Contents

1 M^2/x_n

For the rational difference equation

$$x_{n+1} = M^2/x_n$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{M\}$.
For the parameters $\{M = 651/100\}$: First we check that the equilibrium 651/100 is LAS.
The equilibrium $\bar{x} = 651/100$ is not LAS
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For the parameters $\{M = 983/100\}$: First we check that the equilibrium 983/100 is LAS.
The equilibrium $\bar{x} = 983/100$ is not LAS
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For the parameters $\{M = 49/20\}$: First we check that the equilibrium 49/20 is LAS.
The equilibrium $\bar{x} = 49/20$ is not LAS
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For the parameters $\{M = 427/50\}$: First we check that the equilibrium 427/50 is LAS.
The equilibrium $\bar{x} = 427/50$ is not LAS
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For the parameters $\{M = 43/25\}$: First we check that the equilibrium 43/25 is LAS.
The equilibrium $\bar{x} = 43/25$ is not LAS
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For the parameters $\{M = 251/100\}$: First we check that the equilibrium 251/100 is LAS.

The equilibrium $\bar{x} = 251/100$ is not LAS
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For the parameters $\{M = 9/25\}$: First we check that the equilibrium 9/25 is LAS.
The equilibrium $\bar{x} = 9/25$ is not LAS
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For the parameters $\{M = 41/100\}$: First we check that the equilibrium 41/100 is LAS.
The equilibrium $\bar{x} = 41/100$ is not LAS
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For the parameters $\{M = 319/100\}$: First we check that the equilibrium 319/100 is LAS.
The equilibrium $\bar{x} = 319/100$ is not LAS
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For the parameters $\{M = 7/20\}$: First we check that the equilibrium 7/20 is LAS.
The equilibrium $\bar{x} = 7/20$ is not LAS
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For the parameters $\{M = 193/100\}$: First we check that the equilibrium 193/100 is LAS.
The equilibrium $\bar{x} = 193/100$ is not LAS
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For the parameters $\{M = 163/25\}$: First we check that the equilibrium 163/25 is LAS.
The equilibrium $\bar{x} = 163/25$ is not LAS
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For the parameters $\{M = 43/5\}$: First we check that the equilibrium 43/5 is LAS.
The equilibrium $\bar{x} = 43/5$ is not LAS
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For the parameters $\{M = 177/50\}$: First we check that the equilibrium 177/50 is LAS.
The equilibrium $\bar{x} = 177/50$ is not LAS
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For the parameters $\{M = 283/100\}$: First we check that the equilibrium 283/100 is LAS.
The equilibrium $\bar{x} = 283/100$ is not LAS
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For the parameters $\{M = 463/100\}$: First we check that the equilibrium 463/100 is LAS.
The equilibrium $\bar{x} = 463/100$ is not LAS
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For the parameters $\{M = 33/10\}$: First we check that the equilibrium 33/10 is LAS.
The equilibrium $\bar{x} = 33/10$ is not LAS
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For the parameters $\{M = 143/100\}$: First we check that the equilibrium 143/100 is LAS.
The equilibrium $\bar{x} = 143/100$ is not LAS
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For the parameters $\{M = 353/50\}$: First we check that the equilibrium 353/50 is LAS.
The equilibrium $\bar{x} = 353/50$ is not LAS
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For the parameters $\{M = 1/50\}$: First we check that the equilibrium 1/50 is LAS.
The equilibrium $\bar{x} = 1/50$ is not LAS
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For the parameters $\{M = 823/100\}$: First we check that the equilibrium 823/100 is LAS.
The equilibrium $\bar{x} = 823/100$ is not LAS
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For the parameters $\{M = 29/4\}$: First we check that the equilibrium 29/4 is LAS.
The equilibrium $\bar{x} = 29/4$ is not LAS
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For the parameters $\{M = 343/100\}$: First we check that the equilibrium 343/100 is LAS.
The equilibrium $\bar{x} = 343/100$ is not LAS
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For the parameters $\{M = 33/20\}$: First we check that the equilibrium 33/20 is LAS.
The equilibrium $\bar{x} = 33/20$ is not LAS
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For the parameters $\{M = 9/5\}$: First we check that the equilibrium 9/5 is LAS.
The equilibrium $\bar{x} = 9/5$ is not LAS
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For the parameters $\{M = 693/100\}$: First we check that the equilibrium 693/100 is LAS.
The equilibrium $\bar{x} = 693/100$ is not LAS
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For the parameters $\{M = 24/25\}$: First we check that the equilibrium 24/25 is LAS.

The equilibrium $\bar{x} = 24/25$ is not LAS
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For the parameters $\{M = 219/100\}$: First we check that the equilibrium 219/100 is LAS.
The equilibrium $\bar{x} = 219/100$ is not LAS
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For the parameters $\{M = 1/2\}$: First we check that the equilibrium 1/2 is LAS.
The equilibrium $\bar{x} = 1/2$ is not LAS
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For the parameters $\{M = 709/100\}$: First we check that the equilibrium 709/100 is LAS.
The equilibrium $\bar{x} = 709/100$ is not LAS
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For the parameters $\{M = 209/50\}$: First we check that the equilibrium 209/50 is LAS.
The equilibrium $\bar{x} = 209/50$ is not LAS
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For the parameters $\{M = 13/20\}$: First we check that the equilibrium 13/20 is LAS.
The equilibrium $\bar{x} = 13/20$ is not LAS
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For the parameters $\{M = 217/100\}$: First we check that the equilibrium 217/100 is LAS.
The equilibrium $\bar{x} = 217/100$ is not LAS
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For the parameters $\{M = 147/100\}$: First we check that the equilibrium 147/100 is LAS.
The equilibrium $\bar{x} = 147/100$ is not LAS
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For the parameters $\{M = 67/50\}$: First we check that the equilibrium 67/50 is LAS.
The equilibrium $\bar{x} = 67/50$ is not LAS
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For the parameters $\{M = 361/50\}$: First we check that the equilibrium 361/50 is LAS.
The equilibrium $\bar{x} = 361/50$ is not LAS
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For the parameters $\{M = 803/100\}$: First we check that the equilibrium 803/100 is LAS.
The equilibrium $\bar{x} = 803/100$ is not LAS
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For the parameters $\{M = 897/100\}$: First we check that the equilibrium 897/100 is LAS.
The equilibrium $\bar{x} = 897/100$ is not LAS
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For the parameters $\{M = 38/5\}$: First we check that the equilibrium 38/5 is LAS.
The equilibrium $\bar{x} = 38/5$ is not LAS
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For the parameters $\{M = 663/100\}$: First we check that the equilibrium 663/100 is LAS.
The equilibrium $\bar{x} = 663/100$ is not LAS
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For the parameters $\{M = 57/25\}$: First we check that the equilibrium 57/25 is LAS.
The equilibrium $\bar{x} = 57/25$ is not LAS
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For the parameters $\{M = 158/25\}$: First we check that the equilibrium 158/25 is LAS.
The equilibrium $\bar{x} = 158/25$ is not LAS
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For the parameters $\{M = 431/50\}$: First we check that the equilibrium 431/50 is LAS.
The equilibrium $\bar{x} = 431/50$ is not LAS
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For the parameters $\{M = 681/100\}$: First we check that the equilibrium 681/100 is LAS.
The equilibrium $\bar{x} = 681/100$ is not LAS
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For the parameters $\{M = 71/10\}$: First we check that the equilibrium 71/10 is LAS.
The equilibrium $\bar{x} = 71/10$ is not LAS
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For the parameters $\{M = 761/100\}$: First we check that the equilibrium 761/100 is LAS.
The equilibrium $\bar{x} = 761/100$ is not LAS
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For the parameters $\{M = 46/5\}$: First we check that the equilibrium 46/5 is LAS.
The equilibrium $\bar{x} = 46/5$ is not LAS
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For the parameters $\{M = 23/50\}$: First we check that the equilibrium 23/50 is LAS.

The equilibrium $\bar{x} = 23/50$ is not LAS
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For the parameters $\{M = 441/50\}$: First we check that the equilibrium 441/50 is LAS.
The equilibrium $\bar{x} = 441/50$ is not LAS
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For the parameters $\{M = 657/100\}$: First we check that the equilibrium 657/100 is LAS.
The equilibrium $\bar{x} = 657/100$ is not LAS
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The parameter values for which the equilibrium is not LAS are:

$$[\{M = 1/2\}, 1/2], [\{M = 1/50\}, 1/50], [\{M = 7/20\}, 7/20], [\{M = 9/5\}, 9/5], [\{M = 9/25\}, 9/25], [\{M = 13/20\}, 13/20], [\{M = 23/50\}, 23/50], [\{M = 24/25\}, 24/25], [\{M = 29/4\}, 29/4], [\{M = 33/10\}, 33/10], [\{M = 33/20\}, 33/20], [\{M = 38/5\}, 38/5], [\{M = 41/100\}, 41/100], [\{M = 43/5\}, 43/5], [\{M = 43/25\}, 43/25], [\{M = 46/5\}, 46/5], [\{M = 49/20\}, 49/20], [\{M = 57/25\}, 57/25], [\{M = 67/50\}, 67/50], [\{M = 71/10\}, 71/10], [\{M = 143/100\}, 143/100], [\{M = 147/100\}, 147/100], [\{M = 158/25\}, 158/25], [\{M = 163/25\}, 163/25], [\{M = 177/50\}, 177/50], [\{M = 193/100\}, 193/100], [\{M = 209/50\}, 209/50], [\{M = 217/100\}, 217/100], [\{M = 219/100\}, 219/100], [\{M = 251/100\}, 251/100], [\{M = 283/100\}, 283/100], [\{M = 319/100\}, 319/100], [\{M = 343/100\}, 343/100], [\{M = 353/50\}, 353/50], [\{M = 361/50\}, 361/50], [\{M = 427/50\}, 427/50], [\{M = 431/50\}, 431/50], [\{M = 441/50\}, 441/50], [\{M = 463/100\}, 463/100], [\{M = 651/100\}, 651/100], [\{M = 657/100\}, 657/100], [\{M = 663/100\}, 663/100], [\{M = 681/100\}, 681/100], [\{M = 693/100\}, 693/100], [\{M = 709/100\}, 709/100], [\{M = 761/100\}, 761/100], [\{M = 803/100\}, 803/100], [\{M = 823/100\}, 823/100], [\{M = 897/100\}, 897/100], [\{M = 983/100\}, 983/100],$$

Finished investigating difference equation 1 out of 7

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2 $\beta \cdot x_n$

For the rational difference equation

$$x_{n+1} = \beta \cdot x_n$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{\beta\}$.
For the parameters $\{\beta = 529/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 471/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 87/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 317/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 691/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 107/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 191/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 441/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 601/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 487/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 107/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 9/20\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.

Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 2.1. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 9/20 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 319$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 9/20 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 9/20 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 0$$
$$N = 4$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 9/20 \cdot x_n$$

□

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For the parameters $\{\beta = 179/20\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 283/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 203/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 11/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 238/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 141/20\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 327/50\}$: First we check that the equilibrium 0 is LAS.

The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 91/20\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 34/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 21/100\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 2.2. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 21/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 9559$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 21/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 21/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 0$$
$$N = 4$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 21/100 \cdot x[n]$$

□

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For the parameters $\{\beta = 76/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 46/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 3\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 633/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 69/25\}$: First we check that the equilibrium 0 is LAS.

The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 877/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 61/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 463/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 113/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 533/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 189/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 409/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 823/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 67/100\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 2.3. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 67/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 5511$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 67/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 67/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 0$$
$$N = 4$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 67/100 \cdot x[n]$$

□

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For the parameters $\{\beta = 49/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 93/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 377/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 703/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 81/100\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 2.4. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 81/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 3439$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now

wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 81/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 81/100 \cdot z[n]\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 0 \\ N &= 4 \end{aligned}$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 81/100 \cdot x[n]$$

□

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For the parameters $\{\beta = 174/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 221/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
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For the parameters $\{\beta = 191/20\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 38/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 51/100\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 2.5. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 51/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 7399$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 51/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 51/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:
$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 0 \\ N &= 4 \end{aligned}$$
Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation
$$x_{n+1} = 51/100 \cdot x[n]$$
□

==

For the parameters $\{\beta = 117/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 999/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 983/100\}$: First we check that the equilibrium 0 is LAS.

The equilibrium $\bar{x} = 0$ is not LAS
==

For the parameters $\{\beta = 427/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
The parameter values for which the equilibrium is not LAS are:

$$[\{\beta = 3\}, 0], [\{\beta = 11/10\}, 0], [\{\beta = 34/25\}, 0], [\{\beta = 38/25\}, 0], [\{\beta = 46/25\}, 0], [\{\beta = 49/10\}, 0], [\{\beta = 61/10\}, 0], [\{\beta = 69/25\}, 0], [\{\beta = 76/25\}, 0], [\{\beta = 87/10\}, 0], [\{\beta = 91/20\}, 0], [\{\beta = 93/25\}, 0], [\{\beta = 107/50\}, 0], [\{\beta = 107/100\}, 0], [\{\beta = 113/50\}, 0], [\{\beta = 117/50\}, 0], [\{\beta = 141/20\}, 0], [\{\beta = 174/25\}, 0], [\{\beta = 179/20\}, 0], [\{\beta = 189/25\}, 0], [\{\beta = 191/20\}, 0], [\{\beta = 191/50\}, 0], [\{\beta = 203/100\}, 0], [\{\beta = 221/50\}, 0], [\{\beta = 238/25\}, 0], [\{\beta = 283/100\}, 0], [\{\beta = 317/100\}, 0], [\{\beta = 327/50\}, 0], [\{\beta = 377/50\}, 0], [\{\beta = 409/100\}, 0], [\{\beta = 427/50\}, 0], [\{\beta = 441/50\}, 0], [\{\beta = 463/100\}, 0], [\{\beta = 471/50\}, 0], [\{\beta = 487/50\}, 0], [\{\beta = 529/100\}, 0], [\{\beta = 533/100\}, 0], [\{\beta = 601/100\}, 0], [\{\beta = 633/100\}, 0], [\{\beta = 691/100\}, 0], [\{\beta = 703/100\}, 0], [\{\beta = 823/100\}, 0], [\{\beta = 877/100\}, 0], [\{\beta = 983/100\}, 0], [\{\beta = 999/100\}, 0],$$

The parameter values for which the $K = 1$ are:

$$[\{\beta = 9/20\}, 0], [\{\beta = 21/100\}, 0], [\{\beta = 51/100\}, 0], [\{\beta = 67/100\}, 0], [\{\beta = 81/100\}, 0],$$

Finished investigating difference equation 2 out of 7

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3 $(-1/4 + 1/4 \cdot M^2)/(1 + x_n)$

For the rational difference equation

$$x_{n+1} = (-1/4 + 1/4 \cdot M^2)/(1 + x_n)$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{M\}$.
For the parameters $\{M = 37/25\}$: First we check that the equilibrium 6/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 6/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.1. *The equilibrium 6/25 for the rational difference equation*

$$x_{n+1} = 186/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -36 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 6/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-6/25$,. The new difference equation is:

$$z[n+1] = -6 \cdot z[n]/(31 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -6 \cdot z[n]/(31 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -6/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -6/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 6/25$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 6/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 6/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$6/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 6/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 6/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/6 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/6$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 186/625/(1 + x_n)$$

□

==

For the parameters $\{M = 217/25\}$: First we check that the equilibrium 96/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 96/25$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 96/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.2. *The equilibrium 96/25 for the rational difference equation*

$$x_{n+1} =, 11616/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -84934656 + (12241 + 625 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 96/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-96/25$,. The new difference equation is:

$$z[n + 1] = -96 \cdot z[n]/(121 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -96 \cdot z[n]/(121 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -96/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -96/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-96/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 96/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 96/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$96/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 96/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 96/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/96 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/96$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 11616/625/(1 + x_n)$$

□

==

For the parameters $\{M = 711/100\}$: First we check that the equilibrium 611/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 611/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 611/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.3. *The equilibrium 611/200, for the rational difference equation*

$$x_{n+1} = 495521/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -139368569041 + (535521 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 611/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-611/200$,. The new difference equation is:

$$z[n+1] = -611 \cdot z[n]/(811 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -611 \cdot z[n]/(811 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -611/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -611/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 611/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 611/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 611/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$611/200 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 611/200$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 611/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/611 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/611$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 495521/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 227/25\}$: First we check that the equilibrium 101/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 101/25$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 101/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.4. *The equilibrium 101/25 for the rational difference equation*

$$x_{n+1} = \ 12726/625/(1+x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -104060401 + (13351 + 625 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 101/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-101/25$,. The new difference equation is:

$$z[n+1] = -101 \cdot z[n]/(126 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -101 \cdot z[n]/(126 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -101/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -101/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 101/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 101/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 101/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$101/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 101/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 101/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/101 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/101$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 12726/625/(1 + x_n)$$

□

==

For the parameters $\{M = 709/100\}$: First we check that the equilibrium 609/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 609/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 609/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.5. *The equilibrium 609/200, for the rational difference equation*

$$x_{n+1} = 492681/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -137552716161 + (532681 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 609/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-609/200$,. The new difference equation is:

$$z[n + 1] = -609 \cdot z[n]/(809 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -609 \cdot z[n]/(809 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -609/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -609/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 609/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 609/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 609/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$609/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 609/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 609/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/609 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/609$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 492681/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 969/100\}$: First we check that the equilibrium 869/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 869/200$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 869/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.6. *The equilibrium 869/200, for the rational difference equation*

$$x_{n+1} = 928961/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -570268135921 + (968961 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 869/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-869/200$,. The new difference equation is:

$$z[n+1] = -869 \cdot z[n]/(1069 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -869 \cdot z[n]/(1069 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -869/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -869/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 869/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 869/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 869/200$$
$$N = 4$$

Proving $P > 0$ in the region:
$$869/200 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 869/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 869/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/869 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/869$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 928961/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 108/25\}$: First we check that the equilibrium 83/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 83/50$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 83/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.7. *The equilibrium 83/50 for the rational difference equation*

$$x_{n+1} =, 11039/2500/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -47458321 + (13539 + 2500 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 83/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-83/50$,. The new difference equation is:

$$z[n + 1] = -83 \cdot z[n]/(133 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -83 \cdot z[n]/(133 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -83/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -83/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 83/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 83/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 83/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$83/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 83/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 83/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/83 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/83$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 11039/2500/(1 + x_n)$$

□

==

For the parameters $\{M = 149/50\}$: First we check that the equilibrium 99/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 99/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.8. *The equilibrium 99/100 for the rational difference equation*

$$x_{n+1} = \ 19701/10000/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -9801 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 99/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-99/100$,. The new difference equation is:

$$z[n+1] = -99 \cdot z[n]/(199 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -99 \cdot z[n]/(199 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -99/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -99/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 99/100$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 99/100, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 99/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$99/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 99/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 99/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/99 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/99$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 19701/10000/(1 + x_n)$$

□

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For the parameters $\{M = 217/50\}$: First we check that the equilibrium 167/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 167/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 167/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.9. *The equilibrium 167/100, for the rational difference equation*

$$x_{n+1} = 44589/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -777796321 + (54589 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 167/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-167/100$,. The new difference equation is:

$$z[n + 1] = -167 \cdot z[n]/(267 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -167 \cdot z[n]/(267 + 100 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -167/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -167/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 167/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 167/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 167/100$$
$$N = 4$$

Proving $P > 0$ in the region:
$$167/100 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 167/100$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 167/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/167 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/167$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 44589/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 193/100\}$: First we check that the equilibrium 93/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 93/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.10. *The equilibrium 93/200 for the rational difference equation*

$$x_{n+1} = 27249/40000/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -8649 + (200 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 93/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-93/200$,. The new difference equation is:

$$z[n+1] = -93 \cdot z[n]/(293 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -93 \cdot z[n]/(293 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -93/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -93/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 93/200$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 93/200, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 93/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$93/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 93/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 93/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/93 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/93$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 27249/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 121/25\}$: First we check that the equilibrium 48/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 48/25$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 48/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.11. *The equilibrium 48/25 for the rational difference equation*

$$x_{n+1} =, 3504/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -5308416 + (4129 + 625 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 48/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-48/25$,. The new difference equation is:

$$z[n + 1] = -48 \cdot z[n]/(73 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -48 \cdot z[n]/(73 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -48/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -48/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-48/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 48/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 48/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$48/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 48/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 48/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/48 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/48$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 3504/625/(1 + x_n)$$

□

==

For the parameters $\{M = 341/50\}$: First we check that the equilibrium 291/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 291/100$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 291/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.12. *The equilibrium 291/100, for the rational difference equation*

$$x_{n+1} = 113781/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -7170871761 + (123781 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 291/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-291/100$,. The new difference equation is:

$$z[n + 1] = -291 \cdot z[n]/(391 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -291 \cdot z[n]/(391 + 100 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -291/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -291/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 291/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 291/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 291/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$291/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 291/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 291/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/291 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/291$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 113781/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 33/25\}$: First we check that the equilibrium 4/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 4/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.13. *The equilibrium 4/25 for the rational difference equation*

$$x_{n+1} = 116/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -16 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 4/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-4/25$,. The new difference equation is:

$$z[n+1] = -4 \cdot z[n]/(29 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -4 \cdot z[n]/(29 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -4/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -4/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 4/25$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 4/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 4/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$4/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 4/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 4/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/4 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/4$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 116/625/(1 + x_n)$$

□

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For the parameters $\{M = 238/25\}$: First we check that the equilibrium 213/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 213/50$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 213/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.14. *The equilibrium 213/50 for the rational difference equation*

$$x_{n+1} = \ 56019/2500/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -2058346161 + (58519 + 2500 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 213/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-213/50$,. The new difference equation is:

$$z[n + 1] = -213 \cdot z[n]/(263 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -213 \cdot z[n]/(263 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -213/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -213/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 213/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 213/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 213/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$213/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 213/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 213/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/213 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/213$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 56019/2500/(1 + x_n)$$

□

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For the parameters $\{M = 91/50\}$: First we check that the equilibrium 41/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 41/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.15. *The equilibrium 41/100 for the rational difference equation*

$$x_{n+1} = 5781/10000/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1681 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 41/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-41/100$,. The new difference equation is:

$$z[n + 1] = -41 \cdot z[n]/(141 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -41 \cdot z[n]/(141 + 100 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -41/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -41/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 41/100$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 41/100, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 41/100$$
$$N = 4$$

Proving $P > 0$ in the region:
$$41/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 41/100$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 41/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/41 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/41$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 5781/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 177/25\}$: First we check that the equilibrium 76/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 76/25$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 76/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.16. *The equilibrium 76/25 for the rational difference equation*

$$x_{n+1} =, 7676/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -33362176 + (625 \cdot z[1] + 8301)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 76/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-76/25$,. The new difference equation is:

$$z[n+1] = -76 \cdot z[n]/(25 \cdot z[n] + 101)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -76 \cdot z[n]/(25 \cdot z[n] + 101) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -76/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -76/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 76/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 76/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 76/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$76/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 76/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 76/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/76 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/76$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 7676/625/(1 + x_n)$$

