1 The Bernoulli Problem

Let D be a smooth bounded open set. Define

$$E(u;E) = \int_E |\nabla u|^2 + \Lambda 1_{u>0}$$

for any $u \in H^1(E)$. We study the Alt-Caffarelli variational problem (see [1]):

$$\min\{E(u;D) : u \in H^1(D), u = 1 \text{ on } \partial D\}. \tag{1.1}$$

This may be motivated by the following optimal insulation problem: assume that $\mathbb{R}^n \setminus D$ is held at a constant temperature 1, whereas D is at the ambient temperature 0. We place some insulator ($\{u > 0\} \cap D$) inside of D to reduce the heat transfer from the complement to the interior. Then the equilibrium heat distribution will be given by a function u which is 1 on $\mathbb{R}^n \setminus D$, is harmonic within the insulator region, and is 0 on the remainder of D. The amount of heat being lost to D per unit time is given by the heat flux through ∂D:

$$\int_{\partial D} u_\nu d\mathcal{H}^{n-1},$$

where ν is the outward unit normal. We may then set up an optimization problem: choosing a larger insulator will lower the cost of heating the complement of D, but if we have to pay Λ per unit of insulator, we should minimize

$$\int_{\partial D} u_\nu d\mathcal{H}^{n-1} + \Lambda|\{u > 0\} \cap D|$$

over all u harmonic on $\{u > 0\} \cap D$ and satisfying the boundary conditions above. As $u = 1$ on ∂D, we may integrate by parts:

$$\int_{\partial D} u_\nu d\mathcal{H}^{n-1} = \int_{\partial D} u u_\nu d\mathcal{H}^{n-1} = \int_D |\nabla u|^2.$$

If we then minimize

$$\int_D |\nabla u|^2 + \Lambda|\{u > 0\} \cap D|$$

over *all of* $H^1(\mathbb{R}^n)$ with $u = 1$ on $\mathbb{R}^n \setminus D$, this will recover a u satisfying our boundary value problem, making it equivalent to the insulation functional above (see the next section for details).

2 Basic Properties

Theorem 2.1. *The problem* (1.1) *admits a minimizer. Any minimizer to* (1.1) *is a nonnegative function.*

Proof. First, observe that $E(u_+;D) \le E(u;D)$ for any admissible function u; this establishes the second claim and also lets us find a sequence u_k with $u_k \in H^1(D)$, $u_k = 1$ on ∂D, $u_k \ge 0$, with

$$E(u_k;D) \to \inf\{E(u;D) : u \in H^1(D), u|_{\partial D} = 1\}.$$

We may extract a subsequence $u_k \to u$ weakly in $H^1(D)$, and hence strongly in $L^2(D)$ and $L^2(\partial D)$ (the latter from the continuity of the trace map, the former from the compact embedding). Up to passing to a further subsequence, we may also assume that $u_k \to u$ almost everywhere.

Note that $1_{u>0} \le \liminf 1_{u_k>0}$ a.e.: this follows from the fact that $t \mapsto 1_{t>0}$ is lower semicontinuous. Then from Fatou's lemma,

$$\int_D 1_{u>0} \le \liminf \int_D 1_{u_k>0}.$$

From the weak convergence in H^1,

$$\int_D |\nabla u|^2 \leq \liminf \int_D |\nabla u_k|^2.$$

These imply that

$$E(u;D) \leq \liminf E(u_k;D) = \inf\{E(u;D) : u \in H^1(D), u|_{\partial D} = 1\},$$

and so we have found a minimizer. □

As an exercise, try removing the assumption that D is bounded (you will need to use the lower bound we prove later).

Below it will be convenient to consider the notion of *local minimizer* of 1.1:

Definition 2.1. *We say that a nonnegative function $u \in H^1(U)$ is a* local minimizer *on U if for any v with $u - v \in H^1_0(V)$ for some bounded open $V \subset\subset U$, we have*

$$E(u;V) \leq E(v;V).$$

If $U = \mathbb{R}^n$, we say that u is an entire *local minimizer.*

Note that at least if U is bounded and regular and $E(u,U) < \infty$, one may equivalently take $V = U$, but the form above will be more convenient for us.

Lemma 2.2. *Let u be a local minimizer on U, and $B_r(x) \subset\subset U$. Let h be the harmonic replacement of u, i.e. the solution of*

$$\begin{cases} \triangle h = 0 & \text{on } B_r(x) \\ h = u & \text{on } \partial B_r(x). \end{cases} \tag{2.2}$$

Then

$$\int_{B_r(x)} |\nabla(u-h)|^2 \leq \Lambda |\{u = 0\} \cap B_r(x)|.$$

Proof. Let h be as defined, and extend it to coincide with u outside of $B_r(x)$. Then $E(u;B_r(x)) \leq E(h;B_r(x))$, giving

$$\int_{B_r(x)} |\nabla u|^2 + \Lambda|\{u>0\} \cap B_r(x)| \leq E(u;B_r(x)) \leq E(h;B_r(x)) = \int_{B_r(x)} |\nabla h|^2 + \Lambda|B_r(x)|.$$

Hence

$$\int_{B_r(x)} \nabla(u-h)\cdot\nabla(u+h) = \int_{B_r(x)} |\nabla u|^2 - |\nabla h|^2 \leq \Lambda|\{u=0\} \cap B_r|.$$

Now observe that as h is harmonic and $u - h \in H^1_0(B_r(x))$, we have

$$\int_{B_r(x)} 2\nabla h \cdot \nabla(u-h) = 0,$$

giving

$$\int_{B_r(x)} |\nabla(u-h)|^2 \leq \Lambda|\{u=0\} \cap B_r|.$$

□

We now go through a standard argument due to Morrey, which shows that the property in the preceding lemma gives an almost Lipschitz modulus of continuity. This will be improved in the next section, but that improvement will use some subtle aspects of the free boundary problem we are considering. By contrast, the argument here is completely generic.

Let $A(x,r) = \sqrt{\fint_{B_r(x)} |\nabla u|^2}$ below.

