Experimental Mathematics Applied to the Study of Non-linear Recurrences

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Recurrence Definition

Definition

Given $F : \mathbb{R}^k \to \mathbb{R}$, *and* $\{x_1, \ldots, x_k\} \subset \mathbb{R}$, *a* **recurrence** *is defined as follows:*

$$r_n = F(r_{n-1}, \ldots, r_{n-k}), \quad r_1 = x_1, \ldots, r_k = x_k.$$

The **order** *of the recurrence is* k. *Recurrence produces sequence* $\{r_n\}_{n=1}^{\infty}$.

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Example:

- The Fibonacci recurrence, $f_n = f_{n-1} + f_{n-2}$, of order 2 is defined by $F(x, y) = x + y$, and $x_1 = x_2 = 1$.

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Why Focus on Non-Linear?

Definition

If F is a linear function (addition and scalar multiplication) then we say the recurrence is **linear***, otherwise the recurrence is* **non-linear***.*

- Linear recurrences are very well behaved
- Given a linear recurrence we have a closed form formula for the n^{th} term in the sequence
- No such general understanding for non-linear recurrences (not even for quadratic)

Three Interesting Phenomena

Global Asymptotic Stability:

- Sequence produced by recurrence defined from function F, converges for any set $\{x_1, \ldots, x_k\}$ of initial conditions.

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Surprising Rational Sequences:

- $\{r_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ when expected to be complex

Rational Difference Equation

Definition

A **rational difference equation** *is given by the equation*

$$x_{n+1} = R(x_n, x_{n-1}, \ldots, x_{n-k}),$$

where $R : \mathbb{R}^{k+1} \to \mathbb{R}$ *is a rational function (ratio of polynomials).*

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Example:

$$x_{n+1} = \frac{4 + x_n}{1 + x_{n-1}}$$

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If $x_n = \bar{x}$ *for all* $n \geq -k$ *then* $\bar{x}$ *is called an* **equilibrium**.

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For example

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Definition

If $x_n \to \bar{x}$ for all positive initial conditions, then $\bar{x}$ is **globally asymptotically stable (GAS)**.

Approaches

- Current method to prove GAS given an $R : \mathbb{R}^{k+1} \to \mathbb{R}$ is to verify that R satisfies one of many known sufficient conditions
- Given R_1 and R_2, two rational difference equations, their proofs of GAS may be very different

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Goal

Create an algorithm that takes as input a rational difference equation, R, and equilibrium, $\bar{x}$, conjectured to be GAS, and outputs a rigorous proof of its stability.

Notation

Instead of rational difference equation as a map, $R : \mathbb{R}^{k+1} \to \mathbb{R}$, consider vector valued map, $Q : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$, defined from R as follows:

$$Q\left(\begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k} \end{bmatrix}\right) = \begin{bmatrix} R(x_n, \ldots, x_{n-k}) \\ x_n \\ \vdots \\ x_{n-k+1} \end{bmatrix}$$

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New Goal

Create an algorithm that takes as input a vectorized rational difference equation, Q, and equilibrium, $\bar{\mathcal{X}} := \langle \bar{x}, \ldots, \bar{x} \rangle$, conjectured to be GAS, and outputs a rigorous proof of its stability.

A Useful Theorem

Theorem (Kruse, Nesemann 1999)

Suppose for the difference equation

$$\mathcal{X}_{n+1} = Q(\mathcal{X}_n), \quad n = 0, 1, 2, \ldots$$

with unique positive equilibrium $\bar{\mathcal{X}}$, there exists an integer $K \geq 1$ such that the K^{th} iterate of Q satisfies

$$\frac{\|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|}{\|\mathcal{X} - \bar{\mathcal{X}}\|} < 1 \quad \text{for all } \mathcal{X} \neq \bar{\mathcal{X}} \text{ with all coordinates positive.}$$

Then $\bar{\mathcal{X}}$ is GAS.

$\|\cdot\|$ is the Euclidean norm, i.e., $\|\langle x_0, \ldots, x_k \rangle\| = \sqrt{{x_0}^2 + \cdots + {x_k}^2}$

Algorithm Ingredients

- $R : \mathbb{R}^{k+1} \to \mathbb{R}$ - rational difference equation
- $\bar{x}$ - equilibrium, solution to $\bar{x} = R(\bar{x}, \ldots, \bar{x})$

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Find positive integer K so that

$$\frac{\left\| Q^K(\mathcal{X}) - \bar{\mathcal{X}} \right\|}{\left\| \mathcal{X} - \bar{\mathcal{X}} \right\|} < 1 \quad \text{for all } \mathcal{X} \neq \bar{\mathcal{X}} \text{ with all coordinates positive}$$