□

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For the parameters $\{M = 373/100\}$: First we check that the equilibrium 273/200 is LAS. It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 273/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 273/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.17. *The equilibrium 273/200, for the rational difference equation*

$$x_{n+1} = 129129/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -5554571841 + (169129 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 273/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-273/200$,. The new difference equation is:

$$z[n + 1] = -273 \cdot z[n]/(473 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -273 \cdot z[n]/(473 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -273/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -273/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 273/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 273/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 273/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$273/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 273/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 273/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/273 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/273$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 129129/40000/(1 + x_n)$$

□

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For the parameters $\{M = 447/100\}$: First we check that the equilibrium 347/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 347/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 347/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.18. *The equilibrium 347/200, for the rational difference equation*

$$x_{n+1} = 189809/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -14498327281 + (229809 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 347/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-347/200$,. The new difference equation is:

$$z[n+1] = -347 \cdot z[n]/(547 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -347 \cdot z[n]/(547 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -347/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -347/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 347/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 347/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 347/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$347/200 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 347/200$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 347/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/347 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/347$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 189809/40000/(1 + x_n)$$

□

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For the parameters $\{M = 184/25\}$: First we check that the equilibrium 159/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 159/50$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 159/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.19. *The equilibrium 159/50 for the rational difference equation*

$$x_{n+1} = \ 33231/2500/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -639128961 + (35731 + 2500 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 159/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-159/50$,. The new difference equation is:

$$z[n+1] = -159 \cdot z[n]/(209 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -159 \cdot z[n]/(209 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -159/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -159/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 159/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 159/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 159/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$159/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 159/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 159/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/159 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/159$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 33231/2500/(1 + x_n)$$

□

==

For the parameters $\{M = 697/100\}$: First we check that the equilibrium 597/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 597/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 597/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.20. *The equilibrium 597/200, for the rational difference equation*

$$x_{n+1} = 475809/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -127027375281 + (515809 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 597/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-597/200$,. The new difference equation is:

$$z[n + 1] = -597 \cdot z[n]/(797 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -597 \cdot z[n]/(797 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -597/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -597/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 597/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 597/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 597/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$597/200 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 597/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 597/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/597 \le z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/597$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 475809/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 451/50\}$: First we check that the equilibrium 401/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 401/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 401/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.21. *The equilibrium 401/100, for the rational difference equation*

$$x_{n+1} = 200901/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -25856961601 + (210901 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 401/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-401/100$,. The new difference equation is:

$$z[n + 1] = -401 \cdot z[n]/(501 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -401 \cdot z[n]/(501 + 100 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -401/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -401/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 401/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 401/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 401/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$401/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 401/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 401/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/401 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/401$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 200901/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 97/10\}$: First we check that the equilibrium 87/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 87/20$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 87/20$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.22. *The equilibrium 87/20 for the rational difference equation*

$$x_{n+1} =, 9309/400/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -57289761 + (9709 + 400 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 87/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-87/20$,. The new difference equation is:
$$z[n+1] = -87 \cdot z[n]/(107 + 20 \cdot z[n])$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -87 \cdot z[n]/(107 + 20 \cdot z[n])\rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -87/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -87/20 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 87/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 87/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 87/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$87/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 87/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 87/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/87 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/87$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 9309/400/(1 + x_n)$$

□

==

For the parameters $\{M = 43/10\}$: First we check that the equilibrium 33/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 33/20$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 33/20$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.23. *The equilibrium 33/20 for the rational difference equation*

$$x_{n+1} =, 1749/400/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -1185921 + (2149 + 400 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 33/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-33/20$,. The new difference equation is:

$$z[n + 1] = -33 \cdot z[n]/(53 + 20 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -33 \cdot z[n]/(53 + 20 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -33/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -33/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-33/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 33/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 33/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$33/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 33/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 33/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/33 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/33$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 1749/400/(1 + x_n)$$

□

==

For the parameters $\{M = 489/50\}$: First we check that the equilibrium 439/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 439/100$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 439/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.24. *The equilibrium 439/100, for the rational difference equation*

$$x_{n+1} = 236621/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -37141383841 + (246621 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 439/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-439/100$,. The new difference equation is:

$$z[n+1] = -439 \cdot z[n]/(539 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -439 \cdot z[n]/(539 + 100 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -439/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -439/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 439/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 439/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 439/100$$
$$N = 4$$

Proving $P > 0$ in the region:
$$439/100 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 439/100$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 439/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/439 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/439$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 236621/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 23/20\}$: First we check that the equilibrium 3/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 3/40$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.25. *The equilibrium 3/40 for the rational difference equation*

$$x_{n+1} = 129/1600/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -9 + (40 + 40 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 3/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-3/40$,. The new difference equation is:
$$z[n+1] = -3 \cdot z[n]/(43 + 40 \cdot z[n])$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -3 \cdot z[n]/(43 + 40 \cdot z[n]) \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -3/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -3/40 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 3/40$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 3/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 3/40$$
$$N = 4$$

Proving $P > 0$ in the region:
$$3/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 3/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 3/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/3 \le z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/3$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 129/1600/(1 + x_n)$$

□

==

For the parameters $\{M = 987/100\}$: First we check that the equilibrium 887/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 887/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 887/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.26. *The equilibrium 887/200, for the rational difference equation*

$$x_{n+1} = 964169/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -619005459361 + (1004169 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 887/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-887/200$,. The new difference equation is:

$$z[n + 1] = -887 \cdot z[n]/(1087 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -887 \cdot z[n]/(1087 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -887/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -887/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 887/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 887/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 887/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$887/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 887/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 887/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/887 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/887$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 964169/40000/(1 + x_n)$$

□

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For the parameters $\{M = 169/20\}$: First we check that the equilibrium 149/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 149/40$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 149/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.27. *The equilibrium 149/40 for the rational difference equation*

$$x_{n+1} = 28161/1600/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -492884401 + (29761 + 1600 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 149/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-149/40$,. The new difference equation is:

$$z[n+1] = -149 \cdot z[n]/(189 + 40 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -149 \cdot z[n]/(189 + 40 \cdot z[n])\rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -149/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -149/40 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 149/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 149/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 149/40 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$149/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 149/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 149/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/149 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/149$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 28161/1600/(1 + x_n)$$

□

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For the parameters $\{M = 222/25\}$: First we check that the equilibrium 197/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 197/50$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 197/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.28. *The equilibrium 197/50 for the rational difference equation*

$$x_{n+1} = \ 48659/2500/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -1506138481 + (51159 + 2500 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 197/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-197/50$,. The new difference equation is:

$$z[n + 1] = -197 \cdot z[n]/(247 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -197 \cdot z[n]/(247 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -197/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -197/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 197/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 197/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 197/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$197/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 197/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 197/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/197 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/197$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 48659/2500/(1 + x_n)$$

$\square$

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For the parameters $\{M = 307/50\}$: First we check that the equilibrium 257/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 257/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 257/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.29. *The equilibrium 257/100, for the rational difference equation*

$$x_{n+1} = 91749/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -4362470401 + (101749 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 257/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-257/100$,. The new difference equation is:

$$z[n + 1] = -257 \cdot z[n]/(357 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -257 \cdot z[n]/(357 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -257/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -257/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 257/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 257/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 257/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$257/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 257/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 257/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/257 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/257$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 91749/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 81/20\}$: First we check that the equilibrium 61/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 61/40$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 61/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.30. *The equilibrium 61/40 for the rational difference equation*

$$x_{n+1} =, 6161/1600/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -13845841 + (7761 + 1600 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 61/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-61/40$,. The new difference equation is:

$$z[n + 1] = -61 \cdot z[n]/(101 + 40 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -61 \cdot z[n]/(101 + 40 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -61/40$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -61/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 61/40$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 61/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 61/40 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$61/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 61/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 61/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/61 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/61$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 6161/1600/(1 + x_n)$$

□

==

For the parameters $\{M = 72/25\}$: First we check that the equilibrium 47/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 47/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.31. *The equilibrium 47/50 for the rational difference equation*

$$x_{n+1} =, 4559/2500/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2209 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 47/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-47/50$,. The new difference equation is:

$$z[n+1] = -47 \cdot z[n]/(97 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -47 \cdot z[n]/(97 + 50 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -47/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -47/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 47/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 47/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 47/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$47/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 47/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 47/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/47 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/47$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 4559/2500/(1 + x_n)$$

□

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For the parameters $\{M = 347/50\}$: First we check that the equilibrium 297/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 297/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 297/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.32. *The equilibrium 297/100, for the rational difference equation*

$$x_{n+1} = 117909/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -7780827681 + (127909 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 297/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-297/100$,. The new difference equation is:

$$z[n + 1] = -297 \cdot z[n]/(397 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -297 \cdot z[n]/(397 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -297/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -297/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 297/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 297/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 297/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$297/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 297/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 297/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/297 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/297$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 117909/10000/(1 + x_n)$$

□

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For the parameters $\{M = 387/100\}$: First we check that the equilibrium 287/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 287/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 287/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.33. *The equilibrium 287/200, for the rational difference equation*

$$x_{n+1} = 139769/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -6784652161 + (179769 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 287/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-287/200$,. The new difference equation is:

$$z[n + 1] = -287 \cdot z[n]/(487 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -287 \cdot z[n]/(487 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -287/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -287/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 287/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 287/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 287/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$287/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 287/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 287/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/287 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/287$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 139769/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 133/25\}$: First we check that the equilibrium 54/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 54/25$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 54/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.34. *The equilibrium 54/25 for the rational difference equation*

$$x_{n+1} =, 4266/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -8503056 + (4891 + 625 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 54/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-54/25$,. The new difference equation is:

$$z[n+1] = -54 \cdot z[n]/(79 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -54 \cdot z[n]/(79 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -54/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -54/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 54/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 54/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 54/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$54/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 54/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 54/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/54 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/54$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 4266/625/(1 + x_n)$$

□

==

For the parameters $\{M = 15/2\}$: First we check that the equilibrium 13/4 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 13/4$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 13/4$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.35. *The equilibrium 13/4 for the rational difference equation*

$$x_{n+1} = 221/16/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -28561 + (237 + 16 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 13/4$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-13/4$,. The new difference equation is:

$$z[n + 1] = -13 \cdot z[n]/(17 + 4 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -13 \cdot z[n]/(17 + 4 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -13/4$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -13/4 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 13/4$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 13/4, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 13/4$$
$$N = 4$$

Proving $P > 0$ in the region:

$$13/4 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 13/4$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 13/4$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P, z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$4/13 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 4/13$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 221/16/(1 + x_n)$$

□

==

For the parameters $\{M = 767/100\}$: First we check that the equilibrium 667/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 667/200$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 667/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.36. *The equilibrium 667/200, for the rational difference equation*

$$x_{n+1} = 578289/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -197926222321 + (618289 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 667/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-667/200$,. The new difference equation is:

$$z[n+1] = -667 \cdot z[n]/(867 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -667 \cdot z[n]/(867 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -667/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -667/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 667/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 667/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:
$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 667/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$667/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 667/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 667/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/667 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/667$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 578289/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 527/100\}$: First we check that the equilibrium 427/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 427/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 427/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.37. *The equilibrium 427/200, for the rational difference equation*

$$x_{n+1} = 267729/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -33243864241 + (307729 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 427/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-427/200$,. The new difference equation is:

$$z[n + 1] = -427 \cdot z[n]/(627 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -427 \cdot z[n]/(627 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -427/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -427/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 427/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 427/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 427/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$427/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 427/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 427/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/427 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/427$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 267729/40000/(1 + x_n)$$

□

==

For the parameters $\{M = 483/50\}$: First we check that the equilibrium 433/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 433/100$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 433/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.38. *The equilibrium 433/100, for the rational difference equation*

$$x_{n+1} = 230789/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -35152125121 + (240789 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 433/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-433/100$,. The new difference equation is:

$$z[n + 1] = -433 \cdot z[n]/(533 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -433 \cdot z[n]/(533 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -433/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -433/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 433/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 433/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 433/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$433/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 433/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 433/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/433 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/433$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 230789/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 201/25\}$: First we check that the equilibrium 88/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 88/25$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 88/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.39. *The equilibrium 88/25 for the rational difference equation*

$$x_{n+1} =, 9944/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -59969536 + (10569 + 625 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 88/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-88/25$,. The new difference equation is:

$$z[n + 1] = -88 \cdot z[n]/(113 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -88 \cdot z[n]/(113 + 25 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -88/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -88/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 88/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 88/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 88/25$$
$$N = 4$$

Proving $P > 0$ in the region:
$$88/25 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 88/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 88/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/88 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/88$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 9944/625/(1 + x_n)$$

□

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For the parameters $\{M = 89/10\}$: First we check that the equilibrium 79/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 79/20$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 79/20$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.40. *The equilibrium 79/20 for the rational difference equation*

$$x_{n+1} =, 7821/400/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -38950081 + (8221 + 400 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 79/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-79/20$,. The new difference equation is:

$$z[n + 1] = -79 \cdot z[n]/(99 + 20 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -79 \cdot z[n]/(99 + 20 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -79/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -79/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 79/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 79/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 79/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$79/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 79/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 79/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/79 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/79$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 7821/400/(1 + x_n)$$

□

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For the parameters $\{M = 162/25\}$: First we check that the equilibrium 137/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 137/50$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 137/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.41. *The equilibrium 137/50 for the rational difference equation*

$$x_{n+1} = \ 25619/2500/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -352275361 + (28119 + 2500 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 137/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-137/50$,. The new difference equation is:

$$z[n + 1] = -137 \cdot z[n]/(187 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -137 \cdot z[n]/(187 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -137/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -137/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 137/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 137/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 137/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$137/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 137/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 137/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/137 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/137$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 25619/2500/(1 + x_n)$$

□

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For the parameters $\{M = 59/25\}$: First we check that the equilibrium 17/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 17/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.42. *The equilibrium 17/25 for the rational difference equation*

$$x_{n+1} =, 714/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -289 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 17/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-17/25$,. The new difference equation is:

$$z[n + 1] = -17 \cdot z[n]/(42 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -17 \cdot z[n]/(42 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -17/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -17/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 17/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 17/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 17/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$17/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 17/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 17/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/17 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/17$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 714/625/(1 + x_n)$$

□

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For the parameters $\{M = 947/100\}$: First we check that the equilibrium 847/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 847/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 847/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.43. *The equilibrium 847/200, for the rational difference equation*

$$x_{n+1} = 886809/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -514675673281 + (926809 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 847/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-847/200$,. The new difference equation is:

$$z[n+1] = -847 \cdot z[n]/(1047 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -847 \cdot z[n]/(1047 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left| Q^K(\langle z[1] \rangle) \right|^2}{\left| \langle z[1], \rangle \right|^2} < 1$$

for all $z[1] > -847/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left| \langle z[1] \rangle \right|^2 - \left| Q^K(\langle z[1], \rangle) \right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -847/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 847/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 847/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 847/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$847/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 847/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 847/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/847 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/847$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 886809/40000/(1 + x_n)$$

□

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For the parameters $\{M = 189/100\}$: First we check that the equilibrium 89/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 89/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.44. *The equilibrium 89/200 for the rational difference equation*

$$x_{n+1} = \ 25721/40000/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -7921 + (200 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 89/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-89/200$,. The new difference equation is:

$$z[n + 1] = -89 \cdot z[n]/(289 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -89 \cdot z[n]/(289 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -89/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -89/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 89/200$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 89/200, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 89/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$89/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 89/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 89/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/89 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/89$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 25721/40000/(1 + x_n)$$

□

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For the parameters $\{M = 601/100\}$: First we check that the equilibrium 501/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 501/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 501/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.45. *The equilibrium 501/200, for the rational difference equation*

$$x_{n+1} = 351201/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -63001502001 + (391201 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 501/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-501/200$,. The new difference equation is:

$$z[n + 1] = -501 \cdot z[n]/(701 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -501 \cdot z[n]/(701 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -501/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -501/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 501/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 501/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 501/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$501/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 501/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 501/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/501 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/501$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 351201/40000/(1 + x_n)$$

□

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For the parameters $\{M = 227/50\}$: First we check that the equilibrium 177/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 177/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 177/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.46. *The equilibrium 177/100, for the rational difference equation*

$$x_{n+1} = 49029/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -981506241 + (59029 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 177/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-177/100$,. The new difference equation is:

$$z[n+1] = -177 \cdot z[n]/(277 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -177 \cdot z[n]/(277 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -177/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -177/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 177/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 177/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 177/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$177/100 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 177/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 177/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/177 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/177$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 49029/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 413/50\}$: First we check that the equilibrium 363/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 363/100$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 363/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.47. *The equilibrium 363/100, for the rational difference equation*

$$x_{n+1} = 168069/10000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -17363069361 + (178069 + 10000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 363/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-363/100$,. The new difference equation is:

$$z[n + 1] = -363 \cdot z[n]/(463 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -363 \cdot z[n]/(463 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -363/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -363/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 363/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 363/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 363/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$363/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 363/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 363/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/363 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/363$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 168069/10000/(1 + x_n)$$

□

==

For the parameters $\{M = 941/100\}$: First we check that the equilibrium 841/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 841/200$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 841/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.48. *The equilibrium 841/200, for the rational difference equation*

$$x_{n+1} = 875481/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -500246412961 + (915481 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 841/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-841/200$,. The new difference equation is:

$$z[n + 1] = -841 \cdot z[n]/(1041 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -841 \cdot z[n]/(1041 + 200 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -841/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -841/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 841/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 841/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 841/200$$
$$N = 4$$

Proving $P > 0$ in the region:
$$841/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 841/200$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 841/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/841 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/841$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 875481/40000/(1 + x_n)$$

□

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For the parameters $\{M = 41/25\}$: First we check that the equilibrium 8/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 8/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.49. *The equilibrium 8/25 for the rational difference equation*

$$x_{n+1} = 264/625/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -64 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 8/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-8/25$,. The new difference equation is:
$$z[n+1] = -8 \cdot z[n]/(33 + 25 \cdot z[n])$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -8 \cdot z[n]/(33 + 25 \cdot z[n]) \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -8/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -8/25 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 8/25$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 8/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 8/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$8/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 8/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 8/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/8 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/8$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 264/625/(1 + x_n)$$

□

===

For the parameters $\{M = 869/100\}$: First we check that the equilibrium 769/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 769/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 769/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 3.50. *The equilibrium 769/200, for the rational difference equation*

$$x_{n+1} = 745161/40000/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -349707832321 + (785161 + 40000 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 769/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-769/200$,. The new difference equation is:

$$z[n + 1] = -769 \cdot z[n]/(969 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -769 \cdot z[n]/(969 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -769/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -769/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 769/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 769/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 769/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$769/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 769/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 769/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/769 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/769$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 745161/40000/(1 + x_n)$$

□

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The parameter values for which the $K = 1$ are:

$$[\{M = 23/20\}, 3/40], [\{M = 33/25\}, 4/25], [\{M = 37/25\}, 6/25], [\{M = 41/25\}, 8/25], [\{M = 59/25\}, 17/25], [\{M = 72/25\}, 47/50], [\{M = 91/50\}, 41/100], [\{M = 149/50\}, 99/100], [\{M = 189/100\}, 89/200], [\{M = 193/100\}, 93/200],$$

The parameter values for which the $K = 2$ are:

$$[\{M = 15/2\}, 13/4], [\{M = 43/10\}, 33/20], [\{M = 81/20\}, 61/40], [\{M = 89/10\}, 79/20], [\{M = 97/10\}, 87/20], [\{M = 108/25\}, 83/50], [\{M = 121/25\}, 48/25], [\{M = 133/25\}, 54/25], [\{M = 162/25\}, 137/50], [\{M = 169/20\}, 149/40], [\{M = 177/25\}, 76/25], [\{M = 184/25\}, 159/50], [\{M = 201/25\}, 88/25], [\{M = 217/25\}, 96/25], [\{M = 217/50\}, 167/100], [\{M = 222/25\}, 197/50], [\{M = 227/25\}, 101/25], [\{M = 227/50\}, 177/100], [\{M = 238/25\}, 213/50], [\{M = 307/50\}, 257/100], [\{M = 341/50\}, 291/100], [\{M = 347/50\}, 297/100], [\{M = 373/100\}, 273/200], [\{M = 387/100\}, 287/200], [\{M = 413/50\}, 363/100], [\{M = 447/100\}, 347/200], [\{M = 451/50\}, 401/100], [\{M = 483/50\}, 433/100], [\{M = 489/50\}, 439/100], [\{M = 527/100\}, 427/200], [\{M = 601/100\}, 501/200], [\{M = 697/100\}, 597/200], [\{M = 709/100\}, 609/200], [\{M = 711/100\}, 611/200], [\{M = 767/100\}, 667/200], [\{M = 869/100\}, 769/200], [\{M = 941/100\}, 841/200], [\{M = 947/100\}, 847/200], [\{M = 969/100\}, 869/200], [\{M = 987/100\}, 887/200],$$

Finished investigating difference equation 3 out of 7

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4 $\beta \cdot x_n/(1 + x_n)$

For the rational difference equation

$$x_{n+1} = \beta \cdot x_n/(1 + x_n)$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{\beta\}$.
For the parameters $\{\beta = 199/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 149/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 149/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.1. *The equilibrium 149/50 for the rational difference equation*

$$x_{n+1} = \ 199/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 149/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-149/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(199 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(199 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -149/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -149/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 149/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 149/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 149/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$149/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 149/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 149/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/149 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/149$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 199/50 \cdot x[n]/(1 + x_n)$$

□

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For the parameters $\{\beta = 56/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 31/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 31/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.2. *The equilibrium 31/25 for the rational difference equation*

$$x_{n+1} =, 56/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 31/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-31/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(56 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(56 + 25 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -31/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -31/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 31/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 31/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 31/25$$
$$N = 4$$

Proving $P > 0$ in the region:
$$31/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 31/25$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 31/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/31 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/31$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 56/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 83/100\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.3. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 83/100 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -6889 + 10000 \cdot (1 + z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 83/100 \cdot z[n]/(1 + z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 83/100 \cdot z[n]/(1 + z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

   for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

   and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 0 \\ N &= 4 \end{aligned}$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since

$P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 83/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 547/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 447/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 447/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.4. *The equilibrium 447/100, for the rational difference equation*

$$x_{n+1} = 547/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 447/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-447/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(547 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(547 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -447/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -447/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 447/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 447/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 447/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$447/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 447/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 447/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/447 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/447$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 547/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 703/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 603/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 603/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.5. *The equilibrium 603/100, for the rational difference equation*

$$x_{n+1} = 703/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 603/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-603/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(703 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(703 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -603/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -603/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 603/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 603/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 603/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$603/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 603/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 603/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/603 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/603$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 703/100 \cdot x[n]/(1 + x_n)$$

$\square$

==

For the parameters $\{\beta = 31/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 21/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 21/10$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.6. *The equilibrium 21/10 for the rational difference equation*

$$x_{n+1} =, 31/10 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -100 + (10 + 10 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 21/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-21/10$,. The new difference equation is:

$$z[n + 1] = 10 \cdot z[n]/(31 + 10 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 10 \cdot z[n]/(31 + 10 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -21/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -21/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-21/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 21/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 21/10$$
$$N = 4$$