Lemma 2.3. *Let u be a local minimizer on U and $B_r(x) \subset\subset U$. Then $A(x, \frac{1}{2}r) \leq A(x,r) + \sqrt{2^n \Lambda}$.*

Proof. we have from Lemma 2.2 that

$$\fint_{B_r(x)} |\nabla(u-h)|^2 \leq \Lambda.$$

From the subharmonicity of $|\nabla h|^2$, we have that

$$\fint_{B_{r/2}(x)} |\nabla h|^2 \leq \fint_{B_r(x)} |\nabla h|^2.$$

Hence

$$\begin{aligned} \sqrt{\fint_{B_{r/2}(x)} |\nabla u|^2} &\leq sqrt 2^n \fint_{B_r(x)} |\nabla(u-h)|^2 + sqrt \fint_{B_{r/2}(x)} |\nabla h|^2 \\ &\leq \sqrt{2^n \Lambda} + \sqrt{\fint_{B_r(x)} |\nabla h|^2} \\ &\leq \sqrt{2^n \Lambda} + \sqrt{\fint_{B_r(x)} |\nabla u|^2}. \end{aligned}$$

The last step used that h minimizes the Dirichlet energy, and u is a competitor. □

Lemma 2.4. *Let u be a minimizer of (1.1) and $d(x) = d(x, \partial D)$. Then $A(x,r)^2 \leq \frac{C}{d^n(x)} E(u;D) + C\Lambda |\log r/d(x)|^2$.*

Proof. We have that for $r \geq d(x)$,

$$A^2(x,r) \leq \frac{1}{|B_d(x)|} E(u;D) \leq \frac{C}{d^n(x)} E(u;D).$$

Now for $r_k = 2^{-k} d(x)$, we have

$$A(x, r_k) \leq A(x, r_{k-1}) + \sqrt{2^n \Lambda} \leq A(x, d(x)) + k\sqrt{2^n \Lambda}$$

from Lemma 2.3. This gives

$$A(x, r_k) \leq \sqrt{\frac{C}{d^n(x)} E(u;D)} + \sqrt{2^n \Lambda} |\frac{\log r_k / d(x)}{\log 2}|$$

Using that $A(x,r) \leq 2^n A(x, r_k)$ for $r \in (r_{k+1}, r_k)$ and squaring gives the conclusion. □

Theorem 2.5. *With u as above,*

$$|u(x) - u(y)| \leq C(d(x)) \sqrt{E(x;D) + \Lambda} |x-y| (1 + |\log |x-y||)$$

for any $x, y \in D$ with $|x-y| \leq d(x)/2$.

Proof. Set $m(x,r) = \fint_{B_r(x)} u$. Applying Lemma 2.4, we have that

$$\fint_{B_r(x)} |u - m(x,r)|^2 \leq C(d(x)) r^2 (1 + |\log r|^2)[E(u;D) + \Lambda].$$

for $r < d(x)$. This gives

$$\begin{aligned}|m(x,r)-m(x,r/2)|^2 &= \fint_{B_{r/2}(x)} |m(x,r)-m(x,r/2)|^2 \\ &\leq 2^{n+1}\fint_{B_r(x)} |m(x,r)-u(x)|^2 + 2\fint_{B_{r/2}(x)} |u(x)-m(x,r/2)|^2 \\ &\leq C(d(x))r^2(1+|\log r|^2)[E(u;D)+\Lambda].\end{aligned}$$

In particular, this means that

$$|m(x,2^{-k}d(x)) - m(x,2^{-k-1}d(x))| \leq [E(u;D)+\Lambda]C(d(x))2^{-k}(1+k),$$

meaning it is a Cauchy sequence. Summing gives

$$|m(x,0+) - m(x,2^{-k}d(x))| \leq [E(u;D)+\Lambda]C(d(x))2^{-k}(1+k),$$

noting carefully that no further powers of k are gained when computing the sum. Taking $u(x) = m(x,0+)$ gives a Lebesgue representative for u for which every point is a Lebesgue point.

We may then check that for any $y \in B_{r/2}(x)$, with $r < d(x)$,

$$|u(y)-u(x)| \leq |u(y)-m(y,r/2)|+|m(y,r/2)-m(x,r)|+|m(x,r)-u(x)| \leq [E(u;D)+\Lambda]C(d(x))r(1+|\log r|).$$

Indeed, the first and third terms follow from the above summation property, while the second similarly to our estimate on $m(x,r)-m(x,r/2)$. This establishes the conclusion. ☐

In particular, u is continuous on D. The $d(x)$ dependence may be removed by combining with boundary regularity results for harmonic functions, in light of the following observation:

Theorem 2.6. *Let u be a continuous local minimizer on D. Then u is subharmonic on D, and harmonic on $\{u > 0\} \cap D$.*

Proof. That u is harmonic on $\{u > 0\}$ is immediate from its continuity and Lemma 2.2: it coincides with its harmonic replacement on a small ball where $u > 0$. Any continuous function $u \geq 0$ which is harmonic where positive is subharmonic; this may be checked for each of our notions of subharmonic directly. ☐

This means that $\triangle u$ is representable as a positive Borel measure. Unlike for the obstacle problem, we will see that this measure is not absolutely continuous with respect to Lebesgue measure, and instead is concentrated on the free boundary $\partial\{u > 0\}$.

3 Optimal Regularity

The following sequence of lemmas aims to show that u is Lipschitz continuous. This is an improvement over the results of the previous section, which are basically optimal without the additional special structure of our variational problem. The argument here is inspired by work of Alt, Caffarelli, and Friedman [2] for the two-phase problem; there is a simpler but more specialized argument available in the one-phase case (and can be found in Alt-Caffarelli [1]), but this is more robust and gives some useful extra information.