Algorithm Idea

Given a positive integer K we create a polynomial:

$$P_{Q,\bar{\mathcal{X}},K}(\mathcal{X}) = \text{numerator}\left(\left\|\mathcal{X} - \bar{\mathcal{X}}\right\|^2 - \left\|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\right\|^2\right)$$

If $P_{Q,\bar{\mathcal{X}},K}(\mathcal{X}) > 0$ for all $\mathcal{X} \neq \bar{\mathcal{X}}$ with all coordinates positive (i.e., in the positive *orthant*, $\mathbb{R}_+^{k+1}$) then:

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Polynomial Positivity

New Goal

Create an algorithm that takes as input a polynomial $P(\mathcal{X})$ in m variables ($\mathcal{X} = \langle x_1, \ldots, x_m \rangle$), and outputs a proof that $P(\mathcal{X}) \geq 0$ for $\mathcal{X} \in \mathbb{R}_+^m$.

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Trivial algorithm ("PosCoeffs"): If all coefficients in $P(\mathcal{X})$ are positive then $P(\mathcal{X}) \geq 0$ for $\mathcal{X}$ in the positive orthant.

Polynomial Positivity (cont.)

Less trivial algorithm ("SubP"): If the only negative coefficients are on terms of the form $x_i \cdot x_j$ then:

Example:

$$P := x^2 - xy + y^2 + x + y + 1$$

Polynomial Positivity (cont.)

Less trivial algorithm ("SubP"): If the only negative coefficients are on terms of the form $x_i \cdot x_j$ then:

- Consider the "sub-polynomial", $\overline{P}$, consisting of the terms x_h^2 and $x_i \cdot x_j$ for all $1 \leq h, i, j \leq m$, and their coefficients in P

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- Consider $\overline{P}$ as quadratic form and show it is positive definite (in 2 variables use discriminant)
- If $\overline{P}$ is positive, then $P(\mathcal{X}) \geq 0$ for $\mathcal{X}$ in the positive orthant (since all other coefficients are positive)

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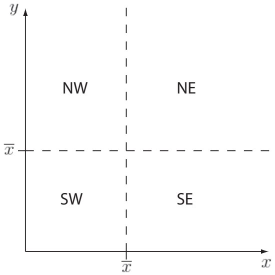
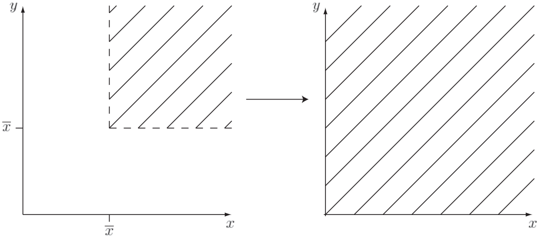
My Algorithm in Two Dimensions

Cut the positive quadrant into 4 regions using $\bar{x}$ as the cut point:



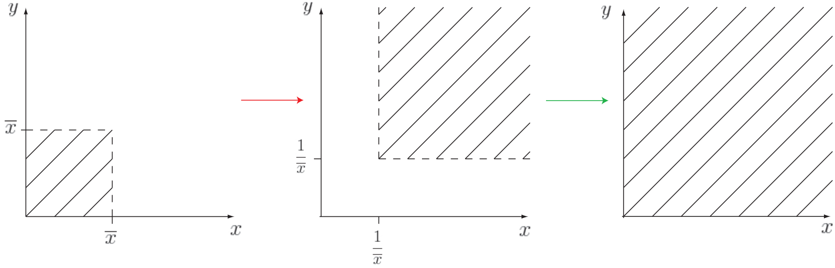
Create 4 new polynomials, from P, by transforming each of the 4 regions into the positive quadrant. The new polynomials will each be defined on the entire positive quadrant.

$P_{NE}(x, y)$



$$P_{NE}(x, y) = P(x + \bar{x}, y + \bar{x})$$

$P_{SW}(x, y)$



$$P_{SW}(x, y) = P\left(\quad , \quad \right)$$

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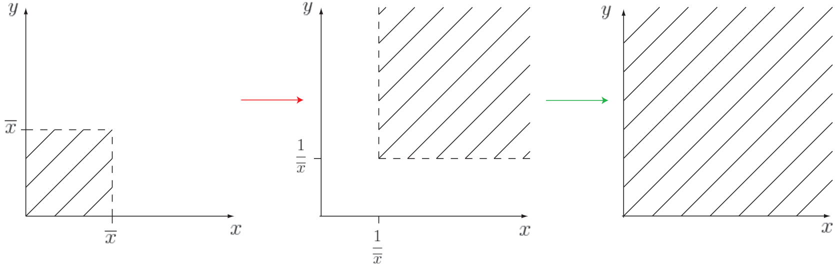
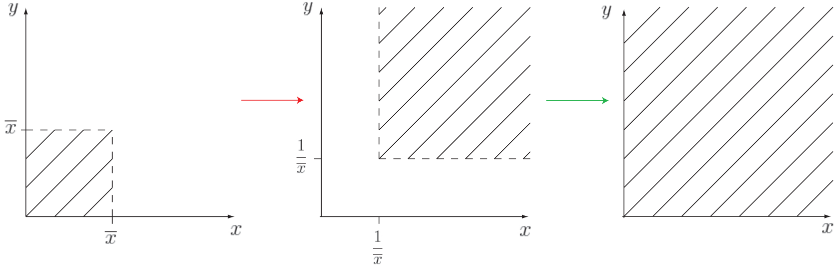