Proving $P > 0$ in the region:

$$21/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 21/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 21/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/21 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/21$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 31/10 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 381/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 331/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 331/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.7. *The equilibrium 331/50 for the rational difference equation*

$$x_{n+1} = 381/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 331/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-331/50$,. The new difference equation is:

$$z[n+1] = 50 \cdot z[n]/(381 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(381 + 50 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -331/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -331/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 331/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 331/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 331/50$$
$$N = 4$$

Proving $P > 0$ in the region:
$$331/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 331/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 331/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/331 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/331$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 381/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 229/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 179/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 179/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.8. *The equilibrium 179/50 for the rational difference equation*

$$x_{n+1} = \ 229/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 179/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-179/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(229 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 50 \cdot z[n]/(229 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -179/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -179/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 179/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 179/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 179/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$179/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 179/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 179/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/179 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/179$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 229/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 17/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 7/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 7/10$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.9. *The equilibrium 7/10 for the rational difference equation*

$$x_{n+1} = 17/10 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -100 + (10 + 10 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 7/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-7/10$,. The new difference equation is:

$$z[n + 1] = 10 \cdot z[n]/(17 + 10 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 10 \cdot z[n]/(17 + 10 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -7/10$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -7/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 7/10$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 7/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 7/10 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$7/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 7/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 7/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/7 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/7$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 17/10 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 149/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 124/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 124/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.10. *The equilibrium 124/25 for the rational difference equation*

$$x_{n+1} = 149/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 124/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-124/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(149 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(149 + 25 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -124/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -124/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 124/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 124/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 124/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$124/25 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 124/25$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 124/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/124 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/124$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 149/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 119/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 69/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 69/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.11. *The equilibrium 69/50 for the rational difference equation*

$$x_{n+1} =, 119/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 69/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-69/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(119 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(119 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -69/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -69/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-69/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 69/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 69/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$69/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 69/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 69/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/69 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/69$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 119/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 889/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 789/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 789/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.12. *The equilibrium 789/100, for the rational difference equation*

$$x_{n+1} = 889/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 789/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-789/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(889 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(889 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -789/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -789/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 789/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 789/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 789/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$789/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 789/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 789/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/789 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/789$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 889/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 481/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 381/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 381/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.13. *The equilibrium 381/100, for the rational difference equation*

$$x_{n+1} = 481/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 381/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-381/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(481 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(481 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -381/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -381/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 381/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 381/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 381/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$381/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 381/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

—————————————————————————-

Proving $P > 0$ in the region:

$$0 < z[1] \leq 381/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/381 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/381$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

—————————————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 481/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 191/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 166/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 166/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.14. *The equilibrium 166/25 for the rational difference equation*

$$x_{n+1} = \ 191/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 166/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-166/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(191 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(191 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -166/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -166/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 166/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 166/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 166/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$166/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 166/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 166/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/166 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/166$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 191/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 331/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 231/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 231/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.15. *The equilibrium 231/100, for the rational difference equation*

$$x_{n+1} = 331/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 231/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-231/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(331 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(331 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -231/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -231/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 231/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 231/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 231/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$231/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 231/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 231/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/231 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/231$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 331/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 109/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 84/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 84/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.16. *The equilibrium 84/25 for the rational difference equation*

$$x_{n+1} =, 109/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 84/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-84/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(109 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(109 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -84/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -84/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 84/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 84/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 84/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$84/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 84/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

————————————————————————-

Proving $P > 0$ in the region:

$$0 < z[1] \leq 84/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/84 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/84$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

————————————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 109/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 99/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 89/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 89/10$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.17. *The equilibrium 89/10 for the rational difference equation*

$$x_{n+1} =, 99/10 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -100 + (10 + 10 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 89/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-89/10$,. The new difference equation is:

$$z[n + 1] = 10 \cdot z[n]/(99 + 10 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 10 \cdot z[n]/(99 + 10 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -89/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -89/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 89/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 89/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 89/10$$
$$N = 4$$

Proving $P > 0$ in the region:

$$89/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 89/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 89/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/89 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/89$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 99/10 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 111/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 86/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 86/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.18. *The equilibrium 86/25 for the rational difference equation*

$$x_{n+1} =, 111/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 86/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-86/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(111 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 25 \cdot z[n]/(111 + 25 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -86/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -86/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-86/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 86/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 86/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$86/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 86/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 86/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/86 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/86$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 111/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 439/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 339/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 339/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.19. *The equilibrium 339/100, for the rational difference equation*

$$x_{n+1} = 439/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 339/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-339/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(439 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(439 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -339/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -339/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 339/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 339/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 339/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$339/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 339/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 339/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/339 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/339$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 439/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 19/4\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 15/4 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 15/4$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.20. *The equilibrium 15/4 for the rational difference equation*

$$x_{n+1} = 19/4 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -16 + (4 + 4 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 15/4$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-15/4$,. The new difference equation is:

$$z[n + 1] = 4 \cdot z[n]/(19 + 4 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 4 \cdot z[n]/(19 + 4 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -15/4$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -15/4 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 15/4$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 15/4, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 15/4$$
$$N = 4$$

Proving $P > 0$ in the region:

$$15/4 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 15/4$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 15/4$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$4/15 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 4/15$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 19/4 \cdot x_n/(1 + x_n)$$

□

==

For the parameters $\{\beta = 101/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 76/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 76/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.21. *The equilibrium 76/25 for the rational difference equation*

$$x_{n+1} =, 101/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 76/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-76/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(101 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(101 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -76/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -76/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-76/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 76/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 76/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$76/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 76/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 76/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/76 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/76$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 101/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 601/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 501/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 501/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.22. *The equilibrium 501/100, for the rational difference equation*

$$x_{n+1} = 601/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 501/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-501/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(601 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(601 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -501/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -501/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 501/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 501/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 501/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$501/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 501/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 501/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/501 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/501$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 601/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 19/20\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.23. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 19/20 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -361 + 400 \cdot (1 + z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n+1] = 19/20 \cdot z[n]/(1 + z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 19/20 \cdot z[n]/(1 + z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 0 \\ N &= 4 \end{aligned}$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since

$P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 19/20 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 433/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 333/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 333/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.24. *The equilibrium 333/100, for the rational difference equation*

$$x_{n+1} = 433/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 333/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-333/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(100 \cdot z[n] + 433)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(100 \cdot z[n] + 433)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -333/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -333/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 333/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 333/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 333/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$333/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 333/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 333/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/333 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/333$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!
—————————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 433/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 143/20\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 123/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 123/20$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.25. *The equilibrium 123/20 for the rational difference equation*

$$x_{n+1} = 143/20 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -400 + (20 + 20 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 123/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-123/20$,. The new difference equation is:

$$z[n+1] = 20 \cdot z[n]/(143 + 20 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 20 \cdot z[n]/(143 + 20 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left| Q^K(\langle z[1] \rangle) \right|^2}{\left| \langle z[1], \rangle \right|^2} < 1$$

for all $z[1] > -123/20$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left| \langle z[1] \rangle \right|^2 - \left| Q^K(\langle z[1], \rangle) \right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -123/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 123/20$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 123/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 123/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$123/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 123/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 123/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/123 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/123$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 143/20 \cdot x[n]/(1 + x_n)$$

$\square$

==

For the parameters $\{\beta = 937/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 837/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 837/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.26. *The equilibrium 837/100, for the rational difference equation*

$$x_{n+1} = 937/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 837/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-837/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(937 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(937 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -837/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -837/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 837/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 837/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 837/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$837/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 837/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 837/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/837 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/837$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 937/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 79/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 69/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 69/10$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.27. *The equilibrium 69/10 for the rational difference equation*

$$x_{n+1} =, 79/10 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -100 + (10 + 10 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 69/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-69/10$,. The new difference equation is:

$$z[n+1] = 10 \cdot z[n]/(79 + 10 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 10 \cdot z[n]/(79 + 10 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -69/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -69/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 69/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 69/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 69/10$$
$$N = 4$$

Proving $P > 0$ in the region:
$$69/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 69/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 69/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/69 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/69$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 79/10 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 68/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 43/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 43/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.28. *The equilibrium 43/25 for the rational difference equation*

$$x_{n+1} =, 68/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 43/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-43/25$,. The new difference equation is:

$$z[n + 1] = 25 \cdot z[n]/(68 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(68 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -43/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -43/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 43/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 43/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 43/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$43/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 43/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 43/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/43 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/43$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 68/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 247/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 197/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 197/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.29. *The equilibrium 197/50 for the rational difference equation*

$$x_{n+1} = \ 247/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 197/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-197/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(247 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 50 \cdot z[n]/(247 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -197/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -197/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 197/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 197/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 197/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$197/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 197/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 197/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/197 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/197$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 247/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 17/20\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.30. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 17/20 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -289 + 400 \cdot (1 + z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n + 1] = 17/20 \cdot z[n]/(1 + z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 17/20 \cdot z[n]/(1 + z[n])\rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:
$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 0 \\ N &= 4 \end{aligned}$$
Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation
$$x_{n+1} = 17/20 \cdot x[n]/(1 + x_n)$$
□

==

For the parameters $\{\beta = 341/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 241/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 241/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.31. *The equilibrium 241/100, for the rational difference equation*

$$x_{n+1} = 341/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 241/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-241/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(341 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(341 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -241/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -241/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 241/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 241/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 241/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$241/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 241/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 241/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/241 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/241$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 341/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 859/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 759/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 759/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.32. *The equilibrium 759/100, for the rational difference equation*

$$x_{n+1} = 859/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 759/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-759/100$,. The new difference equation is:

$$z[n+1] = 100 \cdot z[n]/(859 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(859 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -759/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -759/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 759/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 759/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 759/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$759/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 759/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 759/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/759 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/759$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 859/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 41/20\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 21/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 21/20$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.33. *The equilibrium 21/20 for the rational difference equation*

$$x_{n+1} =, 41/20 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -400 + (20 + 20 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 21/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-21/20$,. The new difference equation is:

$$z[n + 1] = 20 \cdot z[n]/(41 + 20 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 20 \cdot z[n]/(41 + 20 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -21/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -21/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 21/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 21/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 21/20 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$21/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 21/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!
———————————————————————-

Proving $P > 0$ in the region:

$$0 < z[1] \leq 21/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/21 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/21$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!
———————————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 41/20 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 437/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 337/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 337/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.34. *The equilibrium 337/100, for the rational difference equation*

$$x_{n+1} = 437/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 337/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-337/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(437 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 100 \cdot z[n]/(437 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -337/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -337/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 337/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 337/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 337/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$337/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 337/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 337/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/337 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/337$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 437/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 281/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 231/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 231/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.35. *The equilibrium 231/50 for the rational difference equation*

$$x_{n+1} = \ 281/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 231/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-231/50$,. The new difference equation is:

$$z[n+1] = 50 \cdot z[n]/(281 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 50 \cdot z[n]/(281 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -231/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -231/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 231/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 231/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 231/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$231/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 231/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 231/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/231 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/231$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 281/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 999/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 899/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 899/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.36. *The equilibrium 899/100, for the rational difference equation*

$$x_{n+1} = 999/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 899/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-899/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(999 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(999 + 100 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -899/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -899/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 899/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 899/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 899/100$$
$$N = 4$$

Proving $P > 0$ in the region:
$$899/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 899/100$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 899/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/899 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/899$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 999/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 129/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 79/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 79/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.37. *The equilibrium 79/50 for the rational difference equation*

$$x_{n+1} =, 129/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 79/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-79/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(129 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(129 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -79/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -79/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 79/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 79/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 79/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$79/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 79/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 79/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/79 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/79$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 129/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 17/2\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 15/2 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 15/2$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.38. *The equilibrium 15/2 for the rational difference equation*

$$x_{n+1} = 17/2 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -4 + (2 + 2 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 15/2$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-15/2$,. The new difference equation is:

$$z[n + 1] = 2 \cdot z[n]/(17 + 2 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 2 \cdot z[n]/(17 + 2 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -15/2$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -15/2 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-15/2$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 15/2, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 15/2$$
$$N = 4$$

Proving $P > 0$ in the region:

$$15/2 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 15/2$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 15/2$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$2/15 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 2/15$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 17/2 \cdot x_n/(1 + x_n)$$

□

==

For the parameters $\{\beta = 13/25\}$: First we check that the equilibrium 0 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 0$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.39. *The equilibrium 0 for the rational difference equation*

$$x_{n+1} = 13/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -169 + 625 \cdot (1 + z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 0$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than 0. The new difference equation is:

$$z[n + 1] = 13/25 \cdot z[n]/(1 + z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 13/25 \cdot z[n]/(1 + z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > 0$ then we are done (see applicable Theorem in Emi lie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than 0 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 0$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 0, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 0$$
$$N = 4$$

Since $\bar{x} = 0$ we check that all coefficients and the constant term in P are positive All coefficients in P and the constant term are positive so $P > 0$ for all variables ≥ 0 Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 13/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 57/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 7/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 7/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.40. *The equilibrium 7/50 for the rational difference equation*

$$x_{n+1} = 57/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 7/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-7/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(57 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(57 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -7/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -7/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 7/50$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 7/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 7/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$7/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 7/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 7/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/7 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/7$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 57/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 6\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 5 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 5$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.41. *The equilibrium 5 for the rational difference equation*

$$x_{n+1} = 6 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1 + (1 + z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 5$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than -5. The new difference equation is:

$$z[n + 1] = z[n]/(6 + z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle z[n]/(6 + z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -5$ then we are done (see applicable Theorem in Em ilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0 :

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -5 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-5$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 5, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 5 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$5 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 5$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 5$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$1/5 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 1/5$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 6 \cdot x_n/(1 + x_n)$$

□

==

For the parameters $\{\beta = 343/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 293/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 293/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.42. *The equilibrium 293/50 for the rational difference equation*

$$x_{n+1} = 343/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 293/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-293/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(343 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(343 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -293/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -293/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 293/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 293/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 293/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$293/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 293/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 293/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/293 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/293$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 343/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 173/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 73/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 73/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.43. *The equilibrium 73/100 for the rational difference equation*

$$x_{n+1} = \ 173/100 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 73/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-73/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(173 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(173 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -73/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -73/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 73/100$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 73/100, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 73/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$73/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 73/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 73/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/73 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/73$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 173/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 909/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 809/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 809/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.44. *The equilibrium 809/100, for the rational difference equation*

$$x_{n+1} = 909/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 809/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-809/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(909 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(909 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -809/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -809/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 809/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 809/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 809/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$809/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 809/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 809/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/809 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/809$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 909/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 143/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 118/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 118/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.45. *The equilibrium 118/25 for the rational difference equation*

$$x_{n+1} = 143/25 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 118/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-118/25$,. The new difference equation is:

$$z[n+1] = 25 \cdot z[n]/(143 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(143 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -118/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -118/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 118/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 118/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 118/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$118/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 118/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

————————————————————————-

Proving $P > 0$ in the region:

$$0 < z[1] \leq 118/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/118 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/118$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

————————————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 143/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 214/25\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 189/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 189/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.46. *The equilibrium 189/25 for the rational difference equation*

$$x_{n+1} = \ 214/25 \cdot x_n/(1+x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (25 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 189/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-189/25$,. The new difference equation is:

$$z[n+1] = 25 \cdot z[n]/(214 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 25 \cdot z[n]/(214 + 25 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -189/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -189/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 189/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 189/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 189/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$189/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 189/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 189/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/189 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/189$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 214/25 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 821/100\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 721/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 721/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.47. *The equilibrium 721/100, for the rational difference equation*

$$x_{n+1} = 821/100 \cdot x_n/(1 + x_n),$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10000 + (100 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 721/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-721/100$,. The new difference equation is:

$$z[n + 1] = 100 \cdot z[n]/(821 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 100 \cdot z[n]/(821 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -721/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -721/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 721/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 721/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 721/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$721/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 721/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 721/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/721 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/721$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 821/100 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 29/10\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 19/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 19/10$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.48. *The equilibrium 19/10 for the rational difference equation*

$$x_{n+1} =, 29/10 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -100 + (10 + 10 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 19/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-19/10$,. The new difference equation is:

$$z[n + 1] = 10 \cdot z[n]/(29 + 10 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle 10 \cdot z[n]/(29 + 10 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -19/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -19/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 19/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 19/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 19/10$$
$$N = 4$$

Proving $P > 0$ in the region:
$$19/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 19/10$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

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Proving $P > 0$ in the region:

$$0 < z[1] \leq 19/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/19 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/19$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 29/10 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 233/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 183/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 183/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.49. *The equilibrium 183/50 for the rational difference equation*

$$x_{n+1} = \ 233/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 183/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-183/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(233 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 50 \cdot z[n]/(233 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -183/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -183/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 183/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 183/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 183/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$183/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 183/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 183/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/183 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/183$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 233/50 \cdot x[n]/(1 + x_n)$$

□

==

For the parameters $\{\beta = 189/50\}$: First we check that the equilibrium 0 is LAS.
The equilibrium $\bar{x} = 0$ is not LAS
==
First we check that the equilibrium 139/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 139/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 4.50. *The equilibrium 139/50 for the rational difference equation*

$$x_{n+1} = \ 189/50 \cdot x_n/(1 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2500 + (50 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 139/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-139/50$,. The new difference equation is:

$$z[n + 1] = 50 \cdot z[n]/(189 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 50 \cdot z[n]/(189 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -139/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -139/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 139/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 139/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 139/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$139/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 139/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 139/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P, z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/139 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/139$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!
———————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 189/50 \cdot x[n]/(1 + x_n)$$

□

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The parameter values for which the equilibrium is not LAS are:

$$[\{\beta = 6\}, 0], [\{\beta = 17/2\}, 0], [\{\beta = 17/10\}, 0], [\{\beta = 19/4\}, 0], [\{\beta = 29/10\}, 0], [\{\beta = 31/10\}, 0], [\{\beta = 41/20\}, 0], [\{\beta = 56/25\}, 0], [\{\beta = 57/50\}, 0], [\{\beta = 68/25\}, 0], [\{\beta = 79/10\}, 0], [\{\beta = 99/10\}, 0], [\{\beta = 101/25\}, 0], [\{\beta = 109/25\}, 0], [\{\beta = 111/25\}, 0], [\{\beta = 119/50\}, 0], [\{\beta = 129/50\}, 0], [\{\beta = 143/20\}, 0], [\{\beta = 143/25\}, 0], [\{\beta = 149/25\}, 0], [\{\beta = 173/100\}, 0], [\{\beta = 189/50\}, 0], [\{\beta = 191/25\}, 0], [\{\beta = 199/50\}, 0], [\{\beta = 214/25\}, 0], [\{\beta = 229/50\}, 0], [\{\beta = 233/50\}, 0], [\{\beta = 247/50\}, 0], [\{\beta = 281/50\}, 0], [\{\beta = 331/100\}, 0], [\{\beta = 341/100\}, 0], [\{\beta = 343/50\}, 0], [\{\beta = 381/50\}, 0], [\{\beta = 433/100\}, 0], [\{\beta = 437/100\}, 0], [\{\beta = 439/100\}, 0], [\{\beta = 481/100\}, 0], [\{\beta = 547/100\}, 0], [\{\beta = 601/100\}, 0], [\{\beta = 703/100\}, 0], [\{\beta = 821/100\}, 0], [\{\beta = 859/100\}, 0], [\{\beta = 889/100\}, 0], [\{\beta = 909/100\}, 0], [\{\beta = 937/100\}, 0], [\{\beta = 999/100\}, 0],$$

The parameter values for which the $K = 1$ are:

$$[\{\beta = 6\}, 5], [\{\beta = 13/25\}, 0], [\{\beta = 17/2\}, 15/2], [\{\beta = 17/10\}, 7/10], [\{\beta = 17/20\}, 0], [\{\beta = 19/4\}, 15/4], [\{\beta = 19/20\}, 0], [\{\beta = 29/10\}, 19/10], [\{\beta = 31/10\}, 21/10], [\{\beta = 41/20\}, 21/20], [\{\beta = 56/25\}, 31/25], [\{\beta = 57/50\}, 7/50], [\{\beta = 68/25\}, 43/25], [\{\beta = 79/10\}, 69/10], [\{\beta = 83/100\}, 0], [\{\beta = 99/10\}, 89/10], [\{\beta = 101/25\}, 76/25], [\{\beta = 109/25\}, 84/25], [\{\beta = 111/25\}, 86/25], [\{\beta = 119/50\}, 69/50], [\{\beta = 129/50\}, 79/50], [\{\beta = 143/20\}, 123/20], [\{\beta = 143/25\}, 118/25], [\{\beta = 149/25\}, 124/25], [\{\beta = 173/100\}, 73/100], [\{\beta = 189/50\}, 139/50], [\{\beta = 191/25\}, 166/25], [\{\beta = 199/50\}, 149/50], [\{\beta = 214/25\}, 189/25], [\{\beta = 229/50\}, 179/50], [\{\beta = 233/50\}, 183/50], [\{\beta = 247/50\}, 197/50], [\{\beta =$$

$281/50\}, 231/50], [\{\beta = 331/100\}, 231/100], [\{\beta = 341/100\}, 241/100], [\{\beta = 343/50\}, 293/50], [\{\beta = 381/50\}, 331/50], [\{\beta = 433/100\}, 333/100], [\{\beta = 437/100\}, 337/100], [\{\beta = 439/100\}, 339/100], [\{\beta = 481/100\}, 381/100], [\{\beta = 547/100\}, 447/100], [\{\beta = 601/100\}, 501/100], [\{\beta = 703/100\}, 603/100], [\{\beta = 821/100\}, 721/100], [\{\beta = 859/100\}, 759/100], [\{\beta = 889/100\}, 789/100], [\{\beta = 909/100\}, 809/100], [\{\beta = 937/100\}, 837/100], [\{\beta = 999/100\}, 899/100],$

Finished investigating difference equation 4 out of 7

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5 $\alpha + \beta \cdot x_n$

For the rational difference equation

$$x_{n+1} = \alpha + \beta \cdot x_n$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{\alpha, \beta\}$. For the parameters $\{\alpha = 557/100, \beta = 11/100\}$: First we check that the equilibrium 557/89 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 557/89$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.1. *The equilibrium 557/89 for the rational difference equation*

$$x_{n+1} = \ 557/100 + 11/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 9879$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 557/89$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-557/89$,. The new difference equation is:

$$z[n+1] = 11/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 11/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -557/89$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -557/89 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 557/89$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 557/89, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 557/89$$
$$N = 4$$

Proving $P > 0$ in the region:

$$557/89 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 557/89$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 557/89$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$89/557 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 89/557$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 557/100 + 11/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 13/100, \beta = 43/100\}$: First we check that the equilibrium 13/57 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 13/57$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.2. *The equilibrium 13/57 for the rational difference equation*

$$x_{n+1} =, 13/100 + 43/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 8151$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 13/57$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-13/57$,. The new difference equation is:

$$z[n+1] = 43/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 43/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -13/57$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -13/57 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 13/57$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 13/57, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 13/57 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$13/57 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 13/57$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 13/57$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$57/13 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 57/13$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 13/100 + 43/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 713/100, \beta = 47/100\}$: First we check that the equilibrium 713/53 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 713/53$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.3. *The equilibrium 713/53 for the rational difference equation*

$$x_{n+1} = \ 713/100 + 47/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 7791$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 713/53$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-713/53$,. The new difference equation is:

$$z[n + 1] = 47/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 47/100 \cdot z[n]\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

   for all $z[1] > -713/53$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

   and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -713/53 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 713/53$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 713/53, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 713/53$$
$$N = 4$$

