Order 1 Rational Difference Equations Web Book

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Contents

0 Description of Web Book

Each section corresponds to a specific first order rational difference equation (with linear numerator and denominator). For each difference equation we either state that the equilibrium is not LAS, or we prove that it is GAS using the technique in Emilie Hogan's PhD thesis. This Web Book is very informal. It will be made more formal in the near future.

1 $\frac{M^2}{x_n}$

The rational difference equation

$$x_{n+1} = \frac{M^2}{x_n}$$

is not LAS for any parameter value M.

2 βx_n

For the rational difference equation

$$x_{n+1} = \beta x_n$$

the equilibrium is $\bar{x} = 0$. If $\beta \geq 1$ then the equilibrium is not LAS. If $\beta < 1$ then the equilibrium is GAS and $K = 1$ works to prove it. The polynomial for $K = 1$ is

$$1 - \beta^2$$

which is clearly only positive when $\beta < 1$.

3 $\frac{1}{4}\frac{M^2-1}{1+x_n}$

For the rational difference equation

$$x_{n+1} = \frac{1}{4}\frac{M^2-1}{1+x_n}$$

there are two possible equilibria, $\frac{1}{2}(M-1)$ and $-\frac{1}{2}(M+1)$. We require that all coefficients, and the equilibrium point be positive. There are two cases for which the coefficients are positive, each corresponds to one of the two equilibrium points:

$$\begin{aligned} M-1 > 0 \text{ and } M+1 > 0, \bar{x} = \frac{1}{2}(M-1) \\ M-1 < 0 \text{ and } M+1 < 0, \bar{x} = -\frac{1}{2}(M+1) \end{aligned}$$

In the first case, the polynomial for $K = 2$ is

$$P_1 := 16z_n^2 + (8M^2 + 24)z_n + 4(M+1)(M^2 - M + 2)$$

and we require that $M > 1$. Substitute $M = M_1 + 1$ to get the following polynomial

$$16z_n^2 + (8M_1^2 + 16M_1 + 32)z_n + 4(M_1 + 2)(M_1^2 + M_1 + 2).$$

The region that we care about is $\{M_1 > 0, z_n > 0\}$, since all coefficients are positive, and all variables are assumed to be positive, the polynomial is positive.

In the second case, the polynomial for $K = 2$ is

$$P_2 := 16z_n^2 + (8M^2 + 24)z_n - 4(M-1)(M^2 + M + 2)$$

and we require that $M < -1$. Substitute $M = M_1 - 1$, and then $M_1 = -M_2$ to get the polynomial

$$16z_n^2 + (8M_2^2 + 16M_2 + 32)z_n + 4(M_2 + 2)(M_2^2 + M_2 + 2).$$

Again, the region that we care about is $\{M_2 > 0, z_n > 0\}$, and since all coefficients are positive and all variables are assumed to be positive, the polynomial is positive.

4 $\frac{\beta x_n}{1+x_n}$

For the rational difference equation

$$x_{n+1} = \frac{\beta x_n}{1 + x_n}$$

there are two equilibria, 0 and $\beta - 1$. The equilibrium $\bar{x} = 0$ is LAS when $0 \leq \beta < 1$, and the equilibrium $\bar{x} = \beta - 1$ is LAS when $\beta > 1$.

So assume $0 \leq \beta < 1$ with the equilibrium $\bar{x} = 0$. If $K = 1$ the polynomial that must be positive is

$$z_n^2 + 2z_n + 1 - \beta^2.$$

The coefficients on z_n^2 and z_n are positive, and the constant term is positive because $0 \leq \beta < 1$, so $K = 1$ works to prove GAS.

Now assume $1 \leq \beta$ with the equilibrium $\beta - 1$. If $K = 1$ the polynomial that must be positive is

$$z_n^2 + 2z_n.$$

The coefficients are all positive, so the polynomial is positive.

5 $\alpha + \beta x_n$

For the rational difference equation

$$x_{n+1} = \alpha + \beta x_n$$

the equilibrium is $\frac{\alpha}{1-\beta}$, so the equilibrium will be positive when $\beta < 1$. For $K = 1$ the polynomial that must be positive is

$$1 - \beta^2$$

which is clearly positive when $\beta < 1$.

