \times m$ matrix)}$$

Show that, for $A_{2 \times 2}$ and $B_{2 \times 2}$, we always have

$$\text{Tr}(AB) = \text{Tr}(BA)$$

Show this by giving general letter names to the entries of $A$ and $B$ and directly computing the traces of $AB$ and $BA$. (This property is also true in any number of dimensions, by the way).