Proving $P > 0$ in the region:

$$713/53 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 713/53$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 713/53$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$53/713 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 53/713$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 713/100 + 47/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 3/10, \beta = 83/100\}$: First we check that the equilibrium 30/17 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 30/17$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.4. *The equilibrium 30/17 for the rational difference equation*

$$x_{n+1} =, 3/10 + 83/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 3111$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 30/17$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-30/17$,. The new difference equation is:

$$z[n+1] = 83/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 83/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -30/17$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -30/17 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-30/17$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 30/17, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 30/17$$
$$N = 4$$

Proving $P > 0$ in the region:
$$30/17 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 30/17$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 30/17$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$17/30 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 17/30$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 3/10 + 83/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 531/100, \beta = 99/100\}$: First we check that the equilibrium 531 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 531$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.5. *The equilibrium 531 for the rational difference equation*

$$x_{n+1} = 531/100 + 99/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 199$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 531$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when i nitial conditions are greater than -531,. The new difference equation is:

$$z[n+1] = 99/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 99/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -531$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -531 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 531$ so that we have a polynomia l that we wish to prove is positive when all variables are positive (rather tha n when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 531, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 531$$
$$N = 4$$

Proving $P > 0$ in the region:

$$531 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 531$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 531$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$1/531 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 1/531$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 531/100 + 99/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 209/100, \beta = 14/25\}$: First we check that the equilibrium 19/4 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 19/4$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.6. *The equilibrium 19/4 for the rational difference equation*

$$x_{n+1} = 209/100 + 14/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 429$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 19/4$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-19/4$,. The new difference equation is:

$$z[n+1] = 14/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 14/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -19/4$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -19/4 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 19/4$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 19/4, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 19/4$$
$$N = 4$$

Proving $P > 0$ in the region:

$$19/4 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 19/4$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 19/4$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$4/19 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 4/19$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 209/100 + 14/25 \cdot x_n$$

□

==

For the parameters $\{\alpha = 499/50, \beta = 41/100\}$: First we check that the equilibrium 998/59 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 998/59$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.7. *The equilibrium 998/59 for the rational difference equation*

$$x_{n+1} = \; 499/50 + 41/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 8319$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 998/59$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-998/59$,. The new difference equation is:

$$z[n+1] = 41/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 41/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -998/59$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -998/59 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 998/59$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 998/59, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 998/59 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$998/59 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 998/59$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 998/59$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$59/998 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 59/998$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 499/50 + 41/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 34/25, \beta = 39/50\}$: First we check that the equilibrium 68/11 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 68/11$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.8. *The equilibrium 68/11 for the rational difference equation*

$$x_{n+1} =, 34/25 + 39/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 979$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 68/11$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-68/11$,. The new difference equation is:

$$z[n+1] = 39/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 39/50 \cdot z[n]\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -68/11$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -68/11 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 68/11$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 68/11, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 68/11$$
$$N = 4$$

Proving $P > 0$ in the region:

$$68/11 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 68/11$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 68/11$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$11/68 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 11/68$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 34/25 + 39/50 \cdot x_n$$

□

==

For the parameters $\{\alpha = 513/100, \beta = 21/50\}$: First we check that the equilibrium 513/58 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 513/58$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.9. *The equilibrium 513/58 for the rational difference equation*

$$x_{n+1} = \ 513/100 + 21/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 2059$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 513/58$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-513/58$,. The new difference equation is:

$$z[n+1] = 21/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 21/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -513/58$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -513/58 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 513/58$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 513/58, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 513/58$$
$$N = 4$$

Proving $P > 0$ in the region:

$$513/58 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 513/58$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 513/58$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$58/513 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 58/513$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 513/100 + 21/50 \cdot x_n$$

□

==

For the parameters $\{\alpha = 323/50, \beta = 79/100\}$: First we check that the equilibrium 646/21 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 646/21$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.10. *The equilibrium 646/21 for the rational difference equation*

$$x_{n+1} = \ 323/50 + 79/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 3759$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 646/21$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-646/21$,. The new difference equation is:

$$z[n+1] = 79/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 79/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -646/21$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -646/21 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 646/21$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 646/21, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 646/21$$
$$N = 4$$

Proving $P > 0$ in the region:

$$646/21 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 646/21$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 646/21$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$21/646 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 21/646$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 323/50 + 79/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 139/25, \beta = 2/25\}$: First we check that the equilibrium 139/23 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 139/23$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.11. *The equilibrium 139/23 for the rational difference equation*

$$x_{n+1} = 139/25 + 2/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 621$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 139/23$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-139/23$,. The new difference equation is:

$$z[n+1] = 2/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 2/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -139/23$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -139/23 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 139/23$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 139/23, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 139/23 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$139/23 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 139/23$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 139/23$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$23/139 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 23/139$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 139/25 + 2/25 \cdot x_n$$

□

==

For the parameters $\{\alpha = 31/25, \beta = 37/100\}$: First we check that the equilibrium 124/63 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 124/63$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.12. *The equilibrium 124/63 for the rational difference equation*

$$x_{n+1} = \ 31/25 + 37/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 8631$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 124/63$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-124/63$,. The new difference equation is:

$$z[n+1] = 37/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 37/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -124/63$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -124/63 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 124/63$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 124/63, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 124/63$$
$$N = 4$$

Proving $P > 0$ in the region:

$$124/63 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 124/63$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 124/63$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$63/124 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 63/124$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 31/25 + 37/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 221/50, \beta = 43/100\}$: First we check that the equilibrium 442/57 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 442/57$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.13. *The equilibrium 442/57 for the rational difference equation*

$$x_{n+1} = \ 221/50 + 43/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 8151$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 442/57$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-442/57$,. The new difference equation is:

$$z[n+1] = 43/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 43/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -442/57$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -442/57 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 442/57$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 442/57, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 442/57$$
$$N = 4$$

Proving $P > 0$ in the region:

$$442/57 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 442/57$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 442/57$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$57/442 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 57/442$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 221/50 + 43/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 97/10, \beta = 8/25\}$: First we check that the equilibrium 485/34 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 485/34$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.14. *The equilibrium 485/34 for the rational difference equation*

$$x_{n+1} = 97/10 + 8/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 561$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 485/34$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-485/34$,. The new difference equation is:

$$z[n+1] = 8/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 8/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -485/34$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -485/34 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 485/34$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 485/34, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 485/34$$
$$N = 4$$

Proving $P > 0$ in the region:

$$485/34 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 485/34$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 485/34$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$34/485 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 34/485$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 97/10 + 8/25 \cdot x_n$$

□

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For the parameters $\{\alpha = 497/50, \beta = 9/20\}$: First we check that the equilibrium 994/55 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 994/55$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.15. *The equilibrium 994/55 for the rational difference equation*

$$x_{n+1} = \ 497/50 + 9/20 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 319$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 994/55$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-994/55$,. The new difference equation is:

$$z[n+1] = 9/20 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 9/20 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -994/55$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -994/55 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 994/55$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 994/55, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 994/55$$
$$N = 4$$

Proving $P > 0$ in the region:

$$994/55 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 994/55$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 994/55$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$55/994 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 55/994$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 497/50 + 9/20 \cdot x_n$$

□

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For the parameters $\{\alpha = 553/100, \beta = 93/100\}$: First we check that the equilibrium 79 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 79$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.16. *The equilibrium 79 for the rational difference equation*

$$x_{n+1} = 553/100 + 93/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 1351$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 79$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than -79. The new difference equation is:

$$z[n + 1] = 93/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 93/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -79$ then we are done (see applicable Theorem in E milie Hogan's PhD thesis). Simplify this so that it is of the form something $>$ 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -79 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 79$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 79, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 79$$
$$N = 4$$

Proving $P > 0$ in the region:
$$79 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 79$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 79$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$1/79 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 1/79$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 553/100 + 93/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 27/10, \beta = 29/50\}$: First we check that the equilibrium 45/7 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 45/7$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.17. *The equilibrium 45/7 for the rational difference equation*

$$x_{n+1} = 27/10 + 29/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 1659$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 45/7$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-45/7$,. The new difference equation is:
$$z[n+1] = 29/50 \cdot z[n]$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 29/50 \cdot z[n] \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -45/7$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -45/7 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 45/7$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 45/7, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 45/7$$
$$N = 4$$

Proving $P > 0$ in the region:

$$45/7 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 45/7$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 45/7$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$7/45 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 7/45$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 27/10 + 29/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 133/100, \beta = 27/50\}$: First we check that the equilibrium 133/46 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 133/46$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.18. *The equilibrium 133/46 for the rational difference equation*

$$x_{n+1} = 133/100 + 27/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 1771$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 133/46$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-133/46$,. The new difference equation is:

$$z[n+1] = 27/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 27/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -133/46$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -133/46 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 133/46$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 133/46, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 133/46$$
$$N = 4$$

Proving $P > 0$ in the region:

$$133/46 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 133/46$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 133/46$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$46/133 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 46/133$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 133/100 + 27/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 247/50, \beta = 17/25\}$: First we check that the equilibrium 247/16 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 247/16$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.19. *The equilibrium 247/16 for the rational difference equation*

$$x_{n+1} = 247/50 + 17/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 336$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 247/16$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-247/16$,. The new difference equation is:

$$z[n+1] = 17/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 17/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -247/16$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -247/16 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 247/16$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 247/16, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 247/16 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$247/16 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 247/16$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 247/16$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$16/247 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 16/247$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 247/50 + 17/25 \cdot x_n$$

□

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For the parameters $\{\alpha = 67/10, \beta = 2/5\}$: First we check that the equilibrium 67/6 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 67/6$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.20. *The equilibrium 67/6 for the rational difference equation*

$$x_{n+1} = 67/10 + 2/5 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 21$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 67/6$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-67/6$,. The new difference equation is:

$$z[n+1] = 2/5 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 2/5 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -67/6$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -67/6 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 67/6$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 67/6, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 67/6$$
$$N = 4$$

Proving $P > 0$ in the region:

$$67/6 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 67/6$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 67/6$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$6/67 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 6/67$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 67/10 + 2/5 \cdot x_n$$

□

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For the parameters $\{\alpha = 281/50, \beta = 3/50\}$: First we check that the equilibrium 281/47 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 281/47$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.21. *The equilibrium 281/47 for the rational difference equation*

$$x_{n+1} = \ 281/50 + 3/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 2491$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 281/47$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-281/47$,. The new difference equation is:

$$z[n+1] = 3/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 3/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -281/47$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -281/47 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 281/47$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 281/47, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 281/47$$
$$N = 4$$

Proving $P > 0$ in the region:

$$281/47 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 281/47$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 281/47$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$47/281 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 47/281$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 281/50 + 3/50 \cdot x_n$$

□

==

For the parameters $\{\alpha = 419/50, \beta = 1/25\}$: First we check that the equilibrium 419/48 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 419/48$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.22. *The equilibrium 419/48 for the rational difference equation*

$$x_{n+1} = \ 419/50 + 1/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 624$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 419/48$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-419/48$,. The new difference equation is:

$$z[n+1] = 1/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 1/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -419/48$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -419/48 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 419/48$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 419/48, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 419/48 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$419/48 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 419/48$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 419/48$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$48/419 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 48/419$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 419/50 + 1/25 \cdot x_n$$

□

==

For the parameters $\{\alpha = 691/100, \beta = 37/50\}$: First we check that the equilibrium 691/26 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 691/26$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.23. *The equilibrium 691/26 for the rational difference equation*

$$x_{n+1} = 691/100 + 37/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 1131$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 691/26$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-691/26$,. The new difference equation is:
$$z[n+1] = 37/50 \cdot z[n]$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 37/50 \cdot z[n] \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -691/26$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -691/26 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 691/26$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 691/26, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 691/26$$
$$N = 4$$

Proving $P > 0$ in the region:
$$691/26 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 691/26$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 691/26$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$26/691 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 26/691$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 691/100 + 37/50 \cdot x_n$$

□

==

For the parameters $\{\alpha = 227/100, \beta = 33/50\}$: First we check that the equilibrium 227/34 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 227/34$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.24. *The equilibrium 227/34 for the rational difference equation*

$$x_{n+1} = \ 227/100 + 33/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 1411$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 227/34$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-227/34$,. The new difference equation is:

$$z[n+1] = 33/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 33/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -227/34$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -227/34 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 227/34$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 227/34, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 227/34$$
$$N = 4$$

Proving $P > 0$ in the region:

$$227/34 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 227/34$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 227/34$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$34/227 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 34/227$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 227/100 + 33/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 269/100, \beta = 4/5\}$: First we check that the equilibrium 269/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 269/20$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.25. *The equilibrium 269/20 for the rational difference equation*

$$x_{n+1} = \ 269/100 + 4/5 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 9$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 269/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-269/20$,. The new difference equation is:

$$z[n+1] = 4/5 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 4/5 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -269/20$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -269/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 269/20$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 269/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 269/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$269/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 269/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 269/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/269 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/269$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 269/100 + 4/5 \cdot x_n$$

□

==

For the parameters $\{\alpha = 977/100, \beta = 67/100\}$: First we check that the equilibrium 977/33 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 977/33$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.26. *The equilibrium 977/33 for the rational difference equation*

$$x_{n+1} = 977/100 + 67/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 5511$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 977/33$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-977/33$,. The new difference equation is:

$$z[n+1] = 67/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 67/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -977/33$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -977/33 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 977/33$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 977/33, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 977/33 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$977/33 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 977/33$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 977/33$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$33/977 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 33/977$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 977/100 + 67/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 163/100, \beta = 63/100\}$: First we check that the equilibrium 163/37 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 163/37$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.27. *The equilibrium 163/37 for the rational difference equation*

$$x_{n+1} = \ 163/100 + 63/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 6031$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 163/37$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-163/37$,. The new difference equation is:

$$z[n+1] = 63/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 63/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -163/37$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -163/37 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 163/37$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 163/37, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 163/37$$
$$N = 4$$

Proving $P > 0$ in the region:

$$163/37 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 163/37$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 163/37$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$37/163 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 37/163$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 163/100 + 63/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 201/50, \beta = 31/100\}$: First we check that the equilibrium 134/23 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 134/23$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.28. *The equilibrium 134/23 for the rational difference equation*

$$x_{n+1} = 201/50 + 31/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 9039$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 134/23$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-134/23$,. The new difference equation is:

$$z[n+1] = 31/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 31/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -134/23$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -134/23 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 134/23$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 134/23, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 134/23 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$134/23 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 134/23$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 134/23$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$23/134 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 23/134$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 201/50 + 31/100 \cdot x_n$$

□

==

For the parameters $\{\alpha = 69/50, \beta = 7/10\}$: First we check that the equilibrium 23/5 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 23/5$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.29. *The equilibrium 23/5 for the rational difference equation*

$$x_{n+1} = 69/50 + 7/10 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 51$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 23/5$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-23/5$,. The new difference equation is:

$$z[n+1] = 7/10 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 7/10 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -23/5$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -23/5 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 23/5$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 23/5, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 23/5 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$23/5 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 23/5$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 23/5$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$5/23 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 5/23$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 69/50 + 7/10 \cdot x_n$$

□

==

For the parameters $\{\alpha = 783/100, \beta = 41/50\}$: First we check that the equilibrium 87/2 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 87/2$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.30. *The equilibrium 87/2 for the rational difference equation*

$$x_{n+1} = 783/100 + 41/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 819$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 87/2$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-87/2$,. The new difference equation is:

$$z[n+1] = 41/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 41/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -87/2$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -87/2 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 87/2$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 87/2, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 87/2$$
$$N = 4$$

Proving $P > 0$ in the region:

$$87/2 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 87/2$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 87/2$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$2/87 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 2/87$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 783/100 + 41/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 19/25, \beta = 19/20\}$: First we check that the equilibrium 76/5 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 76/5$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.31. *The equilibrium 76/5 for the rational difference equation*

$$x_{n+1} = 19/25 + 19/20 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 39$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 76/5$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-76/5$,. The new difference equation is:

$$z[n+1] = 19/20 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 19/20 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -76/5$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -76/5 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 76/5$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 76/5, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 76/5$$
$$N = 4$$

Proving $P > 0$ in the region:

$$76/5 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 76/5$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 76/5$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$5/76 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 5/76$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 19/25 + 19/20 \cdot x_n$$

□

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For the parameters $\{\alpha = 232/25, \beta = 21/25\}$: First we check that the equilibrium 58 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 58$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.32. *The equilibrium 58 for the rational difference equation*

$$x_{n+1} = 232/25 + 21/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 184$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 58$, in the original difference equation. This new difference equation has 0 as i ts equilibrium, and we now wish to prove that the equilibrium 0 is GAS when ini tial conditions are greater than -58. The new difference equation is:

$$z[n+1] = 21/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 21/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -58$ then we are done (see applicable Theorem in E milie Hogan's PhD thesis). Simplify this so that it is of the form something $>$ 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -58 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 58$ so that we have a polynomial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 58, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 58$$
$$N = 4$$

Proving $P > 0$ in the region:

$$58 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 58$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 58$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$1/58 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 1/58$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 232/25 + 21/25 \cdot x_n$$

□

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For the parameters $\{\alpha = 89/100, \beta = 49/50\}$: First we check that the equilibrium 89/2 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 89/2$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.33. *The equilibrium 89/2 for the rational difference equation*

$$x_{n+1} = 89/100 + 49/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 99$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 89/2$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-89/2$,. The new difference equation is:

$$z[n+1] = 49/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 49/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -89/2$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -89/2 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 89/2$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 89/2, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 89/2 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$89/2 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 89/2$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 89/2$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$2/89 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 2/89$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 89/100 + 49/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 112/25, \beta = 22/25\}$: First we check that the equilibrium 112/3 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 112/3$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.34. *The equilibrium 112/3 for the rational difference equation*

$$x_{n+1} =, 112/25 + 22/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 141$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 112/3$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-112/3$,. The new difference equation is:

$$z[n+1] = 22/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 22/25 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -112/3$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -112/3 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 112/3$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 112/3, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 112/3$$
$$N = 4$$

Proving $P > 0$ in the region:

$$112/3 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 112/3$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 112/3$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$3/112 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 3/112$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 112/25 + 22/25 \cdot x_n$$

□

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For the parameters $\{\alpha = 467/100, \beta = 11/20\}$: First we check that the equilibrium 467/45 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 467/45$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.35. *The equilibrium 467/45 for the rational difference equation*

$$x_{n+1} = 467/100 + 11/20 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 279$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 467/45$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-467/45$,. The new difference equation is:

$$z[n+1] = 11/20 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 11/20 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -467/45$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -467/45 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 467/45$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 467/45, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 467/45$$
$$N = 4$$

Proving $P > 0$ in the region:

$$467/45 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 467/45$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 467/45$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$45/467 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 45/467$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 467/100 + 11/20 \cdot x_n$$

□

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For the parameters $\{\alpha = 397/100, \beta = 1/5\}$: First we check that the equilibrium 397/80 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 397/80$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.36. *The equilibrium 397/80 for the rational difference equation*

$$x_{n+1} = \ 397/100 + 1/5 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 24$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 397/80$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-397/80$,. The new difference equation is:

$$z[n+1] = 1/5 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 1/5 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -397/80$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -397/80 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 397/80$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 397/80, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 397/80$$
$$N = 4$$

Proving $P > 0$ in the region:

$$397/80 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 397/80$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 397/80$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$80/397 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 80/397$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 397/100 + 1/5 \cdot x_n$$

□

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For the parameters $\{\alpha = 869/100, \beta = 27/50\}$: First we check that the equilibrium 869/46 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 869/46$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.37. *The equilibrium 869/46 for the rational difference equation*

$$x_{n+1} = 869/100 + 27/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 1771$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 869/46$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-869/46$,. The new difference equation is:

$$z[n+1] = 27/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 27/50 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -869/46$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -869/46 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 869/46$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 869/46, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 869/46$$
$$N = 4$$

Proving $P > 0$ in the region:
$$869/46 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 869/46$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 869/46$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$46/869 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 46/869$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 869/100 + 27/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 179/100, \beta = 47/50\}$: First we check that the equilibrium 179/6 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 179/6$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.38. *The equilibrium 179/6 for the rational difference equation*

$$x_{n+1} =, 179/100 + 47/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 291$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 179/6$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-179/6$,. The new difference equation is:

$$z[n+1] = 47/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 47/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -179/6$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -179/6 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 179/6$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 179/6, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 179/6 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$179/6 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 179/6$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 179/6$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$6/179 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 6/179$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 179/100 + 47/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 381/50, \beta = 39/100\}$: First we check that the equilibrium 762/61 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 762/61$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.39. *The equilibrium 762/61 for the rational difference equation*

$$x_{n+1} = \ 381/50 + 39/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 8479$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 762/61$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-762/61$,. The new difference equation is:

$$z[n+1] = 39/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 39/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -762/61$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -762/61 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 762/61$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 762/61, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 762/61$$
$$N = 4$$

Proving $P > 0$ in the region:

$$762/61 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 762/61$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 762/61$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$61/762 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 61/762$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 381/50 + 39/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 79/100, \beta = 31/100\}$: First we check that the equilibrium 79/69 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 79/69$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.40. *The equilibrium 79/69 for the rational difference equation*

$$x_{n+1} =, 79/100 + 31/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 9039$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 79/69$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-79/69$,. The new difference equation is:

$$z[n+1] = 31/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 31/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -79/69$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -79/69 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 79/69$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 79/69, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 79/69$$
$$N = 4$$

Proving $P > 0$ in the region:

$$79/69 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 79/69$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 79/69$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$69/79 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 69/79$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 79/100 + 31/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 431/100, \beta = 9/50\}$: First we check that the equilibrium 431/82 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 431/82$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.41. *The equilibrium 431/82 for the rational difference equation*

$$x_{n+1} = 431/100 + 9/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 2419$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 431/82$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-431/82$,. The new difference equation is:
$$z[n+1] = 9/50 \cdot z[n]$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 9/50 \cdot z[n] \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -431/82$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -431/82 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 431/82$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 431/82, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 431/82 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$431/82 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 431/82$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 431/82$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$82/431 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 82/431$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 431/100 + 9/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 109/25, \beta = 7/20\}$: First we check that the equilibrium 436/65 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 436/65$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.42. *The equilibrium 436/65 for the rational difference equation*

$$x_{n+1} = \ 109/25 + 7/20 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 351$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 436/65$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-436/65$,. The new difference equation is:

$$z[n+1] = 7/20 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 7/20 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

   for all $z[1] > -436/65$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

   and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -436/65 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 436/65$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 436/65, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 436/65$$
$$N = 4$$

Proving $P > 0$ in the region:

$$436/65 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 436/65$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 436/65$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$65/436 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 65/436$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 109/25 + 7/20 \cdot x_n$$