$$P_{SW}(x, y) = P\left(\frac{1}{x}, \frac{1}{y}\right)$$

$P_{SW}(x, y)$



$$P_{SW}(x, y) = P\left(\frac{1}{x + \frac{1}{\overline{x}}}, \frac{1}{y + \frac{1}{\overline{x}}}\right)$$

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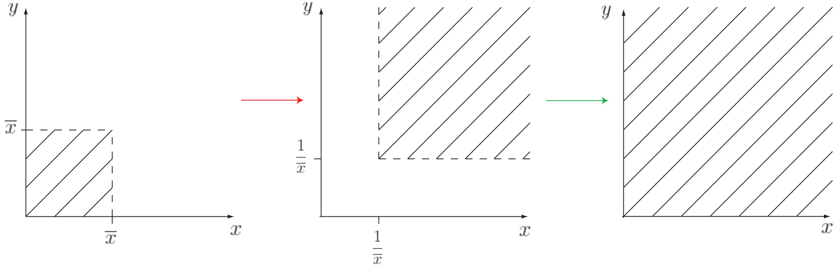
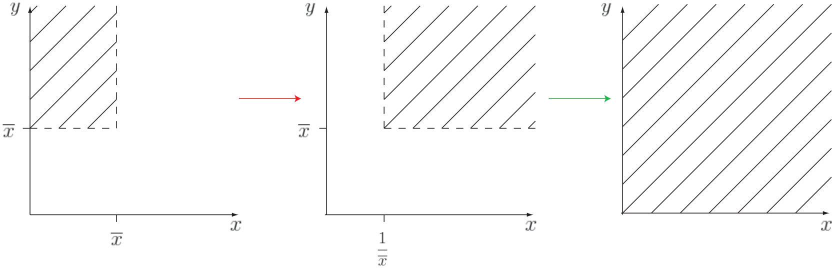


$$P_{SW}(x, y) = P\left(\frac{1}{x + \frac{1}{\overline{x}}}, \frac{1}{y + \frac{1}{\overline{x}}}\right)\left(x + \frac{1}{\overline{x}}\right)^{d_x}\left(y + \frac{1}{\overline{x}}\right)^{d_y}$$

$P_{NW}(x, y)$ and $P_{SE}(x, y)$



$$P_{NW}(x, y) = P\left(\frac{1}{x + \frac{1}{\bar{x}}}, y + \bar{x}\right)\left(x + \frac{1}{\bar{x}}\right)^{d_x}$$

$$P_{SE}(x, y) = P\left(x + \bar{x}, \frac{1}{y + \frac{1}{\bar{x}}}\right)\left(y + \frac{1}{\bar{x}}\right)^{d_y}$$

My Algorithm in Two Dimensions (cont.)

- If all polynomials are ≥ 0 on the positive quadrant then $P(\mathcal{X}) \geq 0$ on the positive quadrant

My Algorithm in Two Dimensions (cont.)

- If all polynomials are ≥ 0 on the positive quadrant then $P(\mathcal{X}) \geq 0$ on the positive quadrant
- Use PosCoeffs and SubP on each of the 4 polynomials to show that they are positive on the positive quadrant
- If PosCoeffs and SubP fail for one of the polynomials then we have to subdivide the associated region and try again

GAS Algorithm Summary

Given a rational difference equation $x_{n+1} = R(x_n, \ldots, x_{n-k})$ and a unique equilibrium $\bar{x}$:

Step 0: Create the function $Q : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$.

Step 1: Conjecture a K value that satisfies

$$\frac{\|Q^K(\mathcal{X}) - \bar{\mathcal{X}}\|}{\|\mathcal{X} - \bar{\mathcal{X}}\|} < 1.$$

Step 2: Create the polynomial $P_{Q,\bar{\mathcal{X}},K}(\mathcal{X})$.

Step 3: Prove $P_{Q,\bar{\mathcal{X}},K}(\mathcal{X}) > 0$ for all $\mathcal{X} \neq \bar{\mathcal{X}}$ in the positive orthant: find subdivision of positive orthant in which all associated polynomials (one for each sub-region) are positive in the positive orthant.

