Overview

This is not all the material that you must know for the midterm! It is only an overview of the main topics in integration, meant to clear up confusion. You should be aware of all other material that was covered too!

- **Double integrals:** \( \iint_{D} f(x, y) \, dA \)
  
  - The significance of this type of integral is the **volume under the surface** that is the graph of \( f \).
  
  - \( D \) is a region of the plane that needs to be described. If it is a rectangle or horizontally simple or vertically simple, the integral becomes respectively
    
    \[
    \int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy \quad \text{or} \quad \int_{a}^{b} \int_{x_{1}(y)}^{x_{2}(y)} f(x, y) \, dx \, dy \quad \text{or} \quad \int_{a}^{b} \int_{y_{1}(x)}^{y_{2}(x)} f(x, y) \, dy \, dx
    \]
  
  - The integral of a constant gives the **area** of the region times that constant:
    \[
    \iint_{D} C \, dA = C \cdot \text{Area}(D)
    \]
  
  - If the region can be easily described in polar coordinates, we need the Jacobian \( r \) after changing the variables. The integral will look like
    \[
    \int_{a}^{b} \int_{r_{1}(	heta)}^{r_{2}(	heta)} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta
    \]

- **Triple integrals:** \( \iiint_{\mathcal{R}} f(x, y, z) \, dV \)
  
  - \( \mathcal{R} \) is a region of the space that needs to be described. If it is a rectangular parallelepiped or vertically simple (between two graphs of functions of \((x, y)\) defined on some region \( D \) of the plane), the integral becomes respectively
    
    \[
    \int_{a}^{b} \int_{c}^{d} \int_{p}^{q} f(x, y, z) \, dx \, dy \, dz \quad \text{or} \quad \iiint_{D} \left( \int_{z_{1}(x,y)}^{z_{2}(x,y)} f(x, y, z) \, dz \right) \, dA
    \]
  
  - The integral of a constant gives the **volume** of the region times that constant:
    \[
    \iiint_{\mathcal{R}} C \, dV = C \cdot \text{Vol}(\mathcal{R})
    \]
  
  - If the region can be easily described in spherical coordinates, we need the Jacobian \( \rho^{2} \sin \phi \) after changing the variables. The integral will look like
    \[
    \int_{a}^{b} \int_{c}^{d} \int_{\rho_{1}(\theta, \phi)}^{\rho_{2}(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^{2} \sin \phi \, d\rho \, d\theta \, d\phi
    \]
• **Line integrals of vector fields:** \[ \int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{or} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} \]

  - \( \mathbf{F} \) is a vector field, either in the plane, \( \mathbf{F}(x, y) = (F_1(x, y), F_2(x, y)) \), or in space, \( \mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \). \( C \) is a curve, and the second symbol above is used when this curve is closed. The significance of this type of integral is the sum over the curve of the component of \( \mathbf{F} \) tangent to it.

  - The symbol \[ \int_C F_1 \, dx + F_2 \, dy \quad \text{or} \quad \oint_C F_1 \, dx + F_2 \, dy + F_3 \, dz \]

    means the same as \( \int_C \mathbf{F} \cdot d\mathbf{r} \). It **does not mean** an integral of the \( x \) derivative of \( F_1 \), plus \( y \) derivative of \( F_2 \), etc.

  - If we have a parametrization of \( C \) given by
    \[ \mathbf{r}(t) = (r_1(t), r_2(t)) \quad \text{or} \quad \mathbf{r}(t) = (r_1(t), r_2(t), r_3(t)) \quad , \quad a \leq t \leq b \]

    then we compute this integral by converting it into the following “Calc 1” integral:
    \[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \]

  - **Fundamental Theorem of Calculus:** When \( \mathbf{F} \) is conservative with potential \( f \) (that is, \( \nabla f = \mathbf{F} \)),
    \[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \]

    In particular, if the curve is also closed:
    \[ \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \]

  - A special type of **scalar line integral** is one that can be used to compute the **length** of the curve:
    \[ \text{Length}(C) = \int_a^b \| \mathbf{r}'(t) \| \, dt \]

• **Surface integrals of vector fields** (flux integrals): \[ \iint_S \mathbf{F} \cdot d\mathbf{S} \quad \text{or} \quad 
\oiint_S \mathbf{F} \cdot d\mathbf{S} \]

  - \( \mathbf{F} \) is a vector field \( \mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \). \( S \) is an oriented surface (a choice of normal vector \( \mathbf{N} \) has been made), and the second symbol above is used when this surface is closed. The significance of this type of integral is the flux (quantity crossing per unit area and unit time) through \( S \) of the quantity whose velocity field is given by \( \mathbf{F} \), in the direction specified by the normal vector \( \mathbf{N} \).