□

==

For the parameters $\{\alpha = 251/100, \beta = 19/100\}$: First we check that the equilibrium 251/81 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 251/81$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.43. *The equilibrium 251/81 for the rational difference equation*

$$x_{n+1} = \ 251/100 + 19/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 9639$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 251/81$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-251/81$,. The new difference equation is:

$$z[n+1] = 19/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 19/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -251/81$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -251/81 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 251/81$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 251/81, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 251/81$$
$$N = 4$$

Proving $P > 0$ in the region:
$$251/81 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 251/81$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 251/81$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$81/251 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 81/251$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 251/100 + 19/100 \cdot x_n$$

□

==

For the parameters $\{\alpha = 94/25, \beta = 1/2\}$: First we check that the equilibrium 188/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 188/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.44. *The equilibrium 188/25 for the rational difference equation*

$$x_{n+1} = 94/25 + 1/2 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 3$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 188/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-188/25$,. The new difference equation is:
$$z[n+1] = 1/2 \cdot z[n]$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 1/2 \cdot z[n] \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -188/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -188/25 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 188/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 188/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 188/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$188/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 188/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 188/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/188 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/188$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 94/25 + 1/2 \cdot x_n$$

□

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For the parameters $\{\alpha = 559/100, \beta = 21/50\}$: First we check that the equilibrium 559/58 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 559/58$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.45. *The equilibrium 559/58 for the rational difference equation*

$$x_{n+1} = \ 559/100 + 21/50 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 2059$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 559/58$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-559/58$,. The new difference equation is:

$$z[n+1] = 21/50 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 21/50 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

   for all $z[1] > -559/58$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

   and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -559/58 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 559/58$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 559/58, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 559/58$$
$$N = 4$$

Proving $P > 0$ in the region:

$$559/58 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 559/58$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 559/58$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$58/559 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 58/559$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 559/100 + 21/50 \cdot x_n$$

□

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For the parameters $\{\alpha = 197/25, \beta = 71/100\}$: First we check that the equilibrium 788/29 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 788/29$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.46. *The equilibrium 788/29 for the rational difference equation*

$$x_{n+1} = \ 197/25 + 71/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 4959$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 788/29$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-788/29$,. The new difference equation is:

$$z[n+1] = 71/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 71/100 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -788/29$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -788/29 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 788/29$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 788/29, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 788/29$$
$$N = 4$$

Proving $P > 0$ in the region:
$$788/29 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 788/29$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 788/29$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$29/788 \le z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 29/788$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 197/25 + 71/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 293/50, \beta = 87/100\}$: First we check that the equilibrium 586/13 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 586/13$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.47. *The equilibrium 586/13 for the rational difference equation*

$$x_{n+1} = 293/50 + 87/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 2431$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 586/13$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-586/13$,. The new difference equation is:
$$z[n+1] = 87/100 \cdot z[n]$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 87/100 \cdot z[n] \rangle$$
3. Notice that if
$$\frac{\left| Q^K(\langle z[1] \rangle) \right|^2}{\left| \langle z[1], \rangle \right|^2} < 1$$
for all $z[1] > -586/13$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left| \langle z[1] \rangle \right|^2 - \left| Q^K(\langle z[1], \rangle) \right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -586/13 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 586/13$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 586/13, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 586/13$$
$$N = 4$$

Proving $P > 0$ in the region:

$$586/13 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 586/13$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 586/13$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$13/586 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 13/586$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 293/50 + 87/100 \cdot x_n$$

$\square$

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For the parameters $\{\alpha = 33/5, \beta = 99/100\}$: First we check that the equilibrium 660 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 660$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.48. *The equilibrium 660 for the rational difference equation*

$$x_{n+1} = 33/5 + 99/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 199$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 660$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when i nitial conditions are greater than -660,. The new difference equation is:

$$z[n+1] = 99/100 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 99/100 \cdot z[n] \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -660$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -660 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 660$ so that we have a polynomia l that we wish to prove is positive when all variables are positive (rather tha n when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 660, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 660$$
$$N = 4$$

Proving $P > 0$ in the region:

$$660 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 660$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 660$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$1/660 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 1/660$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 33/5 + 99/100 \cdot x_n$$

□

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For the parameters $\{\alpha = 47/10, \beta = 8/25\}$: First we check that the equilibrium 235/34 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 235/34$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.49. *The equilibrium 235/34 for the rational difference equation*

$$x_{n+1} = 47/10 + 8/25 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 561$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 235/34$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-235/34$,. The new difference equation is:

$$z[n+1] = 8/25 \cdot z[n]$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 8/25 \cdot z[n] \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -235/34$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -235/34 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 235/34$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 235/34, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 235/34$$
$$N = 4$$

Proving $P > 0$ in the region:
$$235/34 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 235/34$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 235/34$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$34/235 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 34/235$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 47/10 + 8/25 \cdot x_n$$

□

==

For the parameters $\{\alpha = 24/25, \beta = 79/100\}$: First we check that the equilibrium 32/7 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 32/7$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 5.50. *The equilibrium 32/7 for the rational difference equation*

$$x_{n+1} = 24/25 + 79/100 \cdot x_n$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = 3759$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 32/7$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-32/7$,. The new difference equation is:
$$z[n+1] = 79/100 \cdot z[n]$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 79/100 \cdot z[n] \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -32/7$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -32/7 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 32/7$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 32/7, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 32/7$$
$$N = 4$$

Proving $P > 0$ in the region:

$$32/7 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 32/7$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 32/7$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = 1$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$7/32 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 7/32$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = 24/25 + 79/100 \cdot x_n$$

□

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The parameter values for which the $K = 1$ are:

$[\{\alpha = 3/10, \beta = 83/100\}, 30/17], [\{\alpha = 13/100, \beta = 43/100\}, 13/57], [\{\alpha = 19/25, \beta = 19/20\}, 76/5], [\{\alpha = 24/25, \beta = 79/100\}, 32/7], [\{\alpha = 27/10, \beta = 29/50\}, 45/7], [\{\alpha = 31/25, \beta = 37/100\}, 124/63], [\{\alpha = 33/5, \beta = 99/100\}, 660], [\{\alpha = 34/25, \beta = 39/50\}, 68/11], [\{\alpha = 47/10, \beta = 8/25\}, 235/34], [\{\alpha = 67/10, \beta = 2/5\}, 67/6], [\{\alpha = 69/50, \beta = 7/10\}, 23/5], [\{\alpha = 79/100, \beta = 31/100\}, 79/69], [\{\alpha = 89/100, \beta = 49/50\}, 89/2], [\{\alpha = 94/25, \beta = 1/2\}, 188/25], [\{\alpha = 97/10, \beta = 8/25\}, 485/34], [\{\alpha = 109/25, \beta = 7/20\}, 436/65], [\{\alpha = 112/25, \beta = 22/25\}, 112/3], [\{\alpha = 133/100, \beta = 27/50\}, 133/46], [\{\alpha = 139/25, \beta = 2/25\}, 139/23], [\{\alpha = 163/100, \beta = 63/100\}, 163/37], [\{\alpha = 179/100, \beta = 47/50\}, 179/6], [\{\alpha = 197/25, \beta = 71/100\}, 788/29], [\{\alpha = 201/50, \beta = 31/100\}, 134/23], [\{\alpha = 209/100, \beta = 14/25\}, 19/4], [\{\alpha = 221/50, \beta = 43/100\}, 442/57], [\{\alpha = 227/100, \beta = 33/50\}, 227/34], [\{\alpha = 232/25, \beta = 21/25\}, 58], [\{\alpha = 247/50, \beta = 17/25\}, 247/16], [\{\alpha = 251/100, \beta = 19/100\}, 251/81], [\{\alpha = 269/100, \beta = 4/5\}, 269/20], [\{\alpha = 281/50, \beta = 3/50\}, 281/47], [\{\alpha = 293/50, \beta = 87/100\}, 586/13], [\{\alpha = 323/50, \beta = 79/100\}, 646/21], [\{\alpha = 381/50, \beta = 39/100\}, 762/61], [\{\alpha = 397/100, \beta = 1/5\}, 397/80], [\{\alpha = 419/50, \beta = 1/25\}, 419/48], [\{\alpha = 431/100, \beta = 9/50\}, 431/82], [\{\alpha = 467/100, \beta = 11/20\}, 467/45], [\{\alpha = 497/50, \beta = 9/20\}, 994/55], [\{\alpha = 499/50, \beta = 41/100\}, 998/59], [\{\alpha = 513/100, \beta = 21/50\}, 513/58], [\{\alpha = 531/100, \beta = 99/100\}, 531], [\{\alpha = 553/100, \beta = 93/100\}, 79], [\{\alpha = 557/100, \beta = 11/100\}, 557/89], [\{\alpha = 559/100, \beta = 21/50\}, 559/58], [\{\alpha = 691/100, \beta = 37/50\}, 691/26], [\{\alpha = 713/100, \beta = 47/100\}, 713/53], [\{\alpha = 783/100, \beta = 41/50\}, 87/2], [\{\alpha = 869/100, \beta = 27/50\}, 869/46], [\{\alpha = 977/100, \beta = 67/100\}, 977/33],$

Finished investigating difference equation 5 out of 7

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6 $q + (-1/4 \cdot q^2 + 1/4 \cdot M^2)/x_n$

For the rational difference equation

$$x_{n+1} = q + (-1/4 \cdot q^2 + 1/4 \cdot M^2)/x_n$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{M, q\}$.
For the parameters $\{M = 49/10, q = 27/100\}$: First we check that the equilibrium 517/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 517/200$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 517/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.1. *The equilibrium 517/200, for the rational difference equation*

$$x_{n+1} = 27/100 + 239371/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -45954068161 + (10800 \cdot z[1] + 239371)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 517/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-517/200$,. The new difference equation is:

$$z[n+1] = -463 \cdot z[n]/(200 \cdot z[n] + 517)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -463 \cdot z[n]/(200 \cdot z[n] + 517) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -517/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -517/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 517/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 517/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 517/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$517/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 517/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 517/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/517 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/517$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 27/100 + 239371/40000/x_n$$

□

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For the parameters $\{M = 93/50, q = 427/100\}$:
For the parameters $\{M = 401/50, q = 29/4\}$: First we check that the equilibrium 1527/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1527/200$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1527/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.2. *The equilibrium 1527/200, for the rational difference equation*

$$x_{n+1} = 29/4 + 117579/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -35153041 + (290000 \cdot z[1] + 117579)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1527/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1527/200$,. The new difference equation is:

$$z[n+1] = -77 \cdot z[n]/(200 \cdot z[n] + 1527)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -77 \cdot z[n]/(200 \cdot z[n] + 1527)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -1527/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1527/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1527/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1527/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1527/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1527/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1527/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1527/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1527 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1527$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 29/4 + 117579/40000/x_n$$

□

==

For the parameters $\{M = 437/100, q = 1/50\}$: First we check that the equilibrium 439/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 439/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 439/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.3. *The equilibrium 439/200, for the rational difference equation*

$$x_{n+1} = 1/50 + 38193/8000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -35806100625 + (800 \cdot z[1] + 190965)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 439/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-439/200$,. The new difference equation is:

$$z[n+1] = -435 \cdot z[n]/(200 \cdot z[n] + 439)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -435 \cdot z[n]/(200 \cdot z[n] + 439)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -439/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -439/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 439/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 439/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 439/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$439/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 439/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 439/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/439 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/439$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 1/50 + 38193/8000/x_n$$

□

==

For the parameters $\{M = 122/25, q = 957/100\}$:
For the parameters $\{M = 22/5, q = 263/50\}$:
For the parameters $\{M = 226/25, q = 59/50\}$: First we check that the equilibrium 511/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 511/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 511/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.4. *The equilibrium 511/100, for the rational difference equation*

$$x_{n+1} = 59/50 + 200823/10000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -23854493601 + (11800 \cdot z[1] + 200823)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 511/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-511/100$,. The new difference equation is:

$$z[n+1] = -393 \cdot z[n]/(100 \cdot z[n] + 511)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -393 \cdot z[n]/(100 \cdot z[n] + 511)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -511/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -511/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 511/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 511/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 511/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$511/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 511/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 511/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/511 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/511$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 59/50 + 200823/10000/x_n$$

□

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For the parameters $\{M = 54/25, q = 106/25\}$:
For the parameters $\{M = 11/2, q = 62/25\}$: First we check that the equilibrium 399/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 399/100$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 399/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.5. *The equilibrium 399/100, for the rational difference equation*

$$x_{n+1} = 62/25 + 60249/10000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -519885601 + (24800 \cdot z[1] + 60249)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 399/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-399/100$,. The new difference equation is:

$$z[n+1] = -151 \cdot z[n]/(100 \cdot z[n] + 399)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -151 \cdot z[n]/(100 \cdot z[n] + 399) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -399/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -399/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 399/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 399/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 399/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$399/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 399/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 399/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/399 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/399$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 62/25 + 60249/10000/x_n$$

□

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For the parameters $\{M = 251/100, q = 509/100\}$:
For the parameters $\{M = 2/5, q = 559/100\}$:
For the parameters $\{M = 199/100, q = 151/50\}$:
For the parameters $\{M = 381/100, q = 21/5\}$:
For the parameters $\{M = 881/100, q = 843/100\}$: First we check that the equilibrium 431/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 431/50$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 431/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.6. *The equilibrium 431/50 for the rational difference equation*

$$x_{n+1} = 843/100 + 8189/5000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -130321 + 16 \cdot (21075 \cdot z[1] + 8189/2)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 431/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-431/50$,. The new difference equation is:

$$z[n+1] = -19/2 \cdot z[n]/(50 \cdot z[n] + 431)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -19/2 \cdot z[n]/(50 \cdot z[n] + 431)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -431/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -431/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 431/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 431/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 431/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$431/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 431/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

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Proving $P > 0$ in the region:

$$0 < z[1] \leq 431/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/431 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/431$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 843/100 + 8189/5000/x_n$$

□

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For the parameters $\{M = 883/100, q = 209/50\}$: First we check that the equilibrium 1301/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1301/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1301/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.7. *The equilibrium 1301/200, for the rational difference equation*

$$x_{n+1} = 209/50 + 120993/8000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -46753250625 + (167200 \cdot z[1] + 604965)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1301/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1301/200$,. The new difference equation is:

$$z[n+1] = -465 \cdot z[n]/(200 \cdot z[n] + 1301)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -465 \cdot z[n]/(200 \cdot z[n] + 1301) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -1301/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1301/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1301/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1301/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1301/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1301/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1301/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1301/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1301 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1301$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 209/50 + 120993/8000/x_n$$

□

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For the parameters $\{M = 103/25, q = 807/100\}$:
For the parameters $\{M = 413/50, q = 15/2\}$: First we check that the equilibrium 197/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 197/25$
For $K = 1$ we get $\{FAIL, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 197/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.8. *The equilibrium 197/25 for the rational difference equation*

$$x_{n+1} = \ 15/2 + 3743/1250/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -130321 + 4 \cdot (9375 \cdot z[1] + 3743)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 197/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-197/25$,. The new difference equation is:

$$z[n+1] = -19/2 \cdot z[n]/(25 \cdot z[n] + 197)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -19/2 \cdot z[n]/(25 \cdot z[n] + 197)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -197/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -197/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 197/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 197/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 197/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$197/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 197/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 197/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/197 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/197$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 15/2 + 3743/1250/x_n$$

□

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For the parameters $\{M = 999/100, q = 203/50\}$: First we check that the equilibrium 281/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 281/40$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 281/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.9. *The equilibrium 281/40 for the rational difference equation*

$$x_{n+1} = \ 203/50 + 166633/8000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -123657019201 + 625 \cdot (6496 \cdot z[1] + 166633/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 281/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-281/40$,. The new difference equation is:

$$z[n+1] = -593/5 \cdot z[n]/(40 \cdot z[n] + 281)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -593/5 \cdot z[n]/(40 \cdot z[n] + 281)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -281/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -281/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 281/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 281/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 281/40$$
$$N = 4$$

Proving $P > 0$ in the region:

$$281/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 281/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 281/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/281 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/281$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 203/50 + 166633/8000/x_n$$

□

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For the parameters $\{M = 33/5, q = 387/100\}$: First we check that the equilibrium 1047/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1047/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1047/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.10. *The equilibrium 1047/200, for the rational difference equation*

$$x_{n+1} = 387/100 + 285831/40000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -68574961 + (17200 \cdot z[1] + 31759)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1047/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1047/200$,. The new difference equation is:

$$z[n+1] = -273 \cdot z[n]/(200 \cdot z[n] + 1047)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -273 \cdot z[n]/(200 \cdot z[n] + 1047)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -1047/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1047/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1047/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1047/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1047/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1047/200 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1047/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 1047/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1047 \le z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1047$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 387/100 + 285831/40000/x_n$$

□

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For the parameters $\{M = 393/50, q = 499/100\}$: First we check that the equilibrium 257/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 257/40$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 257/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.11. *The equilibrium 257/40 for the rational difference equation*

$$x_{n+1} = 499/100 + 73759/8000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -6784652161 + 625 \cdot (7984 \cdot z[1] + 73759/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 257/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-257/40$,. The new difference equation is:

$$z[n+1] = -287/5 \cdot z[n]/(40 \cdot z[n] + 257)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -287/5 \cdot z[n]/(40 \cdot z[n] + 257) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -257/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -257/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 257/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 257/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 257/40$$
$$N = 4$$

Proving $P > 0$ in the region:

$$257/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 257/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 257/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/257 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/257$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 499/100 + 73759/8000/x_n$$

□

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For the parameters $\{M = 231/50, q = 797/100\}$:
For the parameters $\{M = 221/100, q = 153/25\}$:
For the parameters $\{M = 53/100, q = 209/100\}$:
For the parameters $\{M = 31/100, q = 187/100\}$:
For the parameters $\{M = 983/100, q = 303/50\}$: First we check that the equilibrium 1589/200 is LAS.

It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1589/200$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1589/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.12. *The equilibrium 1589/200, for the rational difference equation*

$$x_{n+1} = 303/50 + 599053/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -20200652641 + (242400 \cdot z[1] + 599053)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1589/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1589/200$,. The new difference equation is:

$$z[n+1] = -377 \cdot z[n]/(200 \cdot z[n] + 1589)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -377 \cdot z[n]/(200 \cdot z[n] + 1589) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -1589/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1589/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1589/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1589/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1589/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1589/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1589/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1589/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1589 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1589$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!
——————————————————————-
Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 303/50 + 599053/40000/x_n$$

□

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For the parameters $\{M = 249/25, q = 1/5\}$: First we check that the equilibrium 127/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 127/25$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 127/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.13. *The equilibrium 127/25 for the rational difference equation*

$$x_{n+1} = \ 1/5 + 15494/625/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -221533456 + (125 \cdot z[1] + 15494)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 127/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-127/25$,. The new difference equation is:

$$z[n+1] = -122 \cdot z[n]/(25 \cdot z[n] + 127)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -122 \cdot z[n]/(25 \cdot z[n] + 127)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -127/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -127/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 127/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 127/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 127/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$127/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 127/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 127/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/127 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/127$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 1/5 + 15494/625/x_n$$

□

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For the parameters $\{M = 139/25, q = 59/50\}$: First we check that the equilibrium 337/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 337/100$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 337/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.14. *The equilibrium 337/100, for the rational difference equation*

$$x_{n+1} = 59/50 + 73803/10000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -2300257521 + (11800 \cdot z[1] + 73803)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 337/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-337/100$,. The new difference equation is:

$$z[n+1] = -219 \cdot z[n]/(100 \cdot z[n] + 337)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -219 \cdot z[n]/(100 \cdot z[n] + 337) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -337/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -337/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 337/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 337/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 337/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$337/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 337/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 337/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/337 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/337$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 59/50 + 73803/10000/x_n$$

□

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For the parameters $\{M = 423/100, q = 1/50\}$: First we check that the equilibrium 17/8 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 17/8$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 17/8$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.15. *The equilibrium 17/8 for the rational difference equation*

$$x_{n+1} = 1/50 + 7157/1600/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -31414372081 + 625 \cdot (32 \cdot z[1] + 7157)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 17/8$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-17/8$,. The new difference equation is:

$$z[n+1] = -421/25 \cdot z[n]/(8 \cdot z[n] + 17)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -421/25 \cdot z[n]/(8 \cdot z[n] + 17) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -17/8$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -17/8 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 17/8$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 17/8, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 17/8$$
$$N = 4$$

Proving $P > 0$ in the region:

$$17/8 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 17/8$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 17/8$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$8/17 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 8/17$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 1/50 + 7157/1600/x_n$$

□

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For the parameters $\{M = 97/20, q = 907/100\}$:
For the parameters $\{M = 149/50, q = 111/100\}$: First we check that the equilibrium 409/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 409/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 409/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.16. *The equilibrium 409/200, for the rational difference equation*

$$x_{n+1} = 111/100 + 76483/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -1222830961 + (44400 \cdot z[1] + 76483)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 409/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-409/200$,. The new difference equation is:

$$z[n+1] = -187 \cdot z[n]/(200 \cdot z[n] + 409)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -187 \cdot z[n]/(200 \cdot z[n] + 409) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -409/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -409/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 409/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 409/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 409/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$409/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 409/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

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Proving $P > 0$ in the region:

$$0 < z[1] \leq 409/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/409 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/409$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 111/100 + 76483/40000/x_n$$

□

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For the parameters $\{M = 919/100, q = 413/50\}$: First we check that the equilibrium 349/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 349/40$
For $K = 1$ we get $\{FAIL, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 349/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.17. *The equilibrium 349/40 for the rational difference equation*

$$x_{n+1} = \ 413/50 + 32457/8000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -74805201 + 625 \cdot (13216 \cdot z[1] + 32457/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 349/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-349/40$,. The new difference equation is:

$$z[n+1] = -93/5 \cdot z[n]/(40 \cdot z[n] + 349)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -93/5 \cdot z[n]/(40 \cdot z[n] + 349) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -349/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -349/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 349/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 349/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 349/40$$
$$N = 4$$

Proving $P > 0$ in the region:

$$349/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 349/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 349/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/349 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/349$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 413/50 + 32457/8000/x_n$$

□

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For the parameters $\{M = 7/2, q = 309/50\}$:
For the parameters $\{M = 24/25, q = 59/20\}$:
For the parameters $\{M = 493/50, q = 949/100\}$: First we check that the equilibrium 387/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 387/40$
For $K = 1$ we get $\{FAIL, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 387/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.18. *The equilibrium 387/40 for the rational difference equation*

$$x_{n+1} = \ 949/100 + 14319/8000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -1874161 + 625 \cdot (15184 \cdot z[1] + 14319/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 387/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-387/40$,. The new difference equation is:

$$z[n+1] = -37/5 \cdot z[n]/(40 \cdot z[n] + 387)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -37/5 \cdot z[n]/(40 \cdot z[n] + 387) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -387/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -387/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 387/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 387/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 387/40 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$387/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 387/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 387/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/387 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/387$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 949/100 + 14319/8000/x_n$$