Demo

Use my maple code to prove GAS of the equilibrium $\bar{x} = 2$ in the running example

$$x_{n+1} = \frac{4 + x_n}{1 + x_{n-1}}$$

Code is available on my website as a link off the page for this project:
`http://math.rutgers.edu/∼eahogan/GAS.html`

Coefficients as Variables

- So far assumed coefficients are numerical, variables are x_n's
- Can prove some cases in which coefficients are parameters as long as equilibrium is rational function of parameters
- Run positivity algorithm with x_n's and parameters as variables

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Example:

If $x_{n+1} = \frac{x_{n-1}}{A+Bx_n+x_{n-1}}$ for $A, B > 0$ then the equilibrium is

$$\bar{x} = \frac{1-A}{1+B}.$$

When we create $P_{Q,K,\bar{\mathcal{X}}}$ it will be a polynomial in x_n, x_{n-1}, A, and B.

Select Results

$x_{n+1} =$	Parameter Values	Findings
$\frac{1}{4}\frac{M^2-1}{1+x_n}$	$M-1>0, M+1>0$ $M-1<0, M+1<0$	$\bar{x}=\frac{1}{2}(M-1)$ is GAS $\bar{x}=-\frac{1}{2}(M+1)$ is GAS
$\frac{\beta x_n}{1+x_n}$	$0<\beta\leq 1$ $1<\beta$	$\bar{x}=0$ is GAS $\bar{x}=\beta-1$ is GAS
$\frac{x_{n-1}}{A+x_n}$	$1<A$	$\bar{x}=0$ is GAS
$\frac{x_{n-1}}{A+x_{n-1}}$	$0<A<1$ $1<A$	$\bar{x}=1-A$ is GAS $\bar{x}=0$ is GAS
$q+\frac{1}{4}\frac{M^2-q^2}{x_n}$	$M-q<0, M+q<0, q>0$ $M-q>0, M+q>0, q>0$	$\bar{x}=-\frac{1}{2}(M-q)$ is GAS $\bar{x}=\frac{1}{2}(M+q)$ is GAS
$\frac{1}{4}\frac{M^2-q^2+4x_n}{1+q+x_n}$	$M-q>0, M+q>0, q>-1$ $M-q<0, M+q<0, q>-1$	$\bar{x}=\frac{1}{2}(M-q)$ is GAS $\bar{x}=-\frac{1}{2}(M+q)$ is GAS
$\frac{x_{n-1}}{A+Bx_n+x_{n-1}}$	$1<A$	$\bar{x}=0$ is GAS

Somos Sequences

Michael Somos, in 1989, conjectured that sequence produced by

$$s_n s_{n-6} = s_{n-1} s_{n-5} + s_{n-2} s_{n-4} + s_{n-3}^2$$

with initial conditions $s_i = 1$ for $1 \leq i \leq 6$, consisted only of integers (A006722).

$$1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527, \ldots$$

Somos' recurrence inspired many similar recurrences that possess the integrality property.

Family of Recurrences Inspired by Somos

Conjecture (Heideman, H. 2008)

Consider the quadratic recurrence

$$x_n x_{n-k} = x_{n-i} x_{n-k+i} + x_{n-j} + x_{n-k+j}$$

with initial conditions $x_m = 1$ *for* $1 \leq m \leq k$. *This recurrence produces a sequence of integers iff one of the following holds:*

1. k *is even,* i *is odd, and* $j = \frac{k}{2}$,
2. k *is even,* i *is even, and* $j = \frac{i}{2}$, $j = \frac{k}{2}$, *or* $j = \frac{k-i}{2}$,
3. k *is odd,* i *is odd, and* $j = \frac{k-i}{2}$,
4. k *is odd,* i *is even, and* $j = \frac{i}{2}$.

Progress on Conjecture

$$x_n x_{n-k} = x_{n-i} x_{n-k+i} + x_{n-j} + x_{n-k+j}$$

In 2008 paper, Heideman and I proved:

- $k = 2K + 1$, $i = 1$, and $j = \frac{k-i}{2} = K$ by showing sequence also satisfies linear recurrence with integer coefficients and initial conditions.

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- Entire conjecture (backwards direction) using “Laurent phenomenon”: if initial conditions are symbolic, sequence consists only of *Laurent polynomials* in initial variables with integer coefficients

Definition

Recall, recurrence of order k defined as:

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- Raise x_n to integer power greater than 1
- To compute x_n must solve degree m equation (so get m possibilities)
- Given initial conditions, produce infinitely many sequences
- Expect to produce complex numbers

Goal

Find *m*-recurrences that generate rational numbers:

- Let $\{g_n\}_{n=1}^{\infty}$ be integer (or rational) sequence
- Consider sequence of ratios of $\{g_n\}$, obviously a rational sequence
- Find *m*-recurrence that annihilates sequence of ratios of $\{g_n\}$
- Generalize the *m*-recurrence

Generalized Somos-4 Ratios

The *generalized Somos-4 recurrence* is given by

$$s_n s_{n-4} = \alpha s_{n-1} s_{n-3} + \beta s_{n-2}^2$$

with initial conditions $s_i = 1$ for $1 \leq i \leq 4$. This recurrence is order 4.

