Partial Hamiltonian formalism, multi-time dynamics and singular theories

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Abstract

We formulate singular (with degenerate Lagrangians) classical theories (for clarity, in local coordinates) without involving constraints. First, we recall the standard action principle (for pedagogical reasons and in order to establish notation). Then, applying it to the action (27)

\[ S = \int \left( p_i dq^i - H_\alpha dq^\alpha - H_0 dt \right), \]

we develop a partial (in the sense that not all velocities are transformed to momenta) Hamiltonian formalism in an initially reduced phase space (with canonical coordinates \( q_i, p_i \), where the number of momenta, see (17),

\[ p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad i = 1, \ldots, n_p, \]

is arbitrary, where \( n \) is the dimension of the configuration space) in terms of a partial Hamiltonian \( H_0(q_i, p_i, q^\alpha, \dot{q}^\alpha) \), see (18)

\[ H_0 = p_i \dot{q}^i + \frac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha - L, \]

and \((n - n_p)\) additional Hamiltonians \( H_\alpha(q_i, p_i, q^\alpha, \dot{q}^\alpha) \), \( \alpha = n_p + 1, \ldots, n \), see (20)

\[ H_\alpha = -\frac{\partial L}{\partial \dot{q}^\alpha}, \quad \alpha = n_p + 1, \ldots n \]

(instead of the remaining momenta \( p_\alpha \) defined in the standard full Hamiltonian formalism (6)). In this way we obtain \((n - n_p + 1)\) Hamilton-Jacobi equations (25)-(26)

\[ \frac{\partial S}{\partial t} + H_0 \left( t, q^i, \frac{\partial S}{\partial q^i}, \dot{q}^\alpha, \dot{q}^\alpha \right) = 0, \]
\[ \frac{\partial S}{\partial q^\alpha} + H_\alpha \left( t, q^i, \frac{\partial S}{\partial q^i}, \dot{q}^\alpha, \dot{q}^\alpha \right) = 0, \]

which fully determine the dynamics. The equations of motion are first-order differential equations (33)-(34)

\[ \dot{q}^i = \{q^i, H_0\} + \{q^i, H_\beta\} \dot{q}^\beta, \]
\[ \dot{p}_i = \{p_i, H_0\} + \{p_i, H_\beta\} \dot{q}^\beta, \]
\[ \{A, B\} = \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q^i} \frac{\partial A}{\partial p_i} \]
with respect to the canonical coordinates $q_i, p_i$ and second-order differential equations (35)

$$\frac{\partial H_\alpha}{\partial q^\beta} \dot{q}^\beta + \frac{d}{dt} \left( \frac{\partial H_0}{\partial q^\alpha} + \frac{\partial H_\alpha}{\partial q^\beta} q^\beta \right) = \left( \frac{\partial H_\beta}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial q^\beta} + \{H_\beta, H_\alpha\} \right) \dot{q}^\beta + \left( \frac{\partial H_0}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial t} + \{H_0, H_\alpha\} \right)$$

in the noncanonical coordinates $q_\alpha$ (which have no corresponding momenta). In the partial Hamiltonian formalism (which describes the same dynamics as the Lagrange equations of motion (3)), the number of momenta $n_p \leq n$ is arbitrary. The limit cases $n_p = n$ and $n_p = 0$ correspond to the standard Hamiltonian and Lagrangian dynamics (discussed in (37)-(41)), respectively.

If the Hamiltonians $H_0(q_i, p_i, q^\alpha), H_\alpha(q_i, p_i, q^\alpha)$ do not depend of the noncanonical velocities $\dot{q}_\alpha$, conditions (42)

$$\frac{\partial H_0}{\partial \dot{q}^\beta} = 0, \quad \frac{\partial H_\alpha}{\partial \dot{q}^\beta} = 0, \quad \alpha, \beta = n_p + 1, \ldots, n,$$

then the second-order differential equations (35) become purely algebraic equations (43)

$$\left( \frac{\partial H_\beta}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial q^\beta} + \{H_\beta, H_\alpha\} \right) \dot{q}^\beta = - \left( \frac{\partial H_0}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial t} + \{H_0, H_\alpha\} \right),$$

with respect to $\dot{q}_\alpha$. In this case we can interpret the noncanonical coordinates $q_\alpha$ as additional times by (45)

$$\tau^\mu = t, \quad H_\mu = H_0, \quad \mu = 0,$n
$$\tau^\mu = q^{\mu+n_p}, \quad H_\mu = H_{\mu+n_p}, \quad \mu = 1, \ldots, (n - n_p),$$

such that the partial Hamiltonian formalism becomes equivalent to multi-time dynamics with action (51)

$$S = \int (p_i dq^i - H_\mu d\tau^\mu)$$

and equations of motion (53)-(54)

$$dq^i = \{q^i, H_\mu\} d\tau^\mu, \quad dp_i = \{p_i, H_\mu\} d\tau^\mu,$$

with additional (integrability) conditions for the Hamiltonians (55)

$$\left( \frac{\partial H_\mu}{\partial \tau^\nu} - \frac{\partial H_\nu}{\partial \tau^\mu} + \{H_\mu, H_\nu\} \right) d\tau^\nu = 0, \quad \mu, \nu = 0, \ldots, (n - n_p).$$