□

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For the parameters $\{M = 103/100, q = 319/50\}$:
For the parameters $\{M = 17/25, q = 357/100\}$:
For the parameters $\{M = 967/100, q = 261/50\}$: First we check that the equilibrium 1489/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1489/200$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1489/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.19. *The equilibrium 1489/200, for the rational difference equation*

$$x_{n+1} = 261/50 + 132521/8000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -39213900625 + (208800 \cdot z[1] + 662605)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1489/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1489/200$,. The new difference equation is:

$$z[n+1] = -445 \cdot z[n]/(200 \cdot z[n] + 1489)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -445 \cdot z[n]/(200 \cdot z[n] + 1489) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -1489/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1489/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1489/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1489/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 1489/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$1489/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1489/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

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Proving $P > 0$ in the region:

$$0 < z[1] \leq 1489/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1489 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1489$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 261/50 + 132521/8000/x_n$$

□

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For the parameters $\{M = 539/100, q = 141/25\}$:
For the parameters $\{M = 91/100, q = 33/20\}$:
For the parameters $\{M = 31/50, q = 108/25\}$:
For the parameters $\{M = 243/25, q = 87/25\}$: First we check that the equilibrium 33/5 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 33/5$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 33/5$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.20. *The equilibrium 33/5 for the rational difference equation*

$$x_{n+1} = 87/25 + 2574/125/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -4112784 + 625 \cdot (29 \cdot z[1] + 858/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 33/5$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-33/5$,. The new difference equation is:

$$z[n+1] = -78/5 \cdot z[n]/(5 \cdot z[n] + 33)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -78/5 \cdot z[n]/(5 \cdot z[n] + 33) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -33/5$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -33/5 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 33/5$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 33/5, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 33/5$$
$$N = 4$$

Proving $P > 0$ in the region:

$$33/5 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 33/5$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 33/5$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$5/33 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 5/33$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 87/25 + 2574/125/x_n$$

□

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For the parameters $\{M = 187/50, q = 747/100\}$:
For the parameters $\{M = 221/100, q = 167/50\}$:
For the parameters $\{M = 87/100, q = 25/4\}$:
For the parameters $\{M = 1, q = 97/50\}$:
For the parameters $\{M = 79/50, q = 381/50\}$:
For the parameters $\{M = 677/100, q = 103/100\}$: First we check that the equilibrium 39/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 39/10$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 39/10$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.21. *The equilibrium 39/10 for the rational difference equation*

$$x_{n+1} =, 103/100 + 11193/1000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -6784652161 + 10000 \cdot (103 \cdot z[1] + 11193/10)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 39/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-39/10$,. The new difference equation is:

$$z[n+1] = -287/10 \cdot z[n]/(10 \cdot z[n] + 39)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -287/10 \cdot z[n]/(10 \cdot z[n] + 39) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -39/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -39/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 39/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 39/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 39/10$$
$$N = 4$$

Proving $P > 0$ in the region:

$$39/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 39/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 39/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/39 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/39$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 103/100 + 11193/1000/x_n$$

□

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For the parameters $\{M = 683/100, q = 139/20\}$:
For the parameters $\{M = 177/25, q = 121/20\}$: First we check that the equilibrium 1313/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1313/200$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1313/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.22. *The equilibrium 1313/200, for the rational difference equation*

$$x_{n+1} = 121/20 + 135239/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -112550881 + (242000 \cdot z[1] + 135239)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1313/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1313/200$,. The new difference equation is:

$$z[n+1] = -103 \cdot z[n]/(200 \cdot z[n] + 1313)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -103 \cdot z[n]/(200 \cdot z[n] + 1313)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -1313/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1313/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1313/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1313/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1313/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1313/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1313/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1313/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1313 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1313$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 121/20 + 135239/40000/x_n$$

□

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For the parameters $\{M = 469/100, q = 239/100\}$: First we check that the equilibrium 177/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 177/50$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 177/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.23. *The equilibrium 177/50 for the rational difference equation*

$$x_{n+1} = 239/100 + 4071/1000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -174900625 + 16 \cdot (5975 \cdot z[1] + 20355/2)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 177/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-177/50$,. The new difference equation is:

$$z[n+1] = -115/2 \cdot z[n]/(50 \cdot z[n] + 177)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -115/2 \cdot z[n]/(50 \cdot z[n] + 177)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -177/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -177/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 177/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 177/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 177/50$$
$$N = 4$$

Proving $P > 0$ in the region:
$$177/50 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 177/50$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 177/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/177 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/177$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 239/100 + 4071/1000/x_n$$

□

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For the parameters $\{M = 91/10, q = 7/10\}$: First we check that the equilibrium 49/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 49/10$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 49/10$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.24. *The equilibrium 49/10 for the rational difference equation*

$$x_{n+1} =, 7/10 + 1029/50/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -63504 + (10 \cdot z[1] + 294)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 49/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-49/10$,. The new difference equation is:

$$z[n+1] = -42 \cdot z[n]/(10 \cdot z[n] + 49)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -42 \cdot z[n]/(10 \cdot z[n] + 49) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -49/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -49/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 49/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 49/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 49/10$$
$$N = 4$$

Proving $P > 0$ in the region:

$$49/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 49/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 49/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/49 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/49$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 7/10 + 1029/50/x_n$$

□

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For the parameters $\{M = 291/100, q = 391/100\}$:
For the parameters $\{M = 319/100, q = 167/100\}$: First we check that the equilibrium 243/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 243/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 243/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.25. *The equilibrium 243/100, for the rational difference equation*

$$x_{n+1} = 167/100 + 4617/2500/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -33362176 + (16700 \cdot z[1] + 18468)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 243/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-243/100$,. The new difference equation is:

$$z[n + 1] = -76 \cdot z[n]/(100 \cdot z[n] + 243)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -76 \cdot z[n]/(100 \cdot z[n] + 243) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -243/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -243/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 243/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 243/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 243/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$243/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 243/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 243/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/243 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/243$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 167/100 + 4617/2500/x_n$$

□

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For the parameters $\{M = 343/100, q = 379/100\}$:
For the parameters $\{M = 239/25, q = 881/100\}$: First we check that the equilibrium 1837/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1837/200$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1837/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.26. *The equilibrium 1837/200, for the rational difference equation*

$$x_{n+1} = 881/100 + 5511/1600/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -31640625 + (352400 \cdot z[1] + 137775)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1837/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1837/200$,. The new difference equation is:

$$z[n+1] = -75 \cdot z[n]/(200 \cdot z[n] + 1837)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -75 \cdot z[n]/(200 \cdot z[n] + 1837)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -1837/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1837/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1837/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1837/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1837/200$$
$$N = 4$$

Proving $P > 0$ in the region:
$$1837/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 1837/200$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1837/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1837 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1837$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 881/100 + 5511/1600/x_n$$

□

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For the parameters $\{M = 461/50, q = 78/25\}$: First we check that the equilibrium 617/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 617/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 617/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.27. *The equilibrium 617/100, for the rational difference equation*

$$x_{n+1} = 78/25 + 37637/2000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -8653650625 + (31200 \cdot z[1] + 188185)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 617/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-617/100$,. The new difference equation is:

$$z[n+1] = -305 \cdot z[n]/(100 \cdot z[n] + 617)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -305 \cdot z[n]/(100 \cdot z[n] + 617) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -617/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -617/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 617/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 617/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 617/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$617/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 617/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 617/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/617 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/617$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 78/25 + 37637/2000/x_n$$

□

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For the parameters $\{M = 16/25, q = 127/100\}$:
For the parameters $\{M = 661/100, q = 263/50\}$: First we check that the equilibrium 1187/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1187/200$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1187/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.28. *The equilibrium 1187/200, for the rational difference equation*

$$x_{n+1} = 263/50 + 32049/8000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -332150625 + (210400 \cdot z[1] + 160245)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1187/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1187/200$,. The new difference equation is:

$$z[n + 1] = -135 \cdot z[n]/(200 \cdot z[n] + 1187)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -135 \cdot z[n]/(200 \cdot z[n] + 1187)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -1187/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1187/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1187/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1187/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1187/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1187/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1187/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1187/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1187 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1187$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 263/50 + 32049/8000/x_n$$

□

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For the parameters $\{M = 639/100, q = 311/100\}$: First we check that the equilibrium 19/4 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 19/4$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 19/4$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.29. *The equilibrium 19/4 for the rational difference equation*

$$x_{n+1} = 311/100 + 779/100/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -723394816 + 625 \cdot (1244 \cdot z[1] + 3116)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 19/4$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-19/4$,. The new difference equation is:

$$z[n+1] = -164/25 \cdot z[n]/(4 \cdot z[n] + 19)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \rightarrow \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -164/25 \cdot z[n]/(4 \cdot z[n] + 19) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -19/4$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -19/4 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 19/4$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 19/4, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 19/4$$
$$N = 4$$

Proving $P > 0$ in the region:

$$19/4 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 19/4$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

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Proving $P > 0$ in the region:

$$0 < z[1] \leq 19/4$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$4/19 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 4/19$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 311/100 + 779/100/x_n$$

□

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For the parameters $\{M = 14/25, q = 283/50\}$:
For the parameters $\{M = 361/50, q = 81/25\}$: First we check that the equilibrium 523/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 523/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 523/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.30. *The equilibrium 523/100, for the rational difference equation*

$$x_{n+1} = 81/25 + 104077/10000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -1568239201 + (32400 \cdot z[1] + 104077)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 523/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-523/100$,. The new difference equation is:

$$z[n+1] = -199 \cdot z[n]/(100 \cdot z[n] + 523)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -199 \cdot z[n]/(100 \cdot z[n] + 523) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -523/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -523/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 523/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 523/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 523/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$523/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 523/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 523/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/523 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/523$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 81/25 + 104077/10000/x_n$$

□

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For the parameters $\{M = 28/5, q = 18/25\}$: First we check that the equilibrium 79/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 79/25$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 79/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.31. *The equilibrium 79/25 for the rational difference equation*

$$x_{n+1} =, 18/25 + 4819/625/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -13845841 + (450 \cdot z[1] + 4819)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 79/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-79/25$,. The new difference equation is:

$$z[n+1] = -61 \cdot z[n]/(25 \cdot z[n] + 79)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -61 \cdot z[n]/(25 \cdot z[n] + 79) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -79/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -79/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-79/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 79/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 79/25 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$79/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 79/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 79/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/79 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/79$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 18/25 + 4819/625/x_n$$

□

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For the parameters $\{M = 859/100, q = 223/100\}$: First we check that the equilibrium 541/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 541/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 541/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.32. *The equilibrium 541/100, for the rational difference equation*

$$x_{n+1} = 223/100 + 86019/5000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -10226063376 + (22300 \cdot z[1] + 172038)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 541/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-541/100$,. The new difference equation is:

$$z[n+1] = -318 \cdot z[n]/(100 \cdot z[n] + 541)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -318 \cdot z[n]/(100 \cdot z[n] + 541)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -541/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -541/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 541/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 541/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 541/100 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$541/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 541/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 541/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/541 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/541$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 223/100 + 86019/5000/x_n$$

□

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For the parameters $\{M = 23/20, q = 92/25\}$:
For the parameters $\{M = 69/100, q = 799/100\}$:
For the parameters $\{M = 249/50, q = 777/100\}$:
For the parameters $\{M = 5/4, q = 18/25\}$: First we check that the equilibrium 197/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 197/200$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 197/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.33. *The equilibrium 197/200, for the rational difference equation*

$$x_{n+1} = 18/25 + 10441/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -7890481 + (28800 \cdot z[1] + 10441)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 197/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-197/200$,. The new difference equation is:

$$z[n+1] = -53 \cdot z[n]/(200 \cdot z[n] + 197)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -53 \cdot z[n]/(200 \cdot z[n] + 197)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -197/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -197/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 197/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 197/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 197/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$197/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 197/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 197/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/197 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/197$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 18/25 + 10441/40000/x_n$$

□

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For the parameters $\{M = 193/25, q = 897/100\}$:
For the parameters $\{M = 31/4, q = 721/100\}$: First we check that the equilibrium 187/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 187/25$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 187/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.34. *The equilibrium 187/25 for the rational difference equation*

$$x_{n+1} = \ 721/100 + 5049/2500/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -531441 + 16 \cdot (18025 \cdot z[1] + 5049)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 187/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-187/25$,. The new difference equation is:

$$z[n+1] = -27/4 \cdot z[n]/(25 \cdot z[n] + 187)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -27/4 \cdot z[n]/(25 \cdot z[n] + 187)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -187/25$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -187/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 187/25$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 187/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 187/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$187/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 187/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 187/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/187 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/187$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 721/100 + 5049/2500/x_n$$

□

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For the parameters $\{M = 67/25, q = 209/50\}$:
For the parameters $\{M = 993/100, q = 53/25\}$: First we check that the equilibrium 241/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 241/40$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 241/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.35. *The equilibrium 241/40 for the rational difference equation*

$$x_{n+1} = \ 53/25 + 188221/8000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -372052421521 + 625 \cdot (3392 \cdot z[1] + 188221/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 241/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-241/40$,. The new difference equation is:

$$z[n+1] = -781/5 \cdot z[n]/(40 \cdot z[n] + 241)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -781/5 \cdot z[n]/(40 \cdot z[n] + 241) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -241/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -241/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 241/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 241/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 241/40 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$241/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 241/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 241/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/241 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/241$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 53/25 + 188221/8000/x_n$$

□

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For the parameters $\{M = 243/100, q = 497/50\}$:
For the parameters $\{M = 789/100, q = 903/100\}$:
For the parameters $\{M = 687/100, q = 13/20\}$: First we check that the equilibrium 94/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 94/25$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 94/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.36. *The equilibrium 94/25 for the rational difference equation*

$$x_{n+1} =, 13/20 + 14617/1250/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -9354951841 + 16 \cdot (1625 \cdot z[1] + 29234)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 94/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-94/25$,. The new difference equation is:

$$z[n+1] = -311/4 \cdot z[n]/(25 \cdot z[n] + 94)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -311/4 \cdot z[n]/(25 \cdot z[n] + 94)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -94/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -94/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 94/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 94/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 94/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$94/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 94/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 94/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/94 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/94$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 13/20 + 14617/1250/x_n$$

□

===

For the parameters $\{M = 471/100, q = 7/5\}$: First we check that the equilibrium 611/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 611/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 611/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.37. *The equilibrium 611/200, for the rational difference equation*

$$x_{n+1} = 7/5 + 202241/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -12003612721 + (56000 \cdot z[1] + 202241)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 611/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-611/200$,. The new difference equation is:

$$z[n+1] = -331 \cdot z[n]/(200 \cdot z[n] + 611)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -331 \cdot z[n]/(200 \cdot z[n] + 611) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -611/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -611/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 611/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 611/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 611/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$611/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 611/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 611/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/611 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/611$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 7/5 + 202241/40000/x_n$$

□

==

For the parameters $\{M = 427/50, q = 64/25\}$: First we check that the equilibrium 111/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 111/20$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 111/20$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.38. *The equilibrium 111/20 for the rational difference equation*

$$x_{n+1} = 64/25 + 33189/2000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -7992538801 + 625 \cdot (1024 \cdot z[1] + 33189/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 111/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-111/20$,. The new difference equation is:

$$z[n+1] = -299/5 \cdot z[n]/(20 \cdot z[n] + 111)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -299/5 \cdot z[n]/(20 \cdot z[n] + 111)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -111/20$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -111/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 111/20$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 111/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 111/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$111/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 111/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 111/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/111 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/111$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 64/25 + 33189/2000/x_n$$

□

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For the parameters $\{M = 242/25, q = 129/20\}$: First we check that the equilibrium 1613/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 1613/200$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 1613/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.39. *The equilibrium 1613/200, for the rational difference equation*

$$x_{n+1} = 129/20 + 520999/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -10884540241 + (258000 \cdot z[1] + 520999)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 1613/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GA S when initial conditions are greater than $-1613/200$,. The new difference equation is:

$$z[n+1] = -323 \cdot z[n]/(200 \cdot z[n] + 1613)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -323 \cdot z[n]/(200 \cdot z[n] + 1613)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -1613/200$ then we are done (see applicable Theore m in Emilie Hogan's PhD thesis). Simplify this so that it is of the form someth ing > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -1613/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 1613/200$ so that we have a poly nomial that we wish to prove is positive when all variables are positive (rathe r than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 1613/ 200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 1613/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$1613/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 1613/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 1613/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/1613 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/1613$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 129/20 + 520999/40000/x_n$$

□

==

For the parameters $\{M = 359/50, q = 13/25\}$: First we check that the equilibrium 77/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 77/20$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 77/20$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.40. *The equilibrium 77/20 for the rational difference equation*

$$x_{n+1} =, 13/25 + 25641/2000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -12296370321 + 625 \cdot (208 \cdot z[1] + 25641/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 77/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-77/20$,. The new difference equation is:

$$z[n+1] = -333/5 \cdot z[n]/(20 \cdot z[n] + 77)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -333/5 \cdot z[n]/(20 \cdot z[n] + 77)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -77/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -77/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-77/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 77/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$
\begin{aligned}
P &= \text{ from above} \\
Z &= [z[1]] \\
\bar{x} &= 77/20 \\
N &= 4
\end{aligned}
$$

Proving $P > 0$ in the region:

$$77/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 77/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 77/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/77 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/77$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 13/25 + 25641/2000/x_n$$

□

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For the parameters $\{M = 2/5, q = 218/25\}$:
For the parameters $\{M = 547/100, q = 13/20\}$: First we check that the equilibrium 153/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 153/50$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 153/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.41. *The equilibrium 153/50 for the rational difference equation*

$$x_{n+1} = 13/20 + 36873/5000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -3373402561 + 16 \cdot (1625 \cdot z[1] + 36873/2)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 153/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-153/50$,. The new difference equation is:

$$z[n+1] = -241/2 \cdot z[n]/(50 \cdot z[n] + 153)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -241/2 \cdot z[n]/(50 \cdot z[n] + 153) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -153/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -153/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 153/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 153/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 153/50$$
$$N = 4$$

Proving $P > 0$ in the region:
$$153/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 153/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 153/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/153 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/153$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 13/20 + 36873/5000/x_n$$

□

==

For the parameters $\{M = 171/50, q = 729/100\}$:
For the parameters $\{M = 179/20, q = 803/100\}$: First we check that the equilibrium 849/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 849/100$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 849/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.42. *The equilibrium 849/100, for the rational difference equation*

$$x_{n+1} = 803/100 + 19527/5000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -4477456 + (80300 \cdot z[1] + 39054)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 849/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-849/100$,. The new difference equation is:
$$z[n+1] = -46 \cdot z[n]/(100 \cdot z[n] + 849)$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -46 \cdot z[n]/(100 \cdot z[n] + 849) \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -849/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -849/100 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 849/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 849/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 849/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$849/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 849/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 849/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/849 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/849$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 803/100 + 19527/5000/x_n$$

□

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For the parameters $\{M = 607/100, q = 23/100\}$: First we check that the equilibrium 63/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 63/20$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 63/20$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.43. *The equilibrium 63/20 for the rational difference equation*

$$x_{n+1} =, 23/100 + 4599/500/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -7269949696 + 625 \cdot (92 \cdot z[1] + 18396/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 63/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-63/20$,. The new difference equation is:

$$z[n+1] = -292/5 \cdot z[n]/(20 \cdot z[n] + 63)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -292/5 \cdot z[n]/(20 \cdot z[n] + 63) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -63/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -63/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-63/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 63/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 63/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$63/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 63/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 63/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/63 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/63$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 23/100 + 4599/500/x_n$$

□

==

For the parameters $\{M = 249/50, q = 237/50\}$: First we check that the equilibrium 243/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 243/50$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 243/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.44. *The equilibrium 243/50 for the rational difference equation*

$$x_{n+1} = \ 237/50 + 729/1250/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -144 + (3950 \cdot z[1] + 486)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 243/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-243/50$,. The new difference equation is:

$$z[n+1] = -6 \cdot z[n]/(50 \cdot z[n] + 243)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -6 \cdot z[n]/(50 \cdot z[n] + 243)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -243/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -243/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 243/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 243/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 243/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$243/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 243/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 243/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/243 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/243$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 237/50 + 729/1250/x_n$$

□

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For the parameters $\{M = 531/100, q = 967/100\}$:
For the parameters $\{M = 379/100, q = 114/25\}$:
For the parameters $\{M = 223/25, q = 391/50\}$: First we check that the equilibrium 837/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 837/100$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 837/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.45. *The equilibrium 837/100, for the rational difference equation*

$$x_{n+1} = 391/50 + 9207/2000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -9150625 + (78200 \cdot z[1] + 46035)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 837/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-837/100$,. The new difference equation is:

$$z[n+1] = -55 \cdot z[n]/(100 \cdot z[n] + 837)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -55 \cdot z[n]/(100 \cdot z[n] + 837) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -837/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -837/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 837/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 837/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 837/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$837/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 837/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 837/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/837 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/837$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 391/50 + 9207/2000/x_n$$

□

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For the parameters $\{M = 471/100, q = 17/100\}$: First we check that the equilibrium 61/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 61/25$
For $K = 1$ we get $\{false, true, \}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 61/25$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.46. *The equilibrium 61/25 for the rational difference equation*

$$x_{n+1} =, 17/100 + 13847/2500/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -2655237841 + 16 \cdot (425 \cdot z[1] + 13847)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 61/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-61/25$,. The new difference equation is:

$$z[n+1] = -227/4 \cdot z[n]/(25 \cdot z[n] + 61)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -227/4 \cdot z[n]/(25 \cdot z[n] + 61) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -61/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -61/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-61/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 61/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$
\begin{aligned}
P &= \text{ from above} \\
Z &= [z[1]] \\
\bar{x} &= 61/25 \\
N &= 4
\end{aligned}
$$

Proving $P > 0$ in the region:

$$61/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 61/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 61/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/61 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/61$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 17/100 + 13847/2500/x_n$$

□

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For the parameters $\{M = 174/25, q = 31/5\}$: First we check that the equilibrium 329/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 329/50$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 329/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.47. *The equilibrium 329/50 for the rational difference equation*

$$x_{n+1} = \ 31/5 + 6251/2500/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -130321 + (15500 \cdot z[1] + 6251)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 329/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-329/50$,. The new difference equation is:

$$z[n+1] = -19 \cdot z[n]/(50 \cdot z[n] + 329)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -19 \cdot z[n]/(50 \cdot z[n] + 329)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$
for all $z[1] > -329/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -329/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 329/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 329/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 329/50$$
$$N = 4$$

Proving $P > 0$ in the region:
$$329/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 329/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 329/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/329 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/329$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 31/5 + 6251/2500/x_n$$

□

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For the parameters $\{M = 389/50, q = 61/100\}$: First we check that the equilibrium 839/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 839/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 839/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.48. *The equilibrium 839/200, for the rational difference equation*

$$x_{n+1} = 61/100 + 601563/40000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -264287499921 + (24400 \cdot z[1] + 601563)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 839/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-839/200$,. The new difference equation is:

$$z[n+1] = -717 \cdot z[n]/(200 \cdot z[n] + 839)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -717 \cdot z[n]/(200 \cdot z[n] + 839) \rangle$$