6 $q + \frac{1}{4}\frac{M^2-q^2}{x_n}$

For the rational difference equation

$$x_{n+1} = q + \frac{1}{4}\frac{M^2 - q^2}{x_n}$$

there are two possible equilibria $\frac{1}{2}(M+q)$ and $\frac{1}{2}(q-M)$. We require that all coefficients, and the equilibrium point be positive. There are two cases for which the coefficients are positive, each corresponds to one of the two equilibrium points:

$$M - q < 0, M + q < 0, \text{ and } q > 0, \text{ with } \bar{x} = -\frac{1}{2}(M - q)$$
$$M - q > 0, M + q > 0, \text{ and } q > 0, \text{ with } \bar{x} = \frac{1}{2}(M + q)$$

In the first case, another way to describe the region is

$$\{0 < q, M < -q\}.$$

The polynomial for $K = 2$ is

$$P_1 := 16q^2 z_n^2 + 8q(M - q)(M + q)z_n - 4Mq(M + q)^2.$$

Substitute $M = M_1 - q$ and then $M_1 = -M_2$ to get the following polynomial

$$16q^2 z_n^2 + 8M_2 q(M2 + 2q)z_n + 4M_2^2 q(M_2 + q).$$

The region that we care about is $\{M_2 > 0, q > 0, z_n > 0\}$, since all coefficients are positive, and all variables are assumed to be positive, the polynomial is positive.

In the second case, another way to describe the region is

$$\{0 < q, q < M\}.$$

The polynomial for $K = 2$ is

$$P_2 := 16q^2 z_n^2 + 8q(M - q)(M + q)z_n + 4Mq(M - q)^2.$$

Substitute $M = M_1 + q$ to get the polynomial

$$16q^2 z_n^2 + 8M_1 q(M_1 + 2q)z_n + 4M_1^2 q(M_1 + q).$$

Again, the region that we care about is $\{M_1 > 0, q > 0, z_n > 0\}$, and since all coefficients are positive and all variables are assumed to be positive, the polynomial is positive.

7 $\frac{1}{4}\frac{M^2-q^2+4x_n}{1+q+x_n}$

For the rational difference equation

$$x_{n+1} = \frac{1}{4}\frac{M^2 - q^2 + 4x_n}{1 + q + x_n}$$

there are two equilibria,

$$\frac{1}{2}(M - q) \text{ and } -\frac{1}{2}(M + q).$$

This is the most complicated of all the order 1 (linear) rational difference equations because of the possibilities for the parameters. We require that all coefficients and the equilibrium are positive. In order for the coefficients to be positive we must have

$$M^2 - q^2 > 0 \text{ and } 1 + q > 0$$
$$(M - q)(M + q) > 0 \text{ and } q > -1$$

Therefore,

$$\{M - q > 0 \text{ and } M + q > 0\} \text{ and } q > -1$$
$$\text{or}$$
$$\{M - q < 0 \text{ and } M + q < 0\} \text{ and } q > -1$$

Now, if we are in the first case then the positive equilibrium is $\frac{1}{2}(M - q)$ since $M - q > 0$. If we are in the second case then the positive equilibrium is $-\frac{1}{2}(M + q)$ since $M + q < 0$.

Let's look at the first case. Another way to state it is

$$\{1 < M \text{ and } -1 < q < M\} \text{ or } \{0 < M \leq 1 \text{ and } -M < q < M\}$$

with equilibrium $\bar{x} = \frac{1}{2}(M - q)$. The polynomial for $K = 2$ is

$$\begin{aligned} P_1 := {} & 16(q+2)^2 z_n^2 + 8(q+2)(M^2 + 3q^2 + 8q + 4)z_n + \\ & + 4(q+2)(M+q)(M^2 - Mq + 2q^2 - 2M + 6q + 4) \end{aligned}$$

Let $\{1 < M \text{ and } -1 < q < M\}$, then perform the substitutions $M = M_1 + 1$ and $q = q_1 - 1$ into P to yield the following polynomial for which all variables (M_1, q_1, and z_n) are positive in the region that we are interested in.

$$\begin{aligned} & 16(q_1+1)^2 z_n^2 + 8(q_1+1)(M_1^2 + 3q_1^2 + 2M_1 + 2q_1)z_n + \\ & \quad + 4(q_1+1)(M_1+q_1)(M_1^2 + 2q_1^2 - q_1M_1 + M_1 + q_1) \end{aligned}$$

Since all coefficients are positive, the polynomial is positive.