  - If we have a parametrization of \( S \) given by
    \[ \mathbf{G}(u, v) = (x(u, v), y(u, v), z(u, v)) \quad , \quad (u, v) \text{ in some region } \mathcal{R} \text{ of the plane} \]

    then we compute this integral by converting it into the following double integral:
    \[ \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{R}} \mathbf{F}(\mathbf{G}(u, v)) \cdot \mathbf{N}(u, v) \, dA \]

  - Before applying this formula, \( \mathbf{N} \) needs to be computed as a cross product of the two tangent vectors to the surface:
    \[ \mathbf{N}(u, v) = \pm \mathbf{T}_u(u, v) \times \mathbf{T}_v(u, v) \quad , \quad \mathbf{T}_u = \frac{\partial \mathbf{G}}{\partial u}, \quad \mathbf{T}_v = \frac{\partial \mathbf{G}}{\partial v} \]

  - A special type of **scalar surface integral** is one that can be used to compute the **area** of the surface:
    \[ \text{Area}(S) = \iint_{\mathcal{R}} \| \mathbf{N}(u, v) \| \, dA \]
• Vector Calculus operations:

  – **Gradient:** Transforms scalar into vector:
    \[
    \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \quad \text{or} \quad \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle
    \]
    This is like a *scalar product* between the symbolic vector \( \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \) and \( f \).

  – **Curl:** Transforms vector into vector; only defined in 3D:
    \[
    \text{curl} \mathbf{F} = \left| \begin{array}{ccc}
    i & j & k \\
    \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
    F_1 & F_2 & F_3
    \end{array} \right| = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle
    \]
    This is like a *cross product* between the symbolic vector \( \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \) and \( \mathbf{F} \).

  – **Divergence:** Transforms vector into scalar:
    \[
    \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad \text{or} \quad \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}
    \]
    This is like a *dot product* between the symbolic vector \( \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \) and \( \mathbf{F} \).

• **Green’s Theorem:** If \( D \) is a region on the plane with a boundary curve or curves \( \partial D \), and \( \mathbf{F} \) is a 2D vector field, then
  \[
  \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA
  \]
  – This theorem relates a line integral on a closed curve on the plane to the double integral, in the region inside, of a certain scalar expression constructed from the vector field.
  – The orientation of the curve \( \partial D \) must be such that, as you walk along the curve in the positive direction, the region \( D \) must lie to your left. So it goes **counterclockwise** if the boundary is exterior and **clockwise** if interior.
  – If \( \mathbf{F} \) is conservative, the expression \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \) is zero, so this theorem proves the fact that a 2D conservative vector field has line integral 0 along any closed curve.

• **Stokes’ Theorem:** If \( S \) is an oriented surface with a boundary curve or curves \( \partial S \), and \( \mathbf{F} \) is a 3D vector field, then
  \[
  \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}
  \]
  – This theorem relates a line integral on a closed curve to the surface integral, over any oriented surface whose boundary is that curve, of the curl of the vector field.
  – The orientation of the curve \( \partial S \) must be such that, as you imagine the vector \( \mathbf{N} \) walking along the curve in the positive direction, the surface \( S \) must lie to its left.
  – If \( \mathbf{F} \) is conservative, then \( \text{curl} \mathbf{F} = 0 \), so this theorem proves the fact that a conservative vector field has line integral 0 along any closed curve.
  – If \( S \) is a closed surface (no boundary), the theorem proves that there is no flux of the curl of any vector field through it.

• **Divergence Theorem:** If \( \mathcal{R} \) is a region in space with boundary surface \( \partial \mathcal{R} \), and \( \mathbf{F} \) is a 3D vector field, then
  \[
  \iiint_{\partial \mathcal{R}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{R}} \text{div} \mathbf{F} \, dV
  \]
  – This theorem relates a surface integral on a closed surface to the triple integral, in the region inside, of the divergence of the vector field.
The orientation of the surface $\partial R$ must be with an **outward-pointing** normal.