Recall that for any subharmonic function

$$\partial_s \fint_{\partial B_s} u = \frac{c(n)}{s^{n-1}} \triangle u(B_s).$$

Integrating gives

$$\fint_{B_{2r}} u - u(0) = c(n)\int_0^{2r} \frac{\triangle u(B_s)}{s^{n-1}} ds. \tag{3.3}$$

Lemma 3.1. *Let u be a local minimizer on D, and $B_r \subseteq D$. Then*

$$\triangle u(B_{r/2}) \cdot \fint_{B_r} u \leq C\Lambda r^n.$$

Proof. From Lemma 2.2, we have that

$$\int_{B_r} |\nabla(u-h)|^2 \leq C\Lambda r^n,$$

where h is the harmonic replacement of u on B_r. As h is harmonic, this gives

$$C\Lambda r^n \geq \int_{B_r} |\nabla(u-h)|^2 = \int_{B_r} \nabla u \cdot \nabla(u-h) = \int_{B_r} (h-u) d\triangle u.$$

From the maximum principle, $h \geq u$ (recall that u is subharmonic). We may therefore split the integral as

$$\int_{B_r} (h-u)d\triangle u = \int_{B_{r/2}\cap\{u=0\}} (h-u)d\triangle u + \int_{(B_{r/2}\cap\{u>0\})\cup B_r\setminus B_{r/2}} (h-u)d\triangle u$$

and observe that the second term is nonnegative; this gives

$$\int_{B_{r/2}\cap\{u=0\}} hd\triangle u \leq C\Lambda r^n.$$

Now, from the mean value property, we have that

$$h(0) = \fint_{\partial B_r} u.$$

From the Harnack inequality, then,

$$\inf_{B_{r/2}} h \geq c(n)h(0) = c\fint_{\partial B_r} u.$$

This gives that

$$C\fint_{\partial B_r} u \triangle u(B_{r/2} \cap \{u=0\}) \leq C\Lambda r^n.$$

To conclude, note that u is harmonic on the open set $\{u>0\} \cap B_{r/2}$, so

$$\triangle u(B_{r/2} \cap \{u>0\}) = 0.$$

□

Lemma 3.2. *There is a constant C_L (depending only on n) such that if $B_{2r} \subseteq D$ and $u(0)=0$, then*

$$\sup_{B_{r/2}} u \leq C\sqrt{\Lambda} r.$$

Proof. First, as u is subharmonic the supremum in question will always be attained on $\partial B_{r/2}$, say at a point x. We will show the following stronger statement, then:

$$u(x) \leq C\sqrt{\Lambda} d(x, \partial\{u>0\}).$$

Set $d = d(x, \partial\{u>0\}) < r/2$.

By definition, $u > 0$ and harmonic on $B_d(x)$, so we may apply the Harnack inequality there to learn that

$$\inf_{B_{d/2}(x)} u \geq c(n)u(x).$$

Let $y \in \partial B_d(x)$ with $u(y) = 0$ below. Define q to be the following radial harmonic function:

$$q(z) = \begin{cases} \frac{|z|^{2-n}-1}{2^{n-2}-1} & n \geq 3 \\ \frac{-\log|z|}{\log 2} & n = 2. \end{cases}$$

This is the unique harmonic function which equals 1 on $\partial B_{1/2}$ and 0 on ∂B_1. Then set $q_u(z) = c(n)u(x)q(x+dz)$; we have that $q_u = 0$ on $\partial B_d(x)$, and that $q \leq u$ on $\partial B_{d/2}(x)$, so $u \geq q_u$ on $B_d \setminus B_{d/2}$. It is straightforward to check that for $|z| \in [1/2, 1]$, $q(z) \geq c(n)(1 - |z|)$ (one may in fact take $c(n) = |\nabla q|_{\partial B_1}$, as q is convex in the radial direction). This translates to

$$u(z) \geq q_u(z) \geq c(n)u(x)(1 - |z - x|/d)$$

for $z \in B_d(x) \setminus B_{d/2}(x)$.

As a consequence, we have that

$$\fint_{\partial B_s(y)} u \geq c(n)u(x)s/d$$

for every $s \leq d$. Indeed, $\partial B_s(y) \cap B_{d-\frac{s}{2}}(x)$ has surface measure at least $c(n)s$, and in this region $u \geq cu(x)s/d$.

Let us apply Lemma 3.1 to balls centered around y: this gives that

$$\triangle u(B_s(y)) \cdot cu(x)s/d \leq C\Lambda s^n.$$

Integrating via (3.3) gives that

$$\frac{cu(x)}{d}[\fint_{\partial B_d(y)} u - u(y)] \leq C\Lambda d;$$

as $u(y) = 0$ and the mean value there is bounded from below by $c(n)u(x)$, this means

$$\frac{cu(x)}{d}u(x) \leq C\Lambda d,$$

or $u(x) \leq C\sqrt{\Lambda}d$. This completes the argument. □

Corollary 3.3. *Let u be a local minimizer on B_1, and $u(0) = 0$. Then*

$$\sup_{B_{1/2}} |\nabla u| \leq C\sqrt{\Lambda}.$$

Proof. It suffices to show this for those x at which $u \neq 0$, for $\nabla u = 0$ a.e. on $\{u = 0\}$ (this is true of all Sobolev functions).

Fix $x \in B_{1/2}$, and let $d = d(x) = d(x, \{u = 0\}) < \frac{1}{2}$. Let $y \in \partial\{u > o\}$ be a point with $|x - y| = d$. Then from Lemma 3.2, $|u| \leq C\sqrt{\Lambda}d$ on $B_{d/8}(y)$.

Now, u is harmonic on $B_d(x)$. Noting that $B_{d/8}(y) \cap B_{9d/10}(x) \neq \emptyset$, apply the Harnack inequality to u to give

$$\sup_{B_{9d/10}} u \leq C \inf_{B_{9d/10}} u \leq C\sqrt{\Lambda}d.$$

Then apply elliptic estimates for u on this ball to give that

$$|\nabla u(x)| \leq C\frac{\sup_{B_{9d/10}} |u|}{d} \leq C\sqrt{\Lambda}.$$

□

There is also a global version of this:

Corollary 3.4. *Let u be a local minimizer on D, with $u \leq 1$. Then $|\nabla u(x)| \leq C\sqrt{\Lambda}(1+1/d(x,\partial D))$ (almost everywhere).*

If this is combined with boundary regularity for harmonic functions, it easily implies global estimates which do not worsen as one approaches ∂D, but again we do not pursue this here.