3. Notice that if

$$\frac{\left| Q^K(\langle z[1] \rangle) \right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -839/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left| Q^K(\langle z[1], \rangle) \right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -839/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 839/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 839/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 839/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$839/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 839/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 839/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/839 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/839$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 61/100 + 601563/40000/x_n$$

□

==

For the parameters $\{M = 207/100, q = 51/100\}$: First we check that the equilibrium 129/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 129/100$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 129/100$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.49. *The equilibrium 129/100, for the rational difference equation*

$$x_{n+1} = 51/100 + 5031/5000/x_n,$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -4112784 + (1700 \cdot z[1] + 3354)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 129/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-129/100$,. The new difference equation is:

$$z[n+1] = -78 \cdot z[n]/(100 \cdot z[n] + 129)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -78 \cdot z[n]/(100 \cdot z[n] + 129)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -129/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -129/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 129/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 129/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$
\begin{aligned}
P &= \text{ from above} \\
Z &= [z[1]] \\
\bar{x} &= 129/100 \\
N &= 4
\end{aligned}
$$

Proving $P > 0$ in the region:

$$129/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 129/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 129/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/129 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/129$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 51/100 + 5031/5000/x_n$$

□

==

For the parameters $\{M = 413/50, q = 369/100\}$: First we check that the equilibrium 239/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 239/40$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 239/40$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 6.50. *The equilibrium 239/40 for the rational difference equation*

$$x_{n+1} = \ 369/100 + 109223/8000/x_n$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -43617904801 + 625 \cdot (5904 \cdot z[1] + 109223/5)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 239/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-239/40$,. The new difference equation is:

$$z[n+1] = -457/5 \cdot z[n]/(40 \cdot z[n] + 239)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -457/5 \cdot z[n]/(40 \cdot z[n] + 239)\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -239/40$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -239/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 239/40$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 239/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 239/40$$
$$N = 4$$

Proving $P > 0$ in the region:
$$239/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 239/40$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 239/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/239 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/239$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = 369/100 + 109223/8000/x_n$$

□

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The parameter values for which the $K = 2$ are:

$$[\{M = 5/4, q = 18/25\}, 197/200], [\{M = 11/2, q = 62/25\}, 399/100], [\{M = 28/5, q = 18/25\}, 79/25], [\{M = 31/4, q = 721/100\}, 187/25], [\{M = 33/5, q = 387/100\}, 1047/200], [\{M = 49/10, q = 27/100\}, 517/200], [\{M = 91/10, q = 7/10\}, 49/10], [\{M = 139/25, q = 59/50\}, 337/100], [\{M = 149/50, q = 111/100\}, 409/200], [\{M = 174/25, q = 31/5\}, 329/50], [\{M = 177/25, q = 121/20\}, 1313/200], [\{M = 179/20, q = 803/100\}, 849/100], [\{M = 207/100, q =$$

$51/100\}, 129/100], [\{M = 223/25, q = 391/50\}, 837/100], [\{M = 226/25, q = 59/50\}, 511/100], [\{M = 239/25, q = 881/100\}, 1837/200], [\{M = 242/25, q = 129/20\}, 1613/200], [\{M = 243/25, q = 87/25\}, 33/5], [\{M = 249/25, q = 1/5\}, 127/25], [\{M = 249/50, q = 237/50\}, 243/50], [\{M = 319/100, q = 167/100\}, 243/100], [\{M = 359/50, q = 13/25\}, 77/20], [\{M = 361/50, q = 81/25\}, 523/100], [\{M = 389/50, q = 61/100\}, 839/200], [\{M = 393/50, q = 499/100\}, 257/40], [\{M = 401/50, q = 29/4\}, 1527/200], [\{M = 413/50, q = 15/2\}, 197/25], [\{M = 413/50, q = 369/100\}, 239/40], [\{M = 423/100, q = 1/50\}, 17/8], [\{M = 427/50, q = 64/25\}, 111/20], [\{M = 437/100, q = 1/50\}, 439/200], [\{M = 461/50, q = 78/25\}, 617/100], [\{M = 469/100, q = 239/100\}, 177/50], [\{M = 471/100, q = 7/5\}, 611/200], [\{M = 471/100, q = 17/100\}, 61/25], [\{M = 493/50, q = 949/100\}, 387/40], [\{M = 547/100, q = 13/20\}, 153/50], [\{M = 607/100, q = 23/100\}, 63/20], [\{M = 639/100, q = 311/100\}, 19/4], [\{M = 661/100, q = 263/50\}, 1187/200], [\{M = 677/100, q = 103/100\}, 39/10], [\{M = 687/100, q = 13/20\}, 94/25], [\{M = 859/100, q = 223/100\}, 541/100], [\{M = 881/100, q = 843/100\}, 431/50], [\{M = 883/100, q = 209/50\}, 1301/200], [\{M = 919/100, q = 413/50\}, 349/40], [\{M = 967/100, q = 261/50\}, 1489/200], [\{M = 983/100, q = 303/50\}, 1589/200], [\{M = 993/100, q = 53/25\}, 241/40], [\{M = 999/100, q = 203/50\}, 281/40],$

Finished investigating difference equation 6 out of 7

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7 $(-1/4 \cdot q^2 + 1/4 \cdot M^2 + x_n)/(q + 1 + x_n)$

For the rational difference equation

$$x_{n+1} = (-1/4 \cdot q^2 + 1/4 \cdot M^2 + x_n)/(q + 1 + x_n)$$

We will try to prove that the equilibrium is GAS for various values of the par ameters $\{M, q\}$. For the parameters $\{M = 789/100, q = 19/10\}$: First we check that the equilibrium 599/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 599/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.1. *The equilibrium 599/200, for the rational difference equation*

$$x_{n+1} = (586421/40000 + x_n)/(29/10 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -159201 + (580 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 599/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-599/200$,. The new difference equation is:

$$z[n+1] = -399 \cdot z[n]/(1179 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -399 \cdot z[n]/(1179 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -599/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -599/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 599/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 599/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 599/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$599/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 599/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 599/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/599 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/599$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (586421/40000 + x_n)/(29/10 + x_n)$$

□

==

For the parameters $\{M = 61/50, q = 71/100\}$: First we check that the equilibrium 51/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 51/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.2. *The equilibrium 51/200 for the rational difference equation*

$$x_{n+1} = \ (9843/40000 + x_n)/(171/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -22201 + (342 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 51/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-51/200$,. The new difference equation is:

$$z[n+1] = 149 \cdot z[n]/(393 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle 149 \cdot z[n]/(393 + 200 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -51/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -51/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 51/200$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 51/200, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 51/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$51/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 51/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 51/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/51 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/51$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

———————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (9843/40000 + x_n)/(171/100 + x_n)$$

□

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For the parameters $\{M = 393/100, q = 109/50\}$: First we check that the equilibrium 7/8 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 7/8$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.3. *The equilibrium 7/8 for the rational difference equation*

$$x_{n+1} = (4277/1600 + x_n)/(159/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (636 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 7/8$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when i nitial conditions are greater than $-7/8$,. The new difference equation is:

$$z[n+1] = 25 \cdot z[n]/(811 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 25 \cdot z[n]/(811 + 200 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -7/8$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -7/8 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 7/8$ so that we have a polynomia l that we wish to prove is positive when all variables are positive (rather tha n when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 7/8, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 7/8 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$7/8 \le z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 7/8$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \le 7/8$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$8/7 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 8/7$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (4277/1600 + x_n)/(159/50 + x_n)$$

□

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For the parameters $\{M = 193/25, q = 37/25\}$: First we check that the equilibrium 78/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 78/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.4. *The equilibrium 78/25 for the rational difference equation*

$$x_{n+1} =, (1794/125 + x_n)/(62/25 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2809 + 25 \cdot (62/5 + 5 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 78/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-78/25$,. The new difference equation is:

$$z[n+1] = -53/5 \cdot z[n]/(28 + 5 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -53/5 \cdot z[n]/(28 + 5 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -78/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -78/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 78/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 78/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 78/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$78/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 78/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 78/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/78 \le z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/78$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (1794/125 + x_n)/(62/25 + x_n)$$

□

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For the parameters $\{M = 179/20, q = 24/5\}$: First we check that the equilibrium 83/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 83/40$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.5. *The equilibrium 83/40 for the rational difference equation*

$$x_{n+1} =, (913/64 + x_n)/(29/5 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1849 + 25 \cdot (232/5 + 8 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 83/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-83/40$,. The new difference equation is:

$$z[n + 1] = -43/5 \cdot z[n]/(63 + 8 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -43/5 \cdot z[n]/(63 + 8 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -83/40$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -83/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 83/40$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 83/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 83/40$$
$$N = 4$$

Proving $P > 0$ in the region:

$$83/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 83/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 83/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/83 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/83$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (913/64 + x[n])/(29/5 + x_n)$$

□

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For the parameters $\{M = 309/50, q = 101/20\}$: First we check that the equilibrium 113/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 113/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.6. *The equilibrium 113/200, for the rational difference equation*

$$x_{n+1} = (126899/40000 + x_n)/(121/20 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -7569 + (1210 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 113/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-113/200$,. The new difference equation is:

$$z[n+1] = 87 \cdot z[n]/(1323 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 87 \cdot z[n]/(1323 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -113/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -113/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 113/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 113/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 113/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$113/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 113/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 113/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/113 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/113$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (126899/40000 + x_n)/(121/20 + x_n)$$

□

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For the parameters $\{M = 181/25, q = 103/20\}$: First we check that the equilibrium 209/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 209/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.7. *The equilibrium 209/200, for the rational difference equation*

$$x_{n+1} = (258951/40000 + x_n)/(123/20 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -81 + (1230 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 209/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-209/200$,. The new difference equation is:

$$z[n+1] = -9 \cdot z[n]/(1439 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -9 \cdot z[n]/(1439 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -209/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -209/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 209/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 209/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 209/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$209/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 209/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 209/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/209 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/209$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (258951/40000 + x_n)/(123/20 + x_n)$$

□

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For the parameters $\{M = 577/100, q = 4/5\}$: First we check that the equilibrium 497/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 497/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.8. *The equilibrium 497/200, for the rational difference equation*

$$x_{n+1} = (326529/40000 + x_n)/(9/5 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -88209 + (360 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 497/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-497/200$,. The new difference equation is:

$$z[n+1] = -297 \cdot z[n]/(857 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -297 \cdot z[n]/(857 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -497/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -497/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 497/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 497/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 497/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$497/200 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 497/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 497/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/497 \le z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/497$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (326529/40000 + x_n)/(9/5 + x_n)$$

□

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For the parameters $\{M = 517/100, q = 247/50\}$: First we check that the equilibrium 23/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 23/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.9. *The equilibrium 23/200 for the rational difference equation*

$$x_{n+1} = (23253/40000 + x_n)/(297/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -31329 + (1188 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 23/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-23/200$,. The new difference equation is:

$$z[n+1] = 177 \cdot z[n]/(1211 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 177 \cdot z[n]/(1211 + 200 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -23/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -23/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 23/200$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 23/200, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 23/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$23/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 23/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 23/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/23 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/23$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (23253/40000 + x_n)/(297/50 + x_n)$$

□

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For the parameters $\{M = 901/100, q = 201/100\}$: First we check that the equilibrium 7/2 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 7/2$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.10. *The equilibrium 7/2 for the rational difference equation*

$$x_{n+1} = (3857/200 + x_n)/(301/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -62500 + (301 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 7/2$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when i nitial conditions are greater than $-7/2$,. The new difference equation is:

$$z[n+1] = -250 \cdot z[n]/(651 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -250 \cdot z[n]/(651 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -7/2$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -7/2 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 7/2$ so that we have a polynomia l that we wish to prove is positive when all variables are positive (rather tha n when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium $7/2$, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 7/2 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$7/2 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 7/2$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 7/2$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$2/7 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 2/7$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (3857/200 + x_n)/(301/100 + x_n)$$

□

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For the parameters $\{M = 35/4, q = 253/100\}$: First we check that the equilibrium 311/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 311/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.11. *The equilibrium 311/100, for the rational difference equation*

$$x_{n+1} = (43851/2500 + x_n)/(353/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -44521 + 16 \cdot (353/4 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 311/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-311/100$,. The new difference equation is:

$$z[n+1] = -211/4 \cdot z[n]/(166 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -211/4 \cdot z[n]/(166 + 25 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -311/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -311/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 311/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 311/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 311/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$311/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 311/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 311/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/311 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/311$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (43851/2500 + x_n)/(353/100 + x_n)$$

□

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For the parameters $\{M = 791/100, q = 67/100\}$: First we check that the equilibrium 181/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 181/50$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 181/50$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.12. *The equilibrium 181/50 for the rational difference equation*

$$x_{n+1} = \ (77649/5000 + x_n)/(167/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -4711998736 + (183187 + 26700 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 181/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-181/50$,. The new difference equation is:

$$z[n+1] = -262 \cdot z[n]/(529 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -262 \cdot z[n]/(529 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -181/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -181/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 181/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 181/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 181/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$181/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 181/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 181/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/181 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/181$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = (77649/5000 + x_n)/(167/100 + x_n)$$

□

==

For the parameters $\{M = 234/25, q = 121/20\}$: First we check that the equilibrium 331/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 331/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.13. *The equilibrium 331/200, for the rational difference equation*

$$x_{n+1} = (510071/40000 + x_n)/(141/20 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -17161 + (1410 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 331/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-331/200$,. The new difference equation is:

$$z[n+1] = -131 \cdot z[n]/(1741 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -131 \cdot z[n]/(1741 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -331/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -331/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 331/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 331/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 331/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$331/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 331/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 331/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/331 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/331$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (510071/40000 + x_n)/(141/20 + x_n)$$

□

==

For the parameters $\{M = 67/20, q = 52/25\}$: First we check that the equilibrium 127/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 127/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.14. *The equilibrium 127/200, for the rational difference equation*

$$x_{n+1} = (68961/40000 + x_n)/(77/25 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -5329 + (616 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 127/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-127/200$,. The new difference equation is:

$$z[n+1] = 73 \cdot z[n]/(743 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 73 \cdot z[n]/(743 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -127/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -127/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 127/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 127/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 127/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$127/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 127/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 127/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/127 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/127$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (68961/40000 + x_n)/(77/25 + x_n)$$

□

==

For the parameters $\{M = 437/100, q = 171/100\}$: First we check that the equilibrium 133/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 133/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.15. *The equilibrium 133/100, for the rational difference equation*

$$x_{n+1} = (2527/625 + x_n)/(271/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1089 + 16 \cdot (271/4 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 133/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-133/100$,. The new difference equation is:

$$z[n+1] = -33/4 \cdot z[n]/(101 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -33/4 \cdot z[n]/(101 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -133/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -133/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 133/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 133/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 133/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$133/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 133/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 133/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/133 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/133$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (2527/625 + x_n)/(271/100 + x_n)$$

□

==

For the parameters $\{M = 261/50, q = 337/100\}$: First we check that the equilibrium 37/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 37/40$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.16. *The equilibrium 37/40 for the rational difference equation*

$$x_{n+1} =, (31783/8000 + x_n)/(437/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -225 + (874 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 37/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-37/40$,. The new difference equation is:

$$z[n+1] = 15 \cdot z[n]/(1059 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 15 \cdot z[n]/(1059 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -37/40$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -37/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 37/40$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 37/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 37/40 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$37/40 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 37/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 37/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/37 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/37$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (31783/8000 + x_n)/(437/100 + x_n)$$

□

==

For the parameters $\{M = 633/100, q = 53/50\}$: First we check that the equilibrium 527/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 527/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.17. *The equilibrium 527/200, for the rational difference equation*

$$x_{n+1} = (389453/40000 + x_n)/(103/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -106929 + (412 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 527/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-527/200$,. The new difference equation is:

$$z[n+1] = -327 \cdot z[n]/(939 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -327 \cdot z[n]/(939 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

   for all $z[1] > -527/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

   and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -527/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 527/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 527/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 527/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$527/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 527/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 527/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/527 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/527$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (389453/40000 + x_n)/(103/50 + x_n)$$

□

==

For the parameters $\{M = 299/50, q = 137/100\}$: First we check that the equilibrium 461/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 461/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.18. *The equilibrium 461/200, for the rational difference equation*

$$x_{n+1} = (67767/8000 + x_n)/(237/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -68121 + 25 \cdot (474/5 + 40 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 461/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-461/200$,. The new difference equation is:

$$z[n+1] = -261/5 \cdot z[n]/(187 + 40 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -261/5 \cdot z[n]/(187 + 40 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -461/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -461/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 461/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 461/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 461/200$$
$$N = 4$$

Proving $P > 0$ in the region:
$$461/200 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 461/200$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 461/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/461 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/461$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (67767/8000 + x_n)/(237/100 + x_n)$$

□

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For the parameters $\{M = 887/100, q = 737/100\}$: First we check that the equilibrium 3/4 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 3/4$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.19. *The equilibrium 3/4 for the rational difference equation*

$$x_{n+1} = (609/100 + x_n)/(837/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + 16 \cdot (837/4 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 3/4$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when i nitial conditions are greater than $-3/4$,. The new difference equation is:

$$z[n+1] = 25/4 \cdot z[n]/(228 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 25/4 \cdot z[n]/(228 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -3/4$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -3/4 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 3/4$ so that we have a polynomia l that we wish to prove is positive when all variables are positive (rather tha n when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 3/4, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 3/4$$
$$N = 4$$

Proving $P > 0$ in the region:

$$3/4 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 3/4$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 3/4$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$4/3 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 4/3$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (609/100 + x_n)/(837/100 + x_n)$$

□

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For the parameters $\{M = 727/100, q = 79/20\}$: First we check that the equilibrium 83/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 83/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.20. *The equilibrium 83/50 for the rational difference equation*

$$x_{n+1} =, (46563/5000 + x_n)/(99/20 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -4356 + (495 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 83/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-83/50$,. The new difference equation is:

$$z[n + 1] = -66 \cdot z[n]/(661 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -66 \cdot z[n]/(661 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -83/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -83/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 83/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 83/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 83/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$83/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 83/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 83/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/83 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/83$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (46563/5000 + x_n)/(99/20 + x_n)$$

□

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For the parameters $\{M = 867/100, q = 307/100\}$: First we check that the equilibrium 14/5 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 14/5$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.21. *The equilibrium 14/5 for the rational difference equation*

$$x_{n+1} = (4109/250 + x_n)/(407/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -32400 + (407 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 14/5$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-14/5$,. The new difference equation is:

$$z[n+1] = -180 \cdot z[n]/(687 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -180 \cdot z[n]/(687 + 100 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -14/5$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -14/5 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 14/5$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 14/5, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:
$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 14/5 \\ N &= 4 \end{aligned}$$
Proving $P > 0$ in the region:
$$14/5 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 14/5$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 14/5$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$5/14 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 5/14$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (4109/250 + x_n)/(407/100 + x_n)$$

□

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For the parameters $\{M = 98/25, q = 251/100\}$: First we check that the equilibrium 141/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 141/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.22. *The equilibrium 141/200, for the rational difference equation*

$$x_{n+1} = (90663/40000 + x_n)/(351/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -3481 + (702 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 141/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-141/200$,. The new difference equation is:

$$z[n+1] = 59 \cdot z[n]/(843 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 59 \cdot z[n]/(843 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -141/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -141/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 141/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 141/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 141/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$141/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 141/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 141/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/141 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/141$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (90663/40000 + x_n)/(351/100 + x_n)$$

□

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For the parameters $\{M = 501/100, q = 177/100\}$: First we check that the equilibrium 81/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 81/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.23. *The equilibrium 81/50 for the rational difference equation*

$$x_{n+1} =, (27459/5000 + x_n)/(277/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -3844 + (277 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 81/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-81/50$,. The new difference equation is:

$$z[n + 1] = -62 \cdot z[n]/(439 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -62 \cdot z[n]/(439 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -81/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -81/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 81/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 81/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 81/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$81/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 81/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 81/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/81 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/81$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!
——————————————————————-
Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (27459/5000 + x_n)/(277/100 + x_n)$$

□

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For the parameters $\{M = 164/25, q = 1\}$: First we check that the equilibrium 139/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 139/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.24. *The equilibrium 139/50 for the rational difference equation*

$$x_{n+1} = \ (26271/2500 + x_n)/(2 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -7921 + (100 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 139/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-139/50$,. The new difference equation is:

$$z[n+1] = -89 \cdot z[n]/(239 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -89 \cdot z[n]/(239 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -139/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -139/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 139/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 139/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 139/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$139/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 139/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 139/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/139 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/139$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (26271/2500 + x_n)/(2 + x_n)$$

□

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For the parameters $\{M = 227/25, q = 142/25\}$: First we check that the equilibrium 17/10 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 17/10$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.25. *The equilibrium 17/10 for the rational difference equation*

$$x_{n+1} =, (6273/500 + x_n)/(167/25 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1225 + (334 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 17/10$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-17/10$,. The new difference equation is:

$$z[n+1] = -35 \cdot z[n]/(419 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -35 \cdot z[n]/(419 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -17/10$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -17/10 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 17/10$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 17/10, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 17/10$$
$$N = 4$$

Proving $P > 0$ in the region:

$$17/10 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 17/10$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 17/10$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$10/17 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 10/17$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (6273/500 + x_n)/(167/25 + x_n)$$

□

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For the parameters $\{M = 849/100, q = 743/100\}$: First we check that the equilibrium 53/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 53/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.26. *The equilibrium 53/100 for the rational difference equation*

$$x_{n+1} = \ (10547/2500 + x_n)/(843/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2209 + 16 \cdot (843/4 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 53/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-53/100$,. The new difference equation is:

$$z[n+1] = 47/4 \cdot z[n]/(224 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 47/4 \cdot z[n]/(224 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -53/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -53/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 53/100$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 53/100, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 53/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$53/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 53/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 53/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/53 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/53$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (10547/2500 + x_n)/(843/100 + x_n)$$

□

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For the parameters $\{M = 631/100, q = 117/25\}$: First we check that the equilibrium 163/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 163/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.27. *The equilibrium 163/200, for the rational difference equation*

$$x_{n+1} = (179137/40000 + x_n)/(142/25 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1369 + (1136 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 163/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-163/200$,. The new difference equation is:

$$z[n+1] = 37 \cdot z[n]/(1299 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 37 \cdot z[n]/(1299 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -163/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -163/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 163/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 163/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 163/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$163/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 163/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 163/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/163 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/163$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (179137/40000 + x_n)/(142/25 + x_n)$$

□

==

For the parameters $\{M = 491/100, q = 79/50\}$: First we check that the equilibrium 333/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 333/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.28. *The equilibrium 333/200, for the rational difference equation*

$$x_{n+1} = (216117/40000 + x_n)/(129/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -17689 + (200 \cdot z[1] + 516)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 333/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-333/200$,. The new difference equation is:

$$z[n+1] = -133 \cdot z[n]/(849 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -133 \cdot z[n]/(849 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -333/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -333/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 333/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 333/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 333/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$333/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 333/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 333/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/333 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/333$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (216117/40000 + x_n)/(129/50 + x_n)$$

□

==

For the parameters $\{M = 206/25, q = 479/100\}$: First we check that the equilibrium 69/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 69/40$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.29. *The equilibrium 69/40 for the rational difference equation*

$$x_{n+1} =, (89907/8000 + x_n)/(579/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -21025 + (1158 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 69/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-69/40$,. The new difference equation is:

$$z[n+1] = -145 \cdot z[n]/(1503 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -145 \cdot z[n]/(1503 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -69/40$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -69/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 69/40$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 69/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 69/40$$
$$N = 4$$

Proving $P > 0$ in the region:

$$69/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 69/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 69/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/69 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/69$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (89907/8000 + x_n)/(579/100 + x_n)$$

□

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For the parameters $\{M = 923/100, q = 39/5\}$: First we check that the equilibrium 143/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 143/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.30. *The equilibrium 143/200, for the rational difference equation*

$$x_{n+1} = (243529/40000 + x_n)/(44/5 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -3249 + (1760 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 143/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-143/200$,. The new difference equation is:

$$z[n+1] = 57 \cdot z[n]/(1903 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 57 \cdot z[n]/(1903 + 200 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -143/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -143/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 143/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 143/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 143/200$$
$$N = 4$$

Proving $P > 0$ in the region:
$$143/200 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 143/200$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 143/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/143 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/143$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (243529/40000 + x_n)/(44/5 + x_n)$$

□

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For the parameters $\{M = 203/25, q = 207/50\}$: First we check that the equilibrium 199/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 199/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.31. *The equilibrium 199/100, for the rational difference equation*

$$x_{n+1} = (121987/10000 + x_n)/(257/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -9801 + (514 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 199/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-199/100$,. The new difference equation is:

$$z[n+1] = -99 \cdot z[n]/(713 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -99 \cdot z[n]/(713 + 100 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -199/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -199/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 199/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 199/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 199/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$199/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 199/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 199/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/199 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/199$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (121987/10000 + x_n)/(257/50 + x_n)$$

□

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For the parameters $\{M = 887/100, q = 349/100\}$: First we check that the equilibrium 269/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 269/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.32. *The equilibrium 269/100, for the rational difference equation*

$$x_{n+1} = (83121/5000 + x_n)/(449/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -28561 + 4 \cdot (449/2 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 269/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-269/100$,. The new difference equation is:

$$z[n+1] = -169/2 \cdot z[n]/(359 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -169/2 \cdot z[n]/(359 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -269/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -269/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 269/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 269/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 269/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$269/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 269/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 269/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/269 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/269$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (83121/5000 + x_n)/(449/100 + x_n)$$

□

===

For the parameters $\{M = 757/100, q = 117/100\}$: First we check that the equilibrium 16/5 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 16/5$
For $K = 1$ we get $\{FAIL, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 16/5$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.33. *The equilibrium 16/5 for the rational difference equation*

$$x_{n+1} = (1748/125 + x_n)/(217/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -2342560000 + (186929 + 31700 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 16/5$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-16/5$,. The new difference equation is:

$$z[n + 1] = -220 \cdot z[n]/(537 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -220 \cdot z[n]/(537 + 100 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$
for all $z[1] > -16/5$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -16/5 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 16/5$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 16/5, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 16/5 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$16/5 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 16/5$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.