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Let $f_n := \frac{s_{n+2} s_n}{s_{n+1}^2}$ be the sequence of ratios of ratios of s_n.

Proposition (H. 2011)

The following 2-recurrence annihilates the sequence $\{f_n\}_{n=1}^{\infty}$*:*

$$f_{n-1}^2 f_n^2 + (\alpha - (2\alpha + \beta + 1) f_{n-1}) f_n + \alpha f_{n-1} + \beta = 0$$

Corollary of result due to Hone and Swart in 2008.

Sequence Tree for $f_{n-1}^2 f_n^2 + (1 - 4f_{n-1}) f_n + f_{n-1} + 1 = 0$

For every f_{n-1} there are two f_n's, so we will store them in a binary tree:

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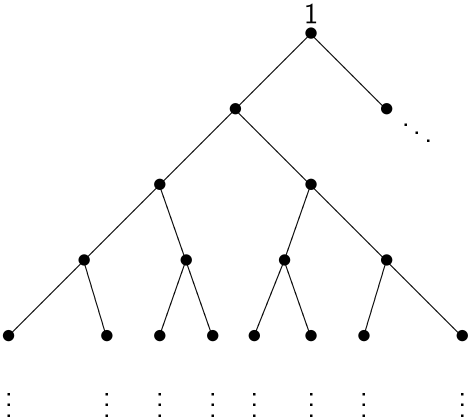

If $f_1 = 1$:

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Then $f_2 = 2$ or 1.

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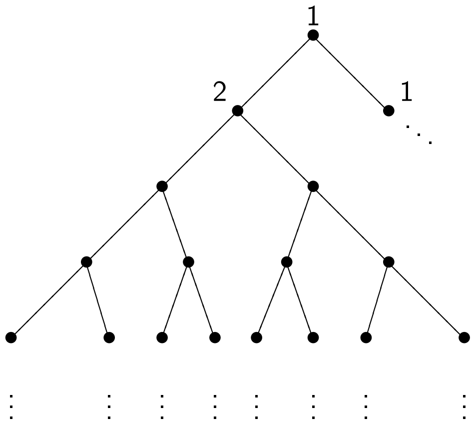

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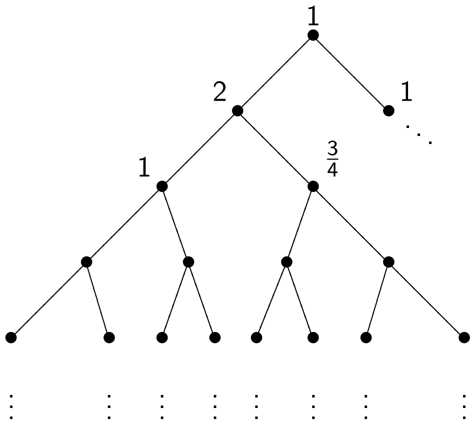

If $f_2 = 2$:

$$2^2 f_3^2 + (1 - 4 \cdot 2)\, f_3 + 2 + 1 = 0$$
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Then $f_3 = 1$ or $\frac{3}{4}$.

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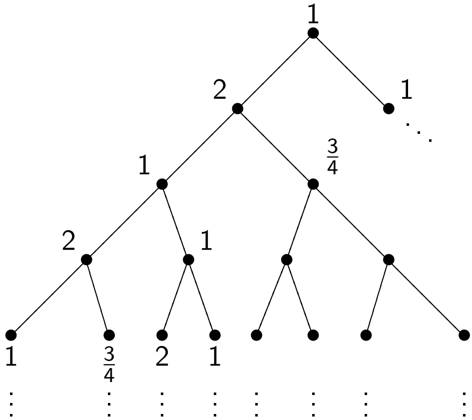

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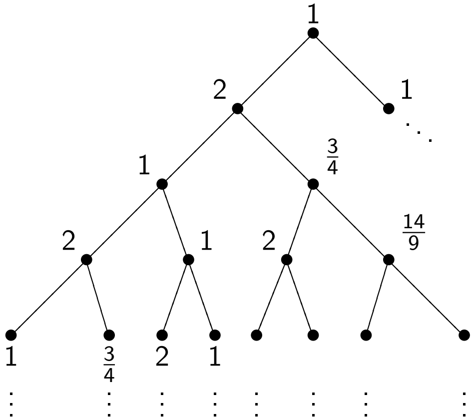

If $f_3 = \frac{3}{4}$:

$$\frac{9}{16} f_4^2 - 2f_4 + \frac{7}{4} = 0.$$

Then $f_4 = 2$ or $\frac{14}{9}$.