Proof. Fix x, and let $d = d(x) = d(x, \{u = 0\}) > 0$ as before. If $8d > d(x, \partial D) := r$, we note that u is harmonic on $B_{r/8}(x)$. Applying standard estimates for harmonic functions gives that

$$|\nabla u(x)| \leq \frac{C(n)}{8r} \operatorname*{osc}_{B_{r/8}(x)} u \leq \frac{C(n)}{r},$$

which is consistent with the conclusion. On the other hand, if this is not the case, there is a point y with $u(y) = 0$, $|x - y| = d$, and $4d < d(y, \partial D)$. Apply Lemma 3.2 to u and $B_{4d}(y)$ to give that $u(x) \leq C\sqrt{\Lambda}d$ on $B_{2d}(y)$, which contains $B_d(x)$. Again we apply standard estimates for harmonic functions on $B_d(x)$, this time with the improved oscillation bound:

$$|\nabla u(x)| \leq \frac{C(n)}{d} \operatorname*{osc}_{B_d(x)} u \leq C(n)\sqrt{\Lambda},$$

□

4 The Lower Bound

There is a second basic estimate available for this problem, which complements the first one. It says that the regularity above is optimal, and that the solution grows linearly away from the zero set. The argument here is taken from [9].

Theorem 4.1. *Let u be a local minimizer on D, and let $B_r \subset\subset D$. Then either*

$$\max_{B_r} u \geq c_L\sqrt{\Lambda}r,$$

where $c_L = c_L(n)$, or $u = 0$ on $B_{r/2}$.

Proof. Assume that

$$\max_{B_r} u \leq \varepsilon\sqrt{\Lambda}r$$

for $\varepsilon < \varepsilon_0$, a constant to be chosen shortly. We will first show that this implies that

$$\max_{B_{r/2}} u \leq \frac{1}{2}\varepsilon\sqrt{\Lambda}\frac{r}{2}.$$

Indeed, first let η_1 be a smooth cutoff function supported on B_r, identically 1 on $B_{9r/10}$, and having $|\nabla \eta_1| \leq C/r$. Using $\eta^2 u$ as a test function for $\triangle u \geq 0$, we have that

$$\int \eta^2 |\nabla u|^2 \leq \frac{C}{r^2} \int_{B_r} u^2 \leq C\Lambda\varepsilon^2 r^n.$$

(This is just the usual Cacciopolli inequality combined with our assumption.)

Now let η be another similar cutoff, this one supported on the complement of $B_{3r/4}$ and identically 1 on $\mathbb{R}^n \setminus B_{9r/10}$. Let $v = \eta u$ and use v as a competitor for the minimization of 1.1: this gives

$$\Lambda|\{u > 0\} \cap B_{3r/4}| \leq \int_{B_{9r/10}} |\nabla v|^2 - |\nabla u|^2 \leq C \int_{B_{9r/10}} |\nabla u|^2 + \frac{u^2}{r^2}.$$

Combining with the estimate on u and $|\nabla u$, this gives

$$|\{u > 0\} \cap B_{3r/4}| \leq C\varepsilon^2 r^n.$$

Then for any $x \in B_{r/2}$, applying the mean value property gives

$$u(x) \leq \fint_{B_{r/4}(x)} u \leq C\sqrt{\Lambda}\varepsilon^3.$$

So long as $C\varepsilon^2 \leq \frac{1}{4}$, this establishes our claim.

Now we show the conclusion of the theorem. If c_L is sufficiently small, we have that for every $x \in B_{1/2}$,

$$\max_{B_{r/2}(x)} u \leq \varepsilon\sqrt{\Lambda}\frac{r}{2}.$$

Now apply the claim inductively on $B_{2^{-k}r}(x)$, each time getting

$$\max_{B_{2^{-k}r}(x)} u \leq \varepsilon 2^{1-k}\sqrt{\Lambda}\frac{r}{2^{-k}}.$$

In particular, this gives that $u(x) = 0$, concluding the proof. □

5 Density Estimates

The theorem below captures some rough geometric information about the size of the free boundary. It is very similar to a similar theorem about minimal surfaces, where it is often referred to as the *uniform density estimates*. In other contexts, it is known as *Ahlfors regularity* or *Ahlfors-David regularity.*

Theorem 5.1. *Let u be a local minimizer on D, $B_r \subset\subset D$, and $0 \in \partial\{u > 0\}$. Then*

$$0 < c_D < \frac{|B_r \cap \{u > 0\}|}{|B_r|} < 1 - c_D < 1$$

and

$$c_D < \frac{\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_{r/16})}{r^{n-1}} < c_D^{-1}.$$

Moreover, $\{u > 0\}$ has locally finite perimeter.

In fact, we will prove a stronger estimate here, which controls the Minkowski content of the free boundary.

Proof. First, we have that $u(x) \geq c_L\sqrt{\Lambda}r$ for some point $x \in \partial B_{r/4}$, from Theorem 4.1. On the other hand, from Lemma 3.2 and its corollary, we have that $|\nabla u| \leq C\sqrt{\Lambda}$ on $B_{r/2}$. In particular, this gives that $u > \frac{c_L}{2}\sqrt{\Lambda}r > 0$ on $B_{\kappa r}(x)$, where $\kappa = \min\{\frac{1}{4}, \frac{c_L}{2C}\}$. Hence

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} \geq \frac{|B_{\kappa r}(x)|}{|B_r|} \geq \kappa^n > 0.$$

For the opposite inequality, recall from Lemma 2.2 that if h is the harmonic replacement of u on $B_{r/4}$, we have that

$$\frac{c}{r^2}\int_{B_{r/4}} |u - h|^2 \leq \int_{B_{r/4}} |\nabla(u - h)|^2 \leq C\Lambda|\{u = 0\} \cap B_{r/4}|. \tag{5.4}$$

Now, we have that

$$h(0) = \fint_{\partial B_{r/4}} h = \fint_{\partial B_{r/4}} u \geq \frac{1}{|\partial B_{r/4}|}\int_{\partial B_{r/4} \cap B_{\kappa r}(x)} u \geq C\kappa^{n-1}\sqrt{\Lambda}r \geq C\sqrt{\Lambda}r.$$

Applying the Harnack inequality to h (which, note, is harmonic and nonnegative on $B_{r/4}$) gives that in fact

$$h \geq C_* \sqrt{\Lambda} r$$

on $B_{r/8}$.