If $\mathbf{F}$ is the curl of some other vector field, then $\text{div}\mathbf{F} = 0$, so this theorem also proves the fact that there is no flux of the curl of any vector field through a closed surface.

- **Divergence Theorem in 2D (vector form of Green’s):** If $\mathcal{D}$ is a region on the plane with a boundary curve or curves $\partial \mathcal{D}$, and $\mathbf{F}$ is a 2D vector field, then
  \[
  \oint_{\partial \mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \text{div} \mathbf{F} \, dA
  \]

  - This theorem is a consequence of Green’s Theorem applied to the vector field being integrated on the curve, which is not $\langle F_1, F_2 \rangle$ but rather $\langle -F_2, F_1 \rangle$. Because this is perpendicular to the original $\mathbf{F}$, this line integral has the interpretation of the flux of $\mathbf{F}$ through the closed curve $\partial \mathcal{D}$, so that this theorem has the same meaning as the regular 3D divergence theorem: flux through a boundary is equal to integral of the divergence inside.

  - The orientation of the curve $\partial \mathcal{D}$ must be such that, as you walk along the curve in the positive direction, the region $\mathcal{D}$ must lie to your left. So it goes **counterclockwise** if the boundary is exterior and **clockwise** if interior.

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**Problems**

*These are not all the problems that you must practice for the midterm!* They are only a collection of some integration problems in a random order, without indication of which topic they pertain to, just like you’ll have it on the exam. You can find hints on the last page. But you should practice many more problems from the book too!

1. **Use Stokes’ Theorem** to compute the flux integral of the curl of the vector field $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$ across the triangle determined by the plane $4x + 2y + z = 8$ in the octant $x \geq 0, y \geq 0, z \geq 0$, with an upward-pointing normal.

2. Let $\mathcal{C}$ denote the piece of the curve $x = y^3 + 5$ from the point $(5, 0)$ to the point $(13, 2)$. Compute

   \[
   \int_{\mathcal{C}} y^2 dx + y(x - 5)dy
   \]

3. Suppose $\mathcal{C}$ is a closed curve in the plane, oriented counterclockwise, and call $\mathcal{R}$ the region that it encloses. Suppose that $\mathbf{F} = \langle F_1, F_2 \rangle$ is a vector field in the plane. What does Stokes’ Theorem have to say about the flux integral of the vector field $\langle F_1, F_2, 0 \rangle$ across the surface $\mathcal{R}$ with an upward pointing normal? What theorem does it prove?

4. Compute the flux integral of $\mathbf{F}(x, y, z) = \langle x + x^2yz, 2y + xy^2z, 3z - 2xyz^2 \rangle$ across the surface of the sphere of radius 7 centered at the origin, with an inward-pointing normal vector.

5. What is the volume of the region of space above the $xy$-plane and below the paraboloid $z = 9 - x^2 - y^2$?

6. Let $\mathcal{C}$ be the curve given by the four sides of the square with vertices $(0, 3, 0), (0, 3, 1), (1, 3, 1), (1, 3, 0)$, oriented clockwise as seen from the side of the origin. Compute

   \[
   \int_{\mathcal{C}} (3x^2 + z)dx + (x - z)dz
   \]

7. Let $S$ be the oriented surface parametrized by

   \[
   \mathbf{G}(u, v) = (u^2 + v^2, u + v, 2u)
   \]

   for $(u, v)$ belonging to the triangle where $u \geq 0, v \geq 0$ and $u + 2v \leq 2$, and with normal vector pointing in the negative $y$ direction. Compute the surface integral of $\mathbf{F}(x, y, z) = \langle y, x + z, x^2 \rangle$ across $S$. 

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8. Consider the vector field
\[ F(x, y) = \left( 2xe^{x^2-y^2}, -2ye^{x^2-y^2} \right) \]

The equation \( 2x^2 + y^2 = 7 \) describes an ellipse centered at the origin. Let \( C \) be the piece of this ellipse going from its leftmost point to its topmost point. Find \( \int_C F \cdot dr \).

9. Calculate:
\[ \int_{-3}^{0} \int_{y}^{0} y^3 \cos(x^5) \, dx \, dy \]

10. Suppose that \( C \) is a planar curve composed of 5 line segments: from \((0, 0)\) to \((2, 2)\), from there to \((2, 4)\), from there to \((5, 4)\), from there to \((3, 2)\), and finally from there to \((3, 0)\) (draw it!). How can we use Green's Theorem, even though \( C \) is not closed, to compute the following?
\[ \int_C (4x - y) \, dx + (3x + 2y) \, dy \]

11. Compute the flux of the vector field \( \langle 2x - y^2, 3y + x^2 \rangle \) across the circle of radius 5 centered at the origin.