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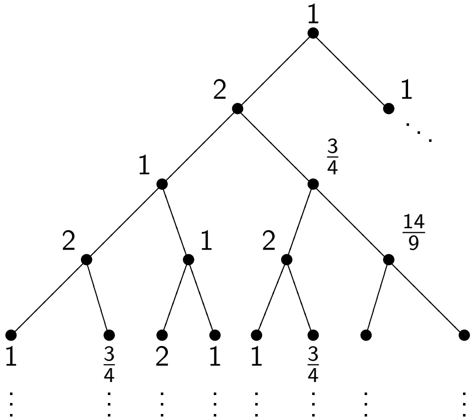

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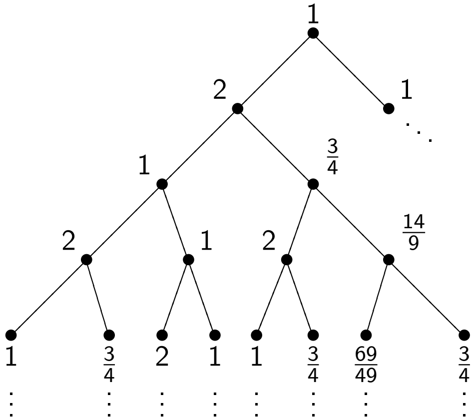

If $f_4 = \frac{14}{9}$:

$$\frac{169}{81} f_5^2 - \frac{47}{9} f_5 + \frac{23}{9} = 0.$$

Then $f_5 = \frac{69}{49}$ or $\frac{3}{4}$.

Generalization of Somos-4 Ratios of Ratios

Proposition (H. 2011)

Let $X := x_n$ *and* $Y := x_{n-1}$*. Consider the 2-recurrence*

$$(Y^2 + A_1 Y + A_0)X^2 + (B_2 Y^2 + B_1 Y + B_0)X + (C_2 Y^2 + C_1 Y + C_0) = 0,$$

with initial condition $x_1 = 1$*. The corresponding sequence tree is rational if*

(i) $A_1 = B_2$, $A_0 = C_2$, $B_0 = C_1$, *and*

(ii) $(B_2 + B_1 + B_0)^2 - 4(A_1 + A_0 + 1)(C_2 + C_1 + C_0) = q^2$, $q \in \mathbb{Q}$.

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- 2-recurrence satisfying (i) known as *Euler-Chasles correspondence*

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- Criteria (i) makes 2-recurrence symmetric in x_n and x_{n-1}
- 2-recurrence satisfying (i) known as *Euler-Chasles correspondence*
- Generalized Somos-4 ratios of ratios: $A_1 = B_2 = A_0 = C_2 = 0$, $B_0 = C_1 = \alpha$, $B_1 = -(2\alpha + \beta + 1)$, and $C_0 = \beta$

$$Y^2 X^2 + (\alpha - (2\alpha + \beta + 1)Y)X + \alpha Y + \beta = 0$$

Proof of Proposition

$X := x_n$, $Y := x_{n-1}$, make substitutions from criteria (i):

$$P(X,Y) := (Y^2 + A_1 Y + A_0)X^2 + (A_1 Y^2 + B_1 Y + B_0)X + (A_0 Y^2 + B_0 Y + C_0).$$

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Base case: Given that $x_1 = 1$, show $x_2 \in \mathbb{Q}$.

Values for x_2 are solutions to $P(X,1) = 0$

$$(1 + A_1 + A_0)X^2 + (A_1 + B_1 + B_0)X + (A_0 + B_0 + C_0) = 0.$$

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Discriminant, $(B_2 + B_1 + B_0)^2 - 4(A_1 + A_0 + 1)(C_2 + C_1 + C_0)$, is assumed to be square of rational in criteria (ii), so both values for x_2 are rational.

Induction Step

- Assume x_{n-2} and its children $x_{n-1}^{(1)}$ and $x_{n-1}^{(2)}$ are rational:

$$\{x_{n-1}^{(1)}, x_{n-1}^{(2)}\} = \{X : P(X, x_{n-2}) = 0\} \subset \mathbb{Q}$$

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- P symmetric, so $P(x_{n-2}, x_{n-1}^{(i)}) = 0$ for both $i = 1, 2$
- Children of $x_{n-1}^{(i)}$ for some $i = 1, 2$ are

$$\{x_n^{(i,1)}, x_n^{(i,2)}\} = \{X : P(X, x_{n-1}^{(i)}) = 0\}$$

- Clearly x_{n-2} is in this set (and is assumed to be rational) so the other root must also be rational

□

"Unfold" Somos-4 ratios

Recall 2-recurrence for $f_n := \frac{s_{n+2}s_n}{s_{n+1}^2}$,

$$f_n^2 f_{n+1}^2 + \left(\alpha - (2\alpha + \beta + 1) f_n\right) f_{n+1} + \alpha f_n + \beta = 0.$$