On the other hand, $u(0) = 0$, so from Lemma 3.2 we have that $\sup_{B_{\tau r}} u \leq C_G \sqrt{\Lambda} \tau r$. Select τ to be $\frac{C_*}{2C_G}$; then on $B_{\tau r}$ we have $|h - u| \geq \frac{1}{2} C_* \sqrt{\Lambda} r$. Substituting into (5.4) gives

$$\Lambda |\{u = 0\} \cap B_{r/4}| \geq \frac{c}{r^2} \int_{B_{\tau r}} |u - h|^2 \geq c \tau^n \Lambda r^n.$$

In particular,

$$\frac{|B_r \cap \{u > 0\}|}{|B_r|} < 1 - C\tau^n < 1.$$

Now for the estimates on the boundary. The lower estimate follows directly from the relative isoperimetric inequality and the estimates we have already shown (or can be shown analogously to the upper one below, but we do not pursue this further), so we focus on the upper one. Recall that $\triangle u$ is a nonnegative Borel measure with the estimate

$$\triangle u(B_{r/8}) \leq C \sqrt{\Lambda} r^{n-1};$$

this follows from Lemma 3.1 and our estimates above. We first aim to prove a matching lower bound on this measure.

Indeed, we have that

$$\fint_{\partial B_{r/8}} u - u(0) = c(n) \int_0^{r/8} \frac{\triangle u(B_s)}{s^{n-1}} ds.$$

The left-hand side is at least $C_1 \sqrt{\Lambda} r$, while the right-hand side's integrand is at most $C_2 \sqrt{\Lambda}$ for every s. If for $s > s_1$ we have that $\triangle u(B_s) \leq C_1 \sqrt{\Lambda} s^{n-1}$, we would have that the right-hand side is at most

$$C_1 \sqrt{\Lambda} (\frac{r}{8} - s_1) + C_2 \sqrt{\Lambda} s_1,$$

which, if $s_1 < \frac{r}{8} \frac{C_1}{C_2 - C_1}$, gives a contradiction:

$$C_1 \sqrt{\Lambda} \frac{r}{8} + [C_2 - C_1] \sqrt{\Lambda} s_1 < \frac{r}{4} \sqrt{\Lambda} C_1,$$

smaller than the left-hand side. Hence there is an $s \geq cr$ such that

$$\triangle u(B_{r/8}) \geq \triangle u(B_s) \geq C \sqrt{\Lambda} s^{n-1} \geq C \sqrt{\Lambda} r^{n-1}.$$

We may now conclude via a standard geometric measure theory argument. Let $N_\delta = \{x \in B_{r/16} : d(x, \partial\{u > 0\}) \leq d\}$. We will show that

$$|N_\delta| \leq C r^{n-1} \delta.$$

The reader may then check that this implies the Hausdorff measure estimate as stated.

Let $\mathcal{U}$ be the set of balls of radius 2δ centered at points in $E := B_{r/16} \cap \partial\{u > 0\}$. This is an open cover of N_δ, and we may find a finite subcover $\mathcal{U}'$ of balls with centers x_i with the property that $|x_i - x_j| \geq \delta$: to do so, continue selecting points in E which are at least δ away from any already selected points; this process will always yield a finite set of points $\{x_i\}$, and terminate when every point x in E has $|x - x_i| < \delta$ for one of the x_i. Then as every point x in N_δ is within δ of E, it has $|x - x_i| < 2\delta$ for some x_i, and so is within one of the balls in $\mathcal{U}'$.

Now the argument hinges on counting the number of balls in $\mathcal{U}'$, which we do with the aid of our estimate on the Laplacian measure. Indeed, we know that

$$\sum_i \triangle u(B_\delta(x_i)) = \triangle u(\cup_i B_\delta(x_i)) \leq \triangle u(B_{r/8}) \leq C \sqrt{\Lambda} r^{n-1}.$$

On the other hand, our estimates on the Laplacian can be applied at every point x_i and radius 8δ, to give that

$$\triangle u(B_\delta(x_i)) \geq C\sqrt{\Lambda}\delta^{n-1}.$$

Combining the two gives

$$\#(\mathcal{U}') \leq C(\frac{r}{\delta})^{n-1}.$$

This lets us estimate

$$|N_\delta| \leq |\cup_i B_{2\delta}(x_i)| \leq \#(\mathcal{U}')|B_{2\delta}| \leq C\delta r^{n-1},$$

as promised.

Finally, to see that $\{u > 0\}$ is a set of locally finite perimeter, we recall a theorem of Federer: if a set E has $\mathcal{H}^{n-1}(\partial E) < \infty$, then it has finite perimeter. □

Worth noting about the last point is that we have actually shown more: while it is true that any set E with $\mathcal{H}^{n-1}(\partial E) < \infty$ has finite perimeter, it need not generally have the property that ∂E is the support of the Gauss-Green measure of E, nor that $\partial\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$. In our case, the set will, in fact, enjoy both of these properties, due to the density estimates on the interior and exterior which we have shown. We will let the more geometric measure theory-inclined readers verify this for themselves, but further discussion may be found in [14].

The approach here is not the only one which can be used to show the Hausdorff measure estimates, but it is fairly robust and generic, and the conclusion it offers is the strongest possible. It is very similar to the original argument of Alt and Caffarelli [1].