Now let $\{0 < M \leq 1 \text{ and } -M < q < M\}$ and perform the substitutions

$$\begin{aligned} M &= \frac{1}{M_1}, \text{ and then multiply by } M_1^3 \\ M_1 &= M_2 + 1 \\ q &= q_1 - \frac{1}{M_2+1} \end{aligned}$$

The region is then $\{0 \leq M_2, 0 < q_1 < \frac{2}{M_2+1}\}$, and the polynomial is

$$\begin{aligned} & 16(M_2+1)(M_2q_1 + 2M_2 + q_1 + 1)^2 z_n^2 + \\ & \quad + 8(M_2q_1 + q_1 + 2M_2 + 1)(3M_2^2q_1^2 + 8M_2^2q_1 + 4M_2^2 + 6M_2q_1^2 + 10M_2q_1 + 3q_1^2 + 2q_1)z_n + \\ & \quad + 4q_1(M_2q_1 + q_1 + 2M_2 + 1)(2M_2^2q_1^2 + 6M_2^2q_1 + 4M_2^2 + 4M_2q_1^2 + 7M_2q_1 + 2q_1^2 + q_1) \end{aligned}$$

In the region all variables are positive, and in the polynomial all coefficients are positive. Therefore, the polynomial is positive as needed.

Then we must do the second case, where the equilibrium is $-\frac{1}{2}(M+q)$, and the parameters satisfy $\{M - q < 0 \text{ and } M + q < 0\}$ and $q > -1$. Again, we can restate this as

$$\{M < -1 \text{ and } -1 < q < -M\} \text{ or } \{-1 \leq M < 0 \text{ and } M < q < -M\}.$$

For $K = 2$ and $\bar{x} = -\frac{1}{2}(M + q)$ the polynomial is

$$\begin{aligned} P_2 := 16(q+2)^2 z_n^2 + 8(q+2)(M^2 + 3q^2 + 8q + 4) z_n - \\ - 4(q+2)(M-q)(M^2 + Mq + 2q^2 + 2M + 6q + 4) \end{aligned}$$

In the first sub-case we assume that $\{M < -1 \text{ and } -1 < q < -M\}$. In order to transform the region into one which all variables are positive we must do the substitutions

$$\begin{aligned} M &= M_1 - 1 \\ q &= q_1 - 1 \\ M_1 &= -M_2 \end{aligned}$$

Then the region is $\{0 < M_2 \text{ and } 0 < q_1 < M_2 + 2\}$, and the polynomial is

$$\begin{aligned} 16(q_1+1)^2 z_n^2 + 8(q_1+1)(M_2^2 + 3q_1^2 + 2M_2 + 2q_1) z_n + \\ + 4(q_1+1)(M_2+q_1)(M_2^2 - M_2 q_1 + 2q_1^2 + M_2 + q_1) \end{aligned}$$

All coefficients are positive, and so the polynomial is positive.

For the second sub-case we assume that $\{-1 \le M < 0 \text{ and } M < q < -M\}$. Again we transform the region into one where all variables are positive using the following substitutions

$$\begin{aligned} M &= -M_1 \\ q &= q_1 - M_1 \\ M_1 &= \frac{1}{M_2}, \text{ and then multiply by } M_2^3 \\ M_2 &= M_3 + 1 \end{aligned}$$

Then the region is $\{0 \le M_3, 0 < q_1 < \frac{2}{M_3+1}\}$, and the polynomial is

$$\begin{aligned} &16(M_3+1)(M_3q_1 + 2M_3 + q_1 + 1)^2 z_n^2 + \\ &\quad + 8(M_3q_1 + 2M_3 + q_1 + 1)(3M_3^2q_1^2 + 8M_3^2q_1 + 6M_3q_1^2 + 4M_3^2 + 10M_3q_1 + 3q_1^2 + 2q_1) z_n + \\ &\quad + 4q_1(M_3q_1 + 2M_3 + q_1 + 1)(2M_3^2q_1^2 + 6M_3^2q_1 + 4M_3q_1^2 + 4M_3^2 + 7M_3q_1 + 2q_1^2 + q_1) \end{aligned}$$

Again, coefficients are positive, and so the polynomial is positive. That is the final case for equation 7, and so for all M and q so that the coefficients are positive, the equilibrium is GAS.