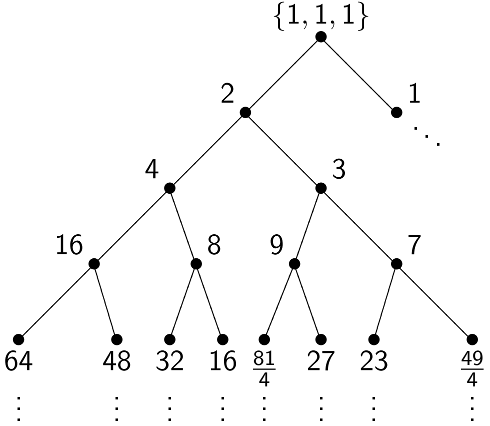
Can make substitution to get 2-recurrence that annihilates generalized Somos-4 sequence:

$$s_n^2 s_{n+3}^2 + \left(\alpha s_{n+1}^3 - (2\alpha + \beta + 1) s_n s_{n+1} s_{n+2}\right) s_{n+3} + \alpha s_n s_{n+2}^3 + \beta s_{n+1}^2 s_{n+2}^2 = 0.$$

This 2-recurrence is order 3.

Sequence Tree: 2-Recurrence of Order 3 for Somos-4

$$s_n^2 s_{n+3}^2 + \left(s_{n+1}^3 - 4 s_n s_{n+1} s_{n+2}\right) s_{n+3} + s_n s_{n+2}^3 + s_{n+1}^2 s_{n+2}^2 = 0.$$



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- 2^{n-3}
- 3^{n-4}
- Non-integer rational numbers

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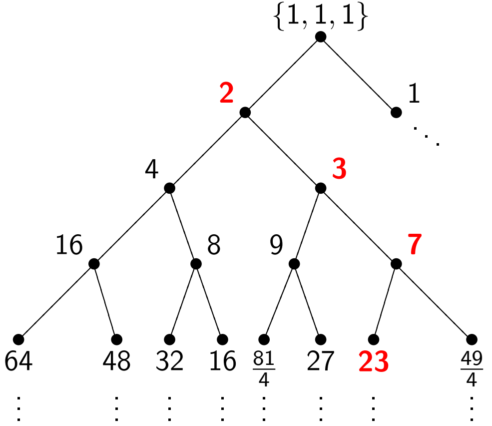

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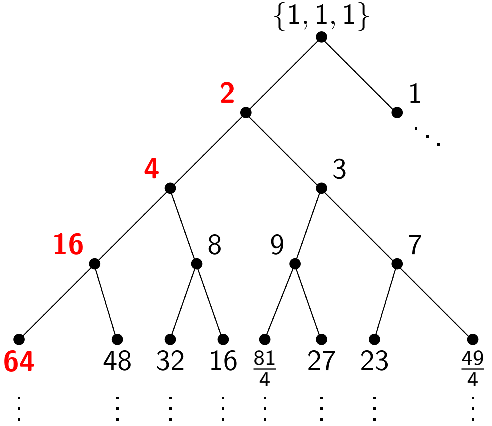

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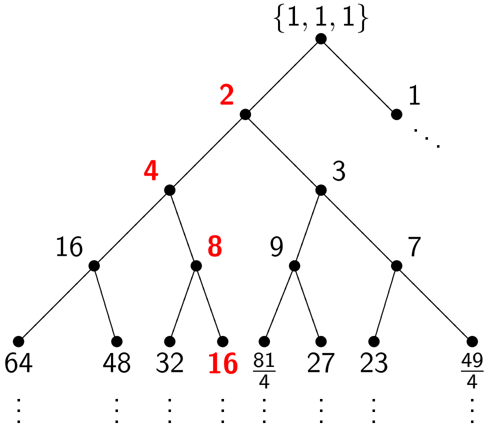

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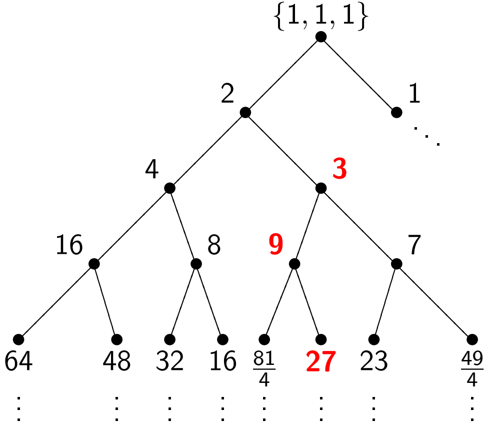