6 Blow-Ups and Convergence

To understand finer properties of minimizers, we will need to go beyond the basic estimates above. The idea will be, roughly, to zoom in at points of the free boundary, then to classify the possible tangent objects obtained in this way, and then to show that the minimizer looks, locally, like that tangent object. The starting point is this:

Remark 6.1. *Let u be a local minimizer on U. Then the rescaling $v(x) = \frac{u(rx)}{r}$ is a local minimizer on U/r.*

Theorem 6.2. *Let $\{u_k\}$ be a sequence of local minimizers on U, with $E(u_k; U)$ uniformly bounded. Then:*

1. *There is a subsequence u_k which converges to a function u_∞ on U strongly in $H^1_{loc}(U)$ and locally uniformly.*
2. *Assume that $u_k \to u_\infty$ as above. Then $\partial\{u_k > 0\}$ converges to $\partial\{u_\infty > 0\}$ in the sense that:*
   (a) *For every sequence $x_k \in \partial\{u_k > 0\}$, if $x = \lim x_k$, then $x \in \{u_\infty > 0\}$.*
   (b) *For every $x \in \partial\{u_\infty > 0\}$ there exists a sequence $x_k \in \partial\{u_k > 0\}$ such that $x = \lim x_k$.*
3. *Assume that $u_k \to u_\infty$ as above. Then u_∞ is a local minimizer.*

Some remarks: the compactness is not surprising in light of our Lipschitz estimate from earlier, except possibly the strong convergence in H^1; this should be thought of as a corollary of (3). On the other hand, (3) is a generic property of well-behaved functionals: sequences of minimizers will converge to minimizers, for similar reasons to why minimizers can be found in the first place. Finally, (2) is a consequence of the density estimates in 5.1; this kind of convergence is close to what is known as Hausdorff convergence (of closed sets). There are various technical points surrounding localizing Hausdorff convergence which we do not pursue here, simply sticking with the notion above.

Proof. Let $\{u_k\}$ be a sequence of local minimizers on U. As all of the statements are local, we may as well assume that U is bounded. We may clearly extract a subsequence $u_k \to u$ weakly in $H^1(U)$, strongly in $L^2(U)$, and locally uniformly (the latter due to the Lipschitz bound of 3.2 CHECK BETTER LEMMA) for some $U \in H^1(U)$. We will now show (3) as well as the final part of (1).

Let $v \in H^1(U)$ with $v = u$ outside of $V \subset\subset U$. Let η be a smooth cutoff function which is supported on U and is 1 on V. Let $v_k = \eta v + (1-\eta)u_k$. This is a valid competitor for u_k, and using it as such gives

$$E(u_k; U) \leq E(v_k; U),$$

which implies

$$\int_U |\nabla u_k|^2 + \Lambda|\{u_k > 0\}| \leq \int_U |\nabla v_k|^2 + \Lambda|\{v_k > 0\}|.$$

Expanding out $|\nabla v_k|^2$ gives the following terms:

$$|\nabla v_k|^2 = |\nabla u_k|^2(1-\eta)^2 + |\nabla v|^2\eta^2 + 2\nabla u_k \cdot \nabla v\eta(1-\eta) + |\nabla\eta|^2(u_k - v)^2 + \nabla\eta(u_k - v) \cdot [\nabla v\eta + \nabla u_k(1-\eta)].$$

To understand what happens to them, we will leave the first three alone for the time being, and integrate the others over U and take the limit as $k \to \infty$:

$$\limsup \int_U |\nabla\eta|^2(u_k - v)^2 + |\nabla\eta||u_k - v|(|\nabla v|\eta + |\nabla u_k|(1-\eta|)) = 0.$$

Indeed, ∇eta is bounded and $(u_k - v)^2 \to (u-v)^2$ in L^1, which is supported on V (away from $\operatorname{supp}|\nabla\eta|$), eliminating the first term. The second works the same way, noting that $|\nabla\eta|$ and $|u-v|$ have disjoint support, while the remaining factors are bounded in L^2. Applying this,

$$\limsup \int_U |\nabla u_k|^2\eta(2-\eta) + \Lambda|\{u_k > 0\} \cap \operatorname{supp}\eta| \leq \liminf \int_U |\nabla v|^2\eta^2 + 2\nabla u_k \cdot \nabla v\eta(1-\eta) + |\{v_k > 0\} \cap \operatorname{supp}\eta|.$$

Now, the second term on the right converges to $\int_U 2\nabla u \cdot \nabla v\eta(1-\eta)$, from weak convergence of $\nabla u_k \to \nabla u$. For the last term on the right, observe that the contribution from V is just $|\{v > 0\}|$, which is independent of k. All of this gives

$$\begin{aligned} E(u;V) &\leq \limsup E(u_k;V) \\ &\leq \limsup \int_U |\nabla u_k|^2\eta(2-\eta) + \Lambda|\{u_k > 0\} \cap \operatorname{supp}\eta| \\ &\leq \int_U |\nabla v|^2\eta^2 + 2\nabla u \cdot \nabla v\eta(1-\eta) + |\{v > 0\} \cap V| + |\operatorname{supp}\eta \setminus V|. \end{aligned}$$

The first inequality used the lower semicontinuity of E. Notice that there are no derivatives of η to be found anywhere in this expression; we may therefore take $\eta \to 1_V$, recovering

$$E(u;V) \leq E(v;V).$$

In particular, this implies (3). We also have (using a middle piece of the above inequality) that

$$\limsup E(u_k;V) \leq E(v;V);$$

taking $u = v$ in this gives that $\lim E(u_k;V) = E(u;V)$. As each term in $E(\cdot, V)$ is lower semicontinuous, this means that

$$\lim \int_V |\nabla u_k|^2 = \int_V |\nabla u|^2,$$

implying that $u_k \to u$ strongly in $H^1(V)$. This proves the remainder of (1).