According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 16/5$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$5/16 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 5/16$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = (1748/125 + x_n)/(217/100 + x_n)$$

□

==

For the parameters $\{M = 29/10, q = 33/50\}$: First we check that the equilibrium 28/25 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 28/25$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.34. *The equilibrium 28/25 for the rational difference equation*

$$x_{n+1} =, (1246/625 + x_n)/(83/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -36 + (50 \cdot z[1] + 83)^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 28/25$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-28/25$,. The new difference equation is:

$$z[n + 1] = -6 \cdot z[n]/(50 \cdot z[n] + 139)$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -6 \cdot z[n]/(50 \cdot z[n] + 139)\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -28/25$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -28/25 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 28/25$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 28/25, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 28/25$$
$$N = 4$$

Proving $P > 0$ in the region:

$$28/25 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 28/25$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 28/25$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$25/28 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 25/28$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (1246/625 + x_n)/(83/50 + x_n)$$

□

==

For the parameters $\{M = 38/5, q = 39/25\}$: First we check that the equilibrium 151/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 151/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.35. *The equilibrium 151/50 for the rational difference equation*

$$x_{n+1} = \ (34579/2500 + x_n)/(64/25 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10201 + (128 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 151/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-151/50$,. The new difference equation is:

$$z[n+1] = -101 \cdot z[n]/(279 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}\ ^1$, which represents the new difference equation:
$$Q(\langle z[n]\rangle) = \langle -101 \cdot z[n]/(279 + 50 \cdot z[n])\rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -151/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -151/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 151/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 151/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 151/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$151/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 151/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 151/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/151 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/151$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (34579/2500 + x_n)/(64/25 + x_n)$$

□

==

For the parameters $\{M = 263/50, q = 117/50\}$: First we check that the equilibrium 73/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 73/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.36. *The equilibrium 73/50 for the rational difference equation*

$$x_{n+1} =, (1387/250 + x_n)/(167/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -529 + 100 \cdot (167/10 + 5 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 73/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-73/50$,. The new difference equation is:

$$z[n + 1] = -23/10 \cdot z[n]/(24 + 5 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -23/10 \cdot z[n]/(24 + 5 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -73/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -73/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-73/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 73/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 73/50 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$73/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 73/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 73/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/73 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/73$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (1387/250 + x_n)/(167/50 + x_n)$$

□

==

For the parameters $\{M = 166/25, q = 17/100\}$: First we check that the equilibrium 647/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 647/200$
For $K = 1$ we get $\{false, true,\}$ output from PolynomialPositive.
Testing $K = 2$ for the equilibrium $\bar{x} = 647/200$
For $K = 2$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.37. *The equilibrium 647/200, for the rational difference equation*

$$x_{n+1} = (440607/40000 + x_n)/(117/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 2,, and the δ value 1 we get a polynomial:
$P = -39923636481 + (495363 + 86800 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 647/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-647/200$,. The new difference equation is:

$$z[n+1] = -447 \cdot z[n]/(881 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -447 \cdot z[n]/(881 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -647/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -647/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 647/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 647/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 647/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$647/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 647/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 647/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/647 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/647$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 2, is proven to work for the rational difference equation

$$x_{n+1} = (440607/40000 + x_n)/(117/100 + x_n)$$

□

==

For the parameters $\{M = 33/4, q = 247/100\}$: First we check that the equilibrium 289/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 289/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.38. *The equilibrium 289/100, for the rational difference equation*

$$x_{n+1} = (19363/1250 + x_n)/(347/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -35721 + 16 \cdot (347/4 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 289/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-289/100$,. The new difference equation is:

$$z[n+1] = -189/4 \cdot z[n]/(159 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -189/4 \cdot z[n]/(159 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -289/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -289/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 289/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 289/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 289/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$289/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 289/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 289/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/289 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/289$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (19363/1250 + x_n)/(347/100 + x_n)$$

□

==

For the parameters $\{M = 537/100, q = 307/100\}$: First we check that the equilibrium 23/20 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 23/20$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.39. *The equilibrium 23/20 for the rational difference equation*

$$x_{n+1} =, (4853/1000 + x_n)/(407/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -225 + 4 \cdot (407/2 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 23/20$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-23/20$,. The new difference equation is:

$$z[n+1] = -15/2 \cdot z[n]/(261 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -15/2 \cdot z[n]/(261 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -23/20$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -23/20 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i]-23/20$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 23/20, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 23/20$$
$$N = 4$$

Proving $P > 0$ in the region:

$$23/20 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 23/20$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 23/20$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$20/23 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 20/23$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (4853/1000 + x_n)/(407/100 + x_n)$$

□

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For the parameters $\{M = 172/25, q = 43/10\}$: First we check that the equilibrium 129/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 129/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.40. *The equilibrium 129/100, for the rational difference equation*

$$x_{n+1} = (72111/10000 + x_n)/(53/10 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -841 + (530 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 129/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-129/100$,. The new difference equation is:

$$z[n+1] = -29 \cdot z[n]/(659 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -29 \cdot z[n]/(659 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -129/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -129/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 129/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 129/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 129/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$129/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 129/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 129/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/129 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/129$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (72111/10000 + x_n)/(53/10 + x_n)$$

□

==

For the parameters $\{M = 727/100, q = 71/25\}$: First we check that the equilibrium 443/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 443/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.41. *The equilibrium 443/200, for the rational difference equation*

$$x_{n+1} = (447873/40000 + x_n)/(96/25 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -59049 + (768 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 443/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-443/200$,. The new difference equation is:

$$z[n+1] = -243 \cdot z[n]/(1211 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -243 \cdot z[n]/(1211 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -443/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -443/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 443/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 443/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 443/200$$
$$N = 4$$

Proving $P > 0$ in the region:

$$443/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 443/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 443/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/443 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/443$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (447873/40000 + x_n)/(96/25 + x_n)$$

□

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For the parameters $\{M = 991/100, q = 177/20\}$: First we check that the equilibrium 53/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 53/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.42. *The equilibrium 53/100 for the rational difference equation*

$$x_{n+1} = (24857/5000 + x_n)/(197/20 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -2209 + 4 \cdot (985/2 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 53/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-53/100$,. The new difference equation is:

$$z[n+1] = 47/2 \cdot z[n]/(519 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 47/2 \cdot z[n]/(519 + 50 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -53/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -53/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 53/100$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 53/100, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 53/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$53/100 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 53/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 53/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/53 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/53$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (24857/5000 + x_n)/(197/20 + x_n)$$

□

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For the parameters $\{M = 102/25, q = 303/100\}$: First we check that the equilibrium 21/40 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 21/40$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.43. *The equilibrium 21/40 for the rational difference equation*

$$x_{n+1} =, (14931/8000 + x_n)/(403/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -9025 + (806 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 21/40$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-21/40$,. The new difference equation is:

$$z[n+1] = 95 \cdot z[n]/(911 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 95 \cdot z[n]/(911 + 200 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{\left|\langle z[1], \rangle\right|^2} < 1$$

for all $z[1] > -21/40$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot \left|\langle z[1] \rangle\right|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -21/40 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 21/40$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 21/40, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 21/40$$
$$N = 4$$

Proving $P > 0$ in the region:

$$21/40 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 21/40$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 21/40$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$40/21 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 40/21$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (14931/8000 + x_n)/(403/100 + x_n)$$

□

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For the parameters $\{M = 837/100, q = 287/100\}$: First we check that the equilibrium 11/4 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 11/4$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.44. *The equilibrium 11/4 for the rational difference equation*

$$x_{n+1} = (3091/200 + x_n)/(387/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -30625 + 4 \cdot (387/2 + 50 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 11/4$ in the original difference equation. This new difference equation has 0 a s its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-11/4$,. The new difference equation is:

$$z[n+1] = -175/2 \cdot z[n]/(331 + 50 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -175/2 \cdot z[n]/(331 + 50 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -11/4$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -11/4 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 11/4$ so that we have a polynomi al that we wish to prove is positive when all variables are positive (rather th an when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 11/4, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 11/4$$
$$N = 4$$

Proving $P > 0$ in the region:

$$11/4 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 11/4$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 11/4$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$4/11 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 4/11$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

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Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (3091/200 + x_n)/(387/100 + x_n)$$

□

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For the parameters $\{M = 349/50, q = 523/100\}$: First we check that the equilibrium 7/8 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 7/8$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.45. *The equilibrium 7/8 for the rational difference equation*

$$x_{n+1} = (8547/1600 + x_n)/(623/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -625 + (1246 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 7/8$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when i nitial conditions are greater than $-7/8$,. The new difference equation is:

$$z[n+1] = 25 \cdot z[n]/(1421 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle 25 \cdot z[n]/(1421 + 200 \cdot z[n]) \rangle$$

3. Notice that if
$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$
for all $z[1] > -7/8$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:
$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -7/8 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 7/8$ so that we have a polynomia l that we wish to prove is positive when all variables are positive (rather tha n when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 7/8, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 7/8 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:
$$7/8 \leq z[1] < \infty$$
Make new polynomial $g[\{\}]$ by substituting
$$z[1] = z[1] + 7/8$$
into P Now need to prove that $g[\{\}] > 0$ in the region:
$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:
$$0 < z[1] \leq 7/8$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$8/7 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 8/7$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (8547/1600 + x_n)/(623/100 + x_n)$$

□

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For the parameters $\{M = 863/100, q = 459/100\}$: First we check that the equilibrium 101/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 101/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.46. *The equilibrium 101/50 for the rational difference equation*

$$x_{n+1} = (66761/5000 + x_n)/(559/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -10404 + (559 + 100 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 101/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-101/50$,. The new difference equation is:

$$z[n+1] = -102 \cdot z[n]/(761 + 100 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\,1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle -102 \cdot z[n]/(761 + 100 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{\left|\langle z[1],\rangle\right|^2} < 1$$

for all $z[1] > -101/50$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot \left|\langle z[1]\rangle\right|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -101/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 101/50$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 101/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 101/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$101/50 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 101/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 101/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/101 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/101$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (66761/5000 + x_n)/(559/100 + x_n)$$

□

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For the parameters $\{M = 751/100, q = 69/10\}$: First we check that the equilibrium 61/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 61/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.47. *The equilibrium 61/200 for the rational difference equation*

$$x_{n+1} = \ (87901/40000 + x_n)/(79/10 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -19321 + (1580 + 200 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 61/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS whe n initial conditions are greater than $-61/200$,. The new difference equation is:

$$z[n+1] = 139 \cdot z[n]/(1641 + 200 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^{\ 1}$, which represents the new difference equation:

$$Q(\langle z[n]\rangle) = \langle 139 \cdot z[n]/(1641 + 200 \cdot z[n])\rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -61/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethin g > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -61/200 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 61/200$ so that we have a polyno mial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 61/200, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$\begin{aligned} P &= \text{ from above} \\ Z &= [z[1]] \\ \bar{x} &= 61/200 \\ N &= 4 \end{aligned}$$

Proving $P > 0$ in the region:

$$61/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 61/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 61/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/61 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/61$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (87901/40000 + x_n)/(79/10 + x_n)$$

□

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For the parameters $\{M = 779/100, q = 281/100\}$: First we check that the equilibrium 249/100 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 249/100$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.48. *The equilibrium 249/100, for the rational difference equation*

$$x_{n+1} = (13197/1000 + x_n)/(381/100 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -22201 + 100 \cdot (381/10 + 10 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 249/100$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-249/100$,. The new difference equation is:

$$z[n+1] = -149/10 \cdot z[n]/(63 + 10 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -149/10 \cdot z[n]/(63 + 10 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1]\rangle)\right|^2}{|\langle z[1],\rangle|^2} < 1$$

for all $z[1] > -249/100$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:

$$0 < 1 \cdot |\langle z[1]\rangle|^2 - \left|Q^K(\langle z[1],\rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -249/100 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 249/100$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 249/100 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 249/100$$
$$N = 4$$

Proving $P > 0$ in the region:

$$249/100 \le z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 249/100$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \le z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \le 249/100$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$100/249 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 100/249$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (13197/1000 + x_n)/(381/100 + x_n)$$

□

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For the parameters $\{M = 773/100, q = 111/50\}$: First we check that the equilibrium 551/200 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 551/200$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.49. *The equilibrium 551/200, for the rational difference equation*

$$x_{n+1} = (109649/8000 + x_n)/(161/50 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -123201 + 25 \cdot (644/5 + 40 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 551/200$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS wh en initial conditions are greater than $-551/200$,. The new difference equation is:
$$z[n+1] = -351/5 \cdot z[n]/(239 + 40 \cdot z[n])$$
2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:
$$Q(\langle z[n] \rangle) = \langle -351/5 \cdot z[n]/(239 + 40 \cdot z[n]) \rangle$$
3. Notice that if
$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$
for all $z[1] > -551/200$ then we are done (see applicable Theorem in Emilie Hogan's PhD thesis). Simplify this so that it is of the form somethi ng > 0:
$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$
and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -551/200 then the whole expression is positive.
4. Finally, replace $z[i]$ by $z[i] - 551/200$ so that we have a polyn omial that we wish to prove is positive when all variables are positive (rather than when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 551/200 is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 551/200$$
$$N = 4$$

Proving $P > 0$ in the region:
$$551/200 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 551/200$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 551/200$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$200/551 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 200/551$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (109649/8000 + x_n)/(161/50 + x_n)$$

□

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For the parameters $\{M = 423/50, q = 49/10\}$: First we check that the equilibrium 89/50 is LAS.
It is LAS, so we continue to test K values.
Testing $K = 1$ for the equilibrium $\bar{x} = 89/50$
For $K = 1$ we get $\{true\}$ output from PolynomialPositive.

Theorem 7.50. *The equilibrium 89/50 for the rational difference equation*

$$x_{n+1} =, (14863/1250 + x_n)/(59/10 + x_n)$$

is GAS.

Proof. From the rational difference equation, the K value 1,, and the δ value 1 we get a polynomial:
$P = -1521 + 4 \cdot (295/2 + 25 \cdot z[1])^2$
The goal is to prove that this polynomial is positive when all variables are p ositive. We create this polynomial by:

1. Create a new rational difference equation, by setting $z[n] = x_n - 89/50$ in the original difference equation. This new difference equation has 0 as its equilibrium, and we now wish to prove that the equilibrium 0 is GAS when initial conditions are greater than $-89/50$,. The new difference equation is:

$$z[n+1] = -39/2 \cdot z[n]/(192 + 25 \cdot z[n])$$

2. Consider the vector valued mapping, $Q : \mathbb{R}^1 \to \mathbb{R}^1$, which represents the new difference equation:

$$Q(\langle z[n] \rangle) = \langle -39/2 \cdot z[n]/(192 + 25 \cdot z[n]) \rangle$$

3. Notice that if

$$\frac{\left|Q^K(\langle z[1] \rangle)\right|^2}{|\langle z[1], \rangle|^2} < 1$$

for all $z[1] > -89/50$ then we are done (see applicable Theorem i n Emilie Hogan's PhD thesis). Simplify this so that it is of the form something > 0:

$$0 < 1 \cdot |\langle z[1] \rangle|^2 - \left|Q^K(\langle z[1], \rangle)\right|^2$$

and then take the numerator. The denominator will always be a product of squar es and sums of squares, so if the numerator is positive when all variables are greater than -89/50 then the whole expression is positive.

4. Finally, replace $z[i]$ by $z[i] - 89/50$ so that we have a polynom ial that we wish to prove is positive when all variables are positive (rather t han when all variables are greater than some negative number).

Now we run the algorithm PolynomialPositive on this polynomial. If the polynom ial is positive when all variables are positive then the equilibrium 89/50, is GAS for the original difference equation. Starting the procedure PolynomialPositive with inputs:

$$P = \text{ from above}$$
$$Z = [z[1]]$$
$$\bar{x} = 89/50$$
$$N = 4$$

Proving $P > 0$ in the region:

$$89/50 \leq z[1] < \infty$$

Make new polynomial $g[\{\}]$ by substituting

$$z[1] = z[1] + 89/50$$

into P Now need to prove that $g[\{\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{\}] > 0$ in the positive orthant!

Proving $P > 0$ in the region:

$$0 < z[1] \leq 89/50$$

Make new polynomial $f[\{1\}]$ by substituting

$$z[1] = 1/z[1]$$

and multiplying by

$$z[1]^{degree(P,z[1])} = z[1]^2$$

Now need to prove that $f[\{1\}] > 0$ in the region:

$$50/89 \leq z[1] < \infty$$

Make new polynomial $g[\{1\}]$ by substituting

$$z[1] = z[1] + 50/89$$

into $f[\{1\}]$ Now need to prove that $g[\{1\}] > 0$ in the region:

$$0 \leq z[1] < \infty$$

except when all variables are simultaneously zero.
Entering IsPos with $g[\{1\}]$
All coefficients are positive, and the constant term is positive so the polynomial is positive.
According to IsPos, $g[\{1\}] > 0$ in the positive orthant!
————————————————————————-

Since $P > 0$ when all variables are positive, the K value 1, is proven to work for the rational difference equation

$$x_{n+1} = (14863/1250 + x_n)/(59/10 + x_n)$$

□

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The parameter values for which the $K = 1$ are:

$$[\{M = 29/10, q = 33/50\}, 28/25], [\{M = 33/4, q = 247/100\}, 289/100], [\{M = 35/4, q = 253/100\}, 311/100], [\{M = 38/5, q = 39/25\}, 151/50], [\{M = 61/50, q = 71/100\}, 51/200], [\{M = 67/20, q = 52/25\}, 127/200], [\{M = 98/25, q = 251/100\}, 141/200], [\{M = 102/25, q = 303/100\}, 21/40], [\{M = 164/25, q = 1\}, 139/50], [\{M = 172/25, q = 43/10\}, 129/100], [\{M = 179/20, q = 24/5\}, 83/40], [\{M = 181/25, q = 103/20\}, 209/200], [\{M = 193/25, q = 37/25\}, 78/25], [\{M = 203/25, q = 207/50\}, 199/100], [\{M = 206/25, q = 479/100\}, 69/40], [\{M = 227/25, q = 142/25\}, 17/10], [\{M = 234/25, q = 121/20\}, 331/200], [\{M = 261/50, q = 337/100\}, 37/40], [\{M = 263/50, q = 117/50\}, 73/50], [\{M = 299/50, q = 137/100\}, 461/200], [\{M = 309/50, q = 101/20\}, 113/200], [\{M = 349/50, q = 523/100\}, 7/8], [\{M = 393/100, q = 109/50\}, 7/8], [\{M = 423/50, q = 49/10\}, 89/50], [\{M = 437/100, q = 171/100\}, 133/100], [\{M = 491/100, q = 79/50\}, 333/200], [\{M = 501/100, q = 177/100\}, 81/50], [\{M = 517/100, q = 247/50\}, 23/200], [\{M = 537/100, q = 307/100\}, 23/20], [\{M = 577/100, q = 4/5\}, 497/200], [\{M = 631/100, q = 117/25\}, 163/200], [\{M = 633/100, q = 53/50\}, 527/200], [\{M = 727/100, q = 71/25\}, 443/200], [\{M = 727/100, q = 79/20\}, 83/50], [\{M = 751/100, q = 69/10\}, 61/200], [\{M = 773/100, q = 111/50\}, 551/200], [\{M = 779/100, q = 281/100\}, 249/100], [\{M = 789/100, q = 19/10\}, 599/200], [\{M = 837/100, q = 287/100\}, 11/4], [\{M = 849/100, q = 743/100\}, 53/100], [\{M = 863/100, q = 459/100\}, 101/50], [\{M = 867/100, q = 307/100\}, 14/5], [\{M = 887/100, q = 349/100\}, 269/100], [\{M = 887/100, q = 737/100\}, 3/4], [\{M = 901/100, q = 201/100\}, 7/2], [\{M = 923/100, q = 39/5\}, 143/200], [\{M = 991/100, q = 177/20\}, 53/100],$$

The parameter values for which the $K = 2$ are:

$$[\{M = 166/25, q = 17/100\}, 647/200], [\{M = 757/100, q = 117/100\}, 16/5], [\{M = 791/100, q = 67/100\}, 181/50],$$

Finished investigating difference equation 7 out of 7