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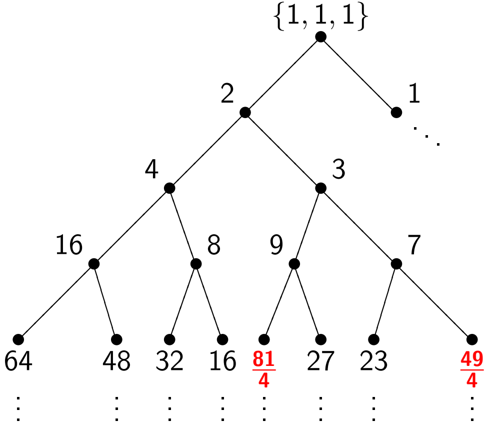

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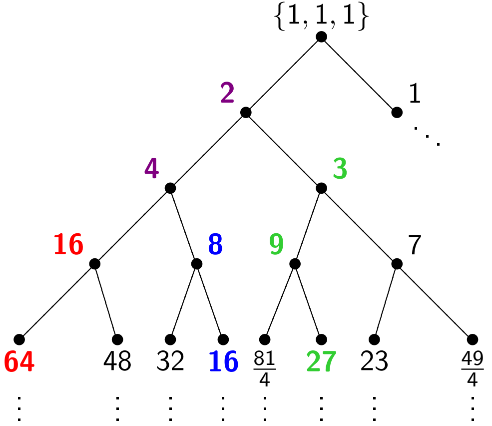

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"Unfold" Euler-Chasles

Substituting $x_n = \frac{a_{n+2}a_n}{a_{n+1}}$ into Euler-Chasles correspondence we can create a more general family of 2-recurrences of order 3:

$$(a_n^2 a_{n+2}^2 + A_1 a_n a_{n+1}^2 a_{n+2} + A_0 a_{n+1}^4) a_{n+1}^2 a_{n+3}^2 +$$
$$+ (A_1 a_n^2 a_{n+2}^2 + B_1 a_n a_{n+1}^2 a_{n+2} + B_0 a_{n+1}^4) a_{n+1} a_{n+2}^2 a_{n+3} +$$
$$+ (A_0 a_n^2 a_{n+2}^2 + B_0 a_n a_{n+1}^2 a_{n+2} + C_0 a_{n+1}^4) a_{n+2}^4 = 0.$$

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Can show

$$a_{n+1} = \frac{a_2^n}{a_1^{n-1}} \left(x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1} \right),$$

so $a_n \in \mathbb{Q}$ for all n since $x_n \in \mathbb{Q}$ and a_1, a_2 assumed to be rational.

$\gamma^{\lfloor n^2/4 \rfloor}$ Sequence Produced

Theorem (H. 2011)

The sequence $a_n = \gamma^{\lfloor n^2/4 \rfloor}$ is annihilated by the more general 2-recurrence of order 3 iff γ is a solution to the following quadratic equation

$$(A_1 + A_0 + 1)\gamma^2 + (A_1 + B_1 + B_0)\gamma + A_0 + B_0 + C_0 = 0.$$

To Prove:

Substitute $a_n = \gamma^{\lfloor n^2/4 \rfloor}$ into 2-recurrence and simplify. The result is exactly the quadratic equation given in the theorem.

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For Somos-4:
Let $A_1 = A_0 = 0$, $B_1 = -4$, $B_0 = 1$, and $C_0 = 1$. Quadratic equation is

$$\gamma^2 - 3\gamma + 2 = 0,$$

so $s_n = 2^{\lfloor n^2/4 \rfloor}$ is annihilated (also $s_n = 1^{\lfloor n^2/4 \rfloor} = 1$).

ψ^n Sequences Produced

Theorem (H. 2011)

For all $\psi \in \mathbb{R}$, $a_n = \psi^n$ is a solution to the more general 2-recurrence of order 3 iff

$$2A_1 + 2A_0 + B_1 + 2B_0 + C_0 + 1 = 0.$$

To Prove:

Substitute $a_n = \psi^n$ and simplify. What remains is exactly the condition on the parameters given in the theorem.

ψ^n Sequences Produced

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For Somos-4:

Let $A_1 = A_0 = 0$, $B_1 = -4$, $B_0 = 1$, and $C_0 = 1$. Criteria in theorem is satisfied:

$$2 \cdot 0 + 2 \cdot 0 - 4 + 2 \cdot 1 + 1 + 1 = 0,$$

so $s_n = \psi^n$ is annihilated for all $\psi \in \mathbb{R}$. In particular, since 2, 4, 8 and 3, 9, 27 appear consecutively, we see 2^{n-3} and 3^{n-4} in sequence tree.

Comparing Order 3 and Order 4 Somos-4 Recurrences

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  Order 4: $s_n \sim \phi^{n^2}$

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- Closed Form

  Order 4: Weierstrass sigma functions

  Order 3: Some branches are elementary functions

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- Rational sequences:
  - Conjectured criteria on coefficients in 2-recurrence of order 1
  - Conjectured existence of exponential branches. Exponential sequences were unexpected given the behavior of the 2-recurrence of order 1.

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Thank You

Any Questions?

To get references to appear!

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