To see (2), we must check several statements. First, let $x \in \partial\{u > 0\}$, and $B_\delta(x)$ a small ball around x. For every k large, we must have that $B_\delta(x)$ has at least one point z_k with $u_k(z_k) > 0$, for otherwise $u_k \equiv 0$

on B_δ and so $u \equiv 0$ on B_δ. We must also have a point y_k with $u_k(y_k) = 0$, for otherwise u_k is harmonic on B_δ, and a uniform limit of harmonic functions is harmonic. Hence there must be points $x_k \in \partial\{u_k > 0\} \cap B_\delta$ for every δ.

On the other hand, take a sequence of points $x_k \to x \; inU$, with $x_k \in \partial\{u_k > 0\}$. We have that for every δ, there is a point x_k^δ in $\partial B_\delta(x_k)$ with $u_k(x_k^\delta) > c\delta_k$, from Theorem 4.1. Hence, letting $x^\delta = \lim x_k^\delta$, we have $u(x_\delta) > 0$ from the uniform convergence. As we also have $u(x) = \lim u_k(x_k) = 0$, this gives that $x \in \partial\{u > 0\}$. □

7 The Free Boundary Condition

We have come rather far without actually attempting to derive any equation which the free boundary $\partial\{u > 0\}$ satisfies in this Bernoulli problem. This is intentional, as the condition is difficult to state in a way which can both be justified rigorously and is useful for any analysis of the free boundary. We aim to correct this here, though. We will go through several ways of understanding the free boundary condition, as well as give a few simple applications. In the subsequent sections, we will (with some effort) show that most of the free boundary is actually given by smooth graphs, where the condition is easy to state and plays a useful role in any further development.

To understand where the free boundary condition comes from, observe that whenever one minimizes a functional $\mathcal{F} : X \to \mathbb{R}$ (with u being a minimizer), it is possible to obtain a large number of differential relations satisfied by u. Indeed, consider any curve $\phi : (-1, 1) \to X$ with $\phi(0) = u$; then if the map $\mathcal{F} \circ \phi : (-1, 1) \to \mathbb{R}$ is differentiable at 0, its derivative must be 0. There is, of course, no reason for it to be differentiable in this manner, so it is important to construct the curves intelligently to maximize the chances of this. We have already done this for some families of curves: the fact that u is harmonic where $u > 0$, and subharmonic everywhere, is based on this kind of argument, where $u_t = u + t\phi$ with ϕ an appropriate test function. However, this is not the full extent of information available: any such curve leaves the free boundary fixed, modifying only u itself away from $\partial\{u > 0\}$. In this section we construct curves which deform $\partial\{u > 0\}$ instead, and obtain information there.

Let u be a local minimizer on U below, and $T \in C_c^\infty(U; \mathbb{R}^n)$ be a vector field. We may construct a one-parameter family of maps $\phi_t(x) = x + tT(x)$, noting that at least when t is small, $\nabla\phi_t = I + t\nabla T$ is invertible, and so ϕ_t is a diffeomorphism which is equal to the identity outside of a compact subset of U. The goal is to use $u_t = u \circ \phi_t^{-1}$ as competitors for u.

To that end, let us compute

$$\nabla u_t = \nabla\phi_t^{-1} \nabla u \circ \phi_t^{-1} = (I - t\nabla T \circ \phi_t^{-1} + O(t^2))\nabla u \circ \phi_t^{-1},$$

so

$$|\nabla u_t|^2 = (|\nabla u|^2 - 2t\nabla u \cdot \nabla T \nabla u) \circ \phi_t^{-1} + O(t^2).$$

The $O(t^2)$ terms here involve $|\nabla u|^2$ multiplied by a function of the entries in ∇T; they do not involve higher derivatives of either T or u.

We will also encounter the Jacobian factor

$$|\det \nabla\phi_t| = 1 + t\mathrm{Tr}(\nabla T) + O(t^2) = 1 + t \operatorname{div} T + O(t^2).$$

This expansion is purely a linear algebra fact, and can be checked either from the definition of the determinant or from identities for exponentials of matrices.

Then

$$\begin{aligned}E(u_t;U) &= \int_U |\nabla u_t|^2 + \Lambda 1_{\{u_t>0\}}\\ &= \int_U (|\nabla u|^2 - 2t\nabla u\cdot\nabla T\nabla u)\circ\phi_t^{-1} + \Lambda 1_{\{u_t>0\}} + O(t^2)\\ &= \int_U (|\nabla u|^2 - 2t\nabla u\cdot\nabla T\nabla u + \Lambda 1_{\{u>0\}})(1+t\operatorname{div}T) + O(t^2)\\ &= \int_U |\nabla u|^2 + \Lambda 1_{\{u>0\}} + t\int_U |\nabla u|^2\operatorname{div}T - 2\nabla u\cdot\nabla T\nabla u + \Lambda\operatorname{div}T 1_{\{u>0\}} + O(t^2).\end{aligned}$$

We know that this should exceed $E(u;U)$, and that is valid for every sufficiently small t. Hence

$$0 \le tc + O(t^2);$$

taking t either positive or negative and sending it to 0 gives

$$\int_U |\nabla u|^2\operatorname{div}T - 2\nabla u\cdot\nabla T\nabla u + \Lambda\operatorname{div}T 1_{\{u>0\}} = 0. \tag{7.5}$$

This is the stationarity condition promised. Unfortunately it is expressed in a rather incomprehensible form like this, and is difficult to understand.