The independence of the Hamiltonians $H_0(q_i, p_i, q^\alpha), H_\alpha(q_i, p_i, q^\alpha)$ of the noncanonical velocities $\dot{q}_\alpha$ (conditions (42)) is satisfied in singular theories

$$\frac{\partial^2 L}{\partial q^\alpha \partial \dot{q}^\beta} = 0, \quad \alpha, \beta = n_p + 1, \ldots, n$$

with degenerate Lagrangians, in the sense that the determinant of the Hessian matrix

$$W_{AB} = \left\| \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right\|, \quad A, B = 1, \ldots, n$$
is zero: \( \det W_{AB} = 0 \), in which case the rank \( r_W \) of the Hessian is less than or equal to the number of momenta: \( r_W \leq n_p \), see (58). If we choose \( n_p > r_W \), then we obtain \( (n_p - r_W) \) primary (and perhaps, higher-level) constraints (as in the Dirac theory), but if we assume the equality \( n_p = r_W \), then there will be no constraints at all. The \( (n - r_W) \) equations for the same number of \( (n - r_W) \) noncanonical velocities \( \dot{q}_\alpha \) (62) \( F_{\alpha\beta} \dot{q}^\beta = G_\alpha \) constitute a standard system of linear algebraic equations, but not constraints, because we do not define the “extra” momenta \( p_\alpha \), and the dynamics is fully described without them by the equations of motion (60)-(62)

\[
\dot{q}^i = \{q^i, H_0\} + \{q^i, H_\beta\} \dot{q}^\beta,
\]

\[
\dot{p}_i = \{p_i, H_0\} + \{p_i, H_\beta\} \dot{q}^\beta,
\]

\[
F_{\alpha\beta} \dot{q}^\beta = G_\alpha,
\]

where \( i = 1, \ldots, r_W, \alpha, \beta = r_W + 1, \ldots, n, \) and

\[
F_{\alpha\beta} = \frac{\partial H_\alpha}{\partial q^\beta} - \frac{\partial H_\beta}{\partial q^\alpha} + \{H_\alpha, H_\beta\},
\]

\[
G_\alpha = D_\alpha H_0 = \frac{\partial H_0}{\partial q^\alpha} - \frac{\partial H_\alpha}{\partial t} + \{H_0, H_\alpha\}.
\]

Classification of singular theories can be made by the analysis of the linear algebraic system (62) in terms of the rank of the tensor \( F_{\alpha\beta} \) (63). If its rank is full, i.e., equal to \( (n - r_W) \), then we can solve the system (62) by

\[
\dot{q}^\alpha = \tilde{F}^{\alpha\beta} G_\beta,
\]

\[
\tilde{F}^{\alpha\beta} F_{\beta\gamma} = F_{\gamma\beta} \tilde{F}^{\beta\alpha} = \delta^\alpha_\gamma,
\]

and there will be no arbitrary parameters (gauge degrees of freedom) in the theory; in other cases, some of the noncanonical velocities \( \dot{q}_\alpha \) will remain arbitrary, which is a sign of gauge theory. In both cases we define new antisymmetric brackets (69)

\[
\{A, B\}_{\text{nongauge}} = \{A, B\} + D_\alpha A \cdot \tilde{F}^{\alpha\beta} \cdot D_\beta B,
\]

and (80)

\[
\{A, B\}_{\text{gauge}} = \{A, B\} + D_{\alpha_1} A \cdot \tilde{F}^{\alpha_1\beta_1} \cdot D_{\beta_1} B,
\]

\[\alpha_1, \beta_1 = r_W + 1, \ldots, r_F,\]

\[r_F = \text{rank} F_{\alpha\beta},\]

which govern time evolution of physical variables (70)and (83) and present the equations of motion in the Hamilton-like form, (67)-(68)

\[
\dot{q}^i = \{q^i, H_0\}_{\text{nongauge}},
\]

\[
\dot{p}_i = \{p_i, H_0\}_{\text{nongauge}},
\]

and (81)-(82)

\[
\dot{q}^i = \{q^i, H_0\}_{\text{gauge}},
\]

\[
\dot{p}_i = \{p_i, H_0\}_{\text{gauge}}.
\]
Finally, we clarify the origin of the Dirac constraints in our framework: if we define \((n - r_W)\) “extra” dynamical variables, that is to say, the momenta \(p_{\alpha}\) by (86)

\[
p_{\alpha} = \frac{\partial L}{\partial \dot{q}_{\alpha}}, \quad \alpha = r_W + 1, \ldots, n,
\]
then we obtain the standard primary constraints (87)

\[
\Phi_{\alpha} = p_{\alpha} + H_{\alpha} = 0, \quad \alpha = r_W + 1, \ldots, n,
\]
and the equations of motion (90)-(91)

\[
\begin{align*}
dq_A &= \left\{q^A, H_{\text{total}}\right\}_{\text{full}} dt, \\
p_A &= \left\{p_A, H_{\text{total}}\right\}_{\text{full}} dt, \quad A = 1, \ldots, n,
\end{align*}
\]
\[
\{A, B\}_{\text{full}} = \frac{\partial A}{\partial q^A} \frac{\partial B}{\partial p_A} - \frac{\partial B}{\partial q^A} \frac{\partial A}{\partial p_A},
\]
in terms of the Dirac total Hamiltonian (88)

\[
H_{\text{total}} = p_i \dot{q}^i + p_{\alpha} \dot{q}^\alpha \cdot L = H_0 + \dot{q}^\alpha \Phi_{\alpha}.
\]

In this case our new brackets (69) and (80) will transform into the Dirac bracket.

At the end of the paper, quantization is discussed briefly.


References


REFERENCES