12. Let \( C \) denote the curve in the plane that starts at the origin, follows the \( y \)-axis up to the point \((0, 1)\), then follows the circle of radius 1 down to the point \((-1, 0)\), then goes back to the origin through the \( x \)-axis (draw it!). Compute
\[ \int_C \left( x^2y + \frac{3}{2}x^2y^2 \cos(x^3) \right) \, dx + \left( \frac{2}{3}x^3 + y \sin(x^3) \right) \, dy \]

13. Find the integral of
\[ f(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \]
on the region of the plane that lies to the left of the line \( x = -1 \) and to the right of the circle \( x^2 + y^2 = 4 \).

14. Let \( S \) denote the part of the surface of the sphere \( x^2 + y^2 + z^2 = 1 \) that lies above the plane \( z = 1/2 \), with an upward-pointing normal. What is the flux of \( \langle 0, 0, 1 \rangle \) across \( S \)?

15. What is the area of the region of the plane between \( y = x \) and \( y = x^2 \)?

16. Use a triple integral to find the volume of a sphere of radius \( R \).

17. Use the Divergence Theorem to compute \( \iiint_{\mathcal{R}} \text{div} F \, dV \), where \( \mathcal{R} \) is the rectangular parallelepiped defined by \( 0 \leq x \leq 1, 0 \leq y \leq 2 \) and \( 0 \leq z \leq 3 \), and
\[ F = (x^2 + y^2, 2y^2 + 2z^2, 3x^2 + 3z^2) \]

18. Compute the surface area of the part of the paraboloid \( z = 9 - x^2 - y^2 \) that lies above the \( z = 0 \) plane.

19. Let \( \mathcal{R} \) denote the region of space below the plane \( z = -5\sqrt{2}/2 \) and above the sphere \( x^2 + y^2 + z^2 = 25 \). Find
\[ \iiint_{\mathcal{R}} \frac{1}{x^2 + y^2 + z^2} \, dV \]

20. Let \( \mathcal{R} \) be the region of the plane between the circles \( x^2 + y^2 = 4 \) and \( x^2 + y^2 = 9 \). Use Green's Theorem to compute
\[ \iint_{\mathcal{R}} x \, dA \]

*Hint: Green's requires a vector field, but we don't have one in this problem. Consider \( F(x, y) = \left( 0, \frac{1}{2}x^2 \right) \).*

What is \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \) for this \( F \)?
Main idea for each problem

1. You’ll need the line integral of \( \mathbf{F} \) on each of the 3 pieces of the boundary of the triangle.
2. Parametrize the curve and compute directly.
3. The line integral on the boundary becomes the line integral of \( \mathbf{F} \); the surface integral becomes the double integral of \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \). This proves Green’s theorem.
4. Use the Divergence Theorem.
5. Find the region \( R \) of the \( xy \)-plane that lies below this solid. The volume is a double integral over \( R \) of the function \( f(x, y) = 9 - x^2 - y^2 \).
6. Use Stokes’ Theorem.
7. Direct computation of a surface integral.
8. The vector field is conservative. Use the Fundamental Theorem of Calculus.
9. Switch the order to make it actually integrable.
10. By considering the line segment from \((3, 0)\) to \((0, 0)\), we have a closed curve. The line integral on this segment is easy to compute, and Green’s allows the computation of the line integral around the entire loop.
11. Use the 2D Divergence Theorem.
12. Use Green’s Theorem.
13. This is just a double integral. Use polar coordinates.
14. Direct computation of a surface integral. Must parametrize \( S \); do that by writing \( z \) in terms of \( x \) and \( y \).
15. Area is the double integral of 1 on that region.
16. Volume is the triple integral of 1 on that region.
17. You will need to compute the flux of \( \mathbf{F} \) across the boundary surface, which is made of 6 pieces. Each needs to be parametrized as a surface.
18. Parametrize the paraboloid and use the formula for area of a parametrized oriented surface.
19. This is just a triple integral. Use spherical coordinates.
20. Green’s says that the integral we want is the line integral of the given \( \mathbf{F} \) on the boundary. Be careful with the orientation of each piece.