We do wish, however, to actually understand it. While all of the previous theory could be interpreted in terms of scaling considerations for the energy and so forth, the same cannot be said of any of the subsequent results; they hinge on understanding the free boundary condition that (7.5) supposedly represents. To that end, we begin with an identity which is attributed to Rellich (though sometimes also to Pohozaev, depending on the context). For a harmonic function u and a vector field T,

$$\begin{aligned}\operatorname{div}(|\nabla u|^2 T - 2\nabla u\cdot T\nabla u) &= |\nabla u|^2\operatorname{div}T + 2\nabla u\cdot D^2uT - 2T\cdot D^2u\nabla u - 2\nabla u\cdot T\triangle u - 2\nabla u\cdot\nabla T\nabla u\\ &= |\nabla u|^2\operatorname{div}T - 2\nabla u\cdot\nabla T\nabla u.\end{aligned}$$

Thus we may rewrite (7.5) as

$$\int_{\{u>0\}} \operatorname{div}(|\nabla u|^2 T - 2\nabla u\cdot T\nabla u + \Lambda T) = 0.$$

One would at this point hope to apply the divergence theorem. This is possible in general, using that $\{u>0\}$ is a set of finite perimeter and that $|\nabla u|$ is bounded, but it is not particularly trivial and we do not pursue the point here (see [8] the relevant result). Let us instead just assume for the moment that $\partial\{u>0\}$ is smooth, and that ν represents the outer unit normal to $\{u>0\}$. Then

$$\int_{\partial\{u>0\}} |\nabla u|^2 T\cdot\nu - 2\nabla u\cdot T\nabla u\cdot\nu + \Lambda T\cdot\nu = 0.$$

Now, $u=0$ along $\partial\{u>0\}$, so $\nabla u = -\nu|\nabla u|$ there. This gives

$$\int_{\partial\{u>0\}} -|\nabla u|^2 T\cdot\nu + \Lambda T\cdot\nu = 0;$$

by letting T converge to a delta function (relative to surface measure) times ν, this implies that

$$|\nabla u|^2 = \Lambda \text{ on } \partial\{u>0\}. \tag{7.6}$$

This is the correct way of thinking about the free boundary condition.

To summarize the above discussion,

Theorem 7.1. *Let u be a local minimizer on U. Then for every $T \in C_c^\infty(U)$, (7.5) holds. Assume in addition that $\partial\{u > 0\}$ is $C^{1,\alpha}$. Then (7.6) holds.*

Let us obtain some corollaries, with an aim of finding some intermediate forms of the free boundary condition between the hard to justify (7.6) and the confusing (7.5).

Corollary 7.2. *Let u be a local minimizer on B_1, and $0 \in \partial^*\{u > 0\}$, where ν is the measure-theoretic outward unit normal to $\{u > 0\}$ at 0. Let*

$$u_r(x) = \frac{u(rx)}{r}.$$

Then u_r converge in the sense of Theorem 6.2 to $u_0 = \sqrt{\Lambda}(x \cdot \nu)_-$.

Recall that $\partial^*\{u > 0\}$ stands for the reduced boundary, i.e. those points x in $\partial\{u > 0\}$ for which the rescalings $(\{u > 0\} - x)/r$ converge to the half-space $x \cdot \nu < 0$ locally in L^1. Theorems of De Giorgi and Federer guarantee that for a set of finite perimeter with density bounds like in Theorem 5.1, $\mathcal{H}^{n-1}(\partial\{u > 0\} \setminus \partial^*\{u > 0\}) = 0$.

The convergence in this theorem is along the entire sequence, as the limit is uniquely determined.

Proof. We know that $u_r \to u_0$, along subsequences, for some entire local minimizer u_0, by applying Theorem 6.2 and the Lipschitz estimate: $\nabla u_r| \leq C$ on $B_{\frac{1}{2r}}$, so $E(u_r; B_S) \leq CS^n$ for r small enough. Note that this also gives that $|\nabla u_0| \leq C\sqrt{\Lambda}$.

By assumption, we have that $\{u_r > 0\} \to \{x \cdot \nu < 0\}$ locally in L^1. From Theorem 5.1, this implies that if $x_k \to x$ and $x_k \in \partial\{u_k > 0\}$, then $x \in H := \{x \cdot \nu = 0\}$. Indeed, if this was not the case we would have that

$$c_D < \frac{|B_{d(x_k,H)/2}(x_k) \cap \{u_r > 0\}|}{|B_{d(x_k,H)/2}} < 1 - c_D;$$

if the radius of the ball remains bounded from below in r, this contradicts the convergence in L^1. Together with Theorem 6.2, (2), we have shown that $\partial\{u_0 > 0\} = H$.

In particular, $\partial\{u_0 > 0\}$ is smooth, so Theorem 7.1 applies to give that $(u_0)_\nu = -\sqrt{\Lambda}$ along H. Let $\bar{u}$ be the odd reflection of u_0 across H; we then have that $\bar{u}$ is harmonic (the odd reflection of any harmonic function on a half-space which vanishes on the boundary is harmonic). From the estimate on ∇u_0, we have from Liouville's theorem that $\bar{u}$ is a linear function. From the free boundary condition, we have that $u_0 = \sqrt{\Lambda}(x \cdot \nu)_-$, as promised. ☐

Corollary 7.3. *Let u be a local minimizer on U. Then*

$$\triangle u = \sqrt{\Lambda}\mathcal{H}^{n-1} \llcorner \partial\{u > 0\}.$$

This may be interpreted as a weak form of (7.6): any nonnegative function satisfying (7.6) and which is harmonic where positive will satisfy the conclusion of this corollary, and conversely if $\partial\{u > 0\}$ is smooth enough.

Proof. From Lemma 3.1 and Theorem 4.1, we have that for any $x \in \partial\{u > 0\}$,

$$\triangle u(B_r(x)) \leq C\sqrt{\Lambda}r^{n-1}.$$

Along with the fact that $\operatorname{supp} \triangle u \subseteq \partial\{u > 0\}$, this implies that $\triangle u$ is absolutely continuous with respect to $\mathcal{H}^{n-1}_{\llcorner}\partial\{u > 0\}$. Applying the Radon-Nikodym theorem lets us write

$$\triangle u = g d\mathcal{H}^{n-1} \llcorner \partial\{u > 0\},$$

where

$$g(x) = \lim_{r \searrow 0} \frac{\triangle u(B_r(x))}{\mathcal{H}^{n-1}(B_r(x) \cap \partial\{u > 0\})}$$

at the $\mathcal{H}^{n-1}$-a.e. set of points at which the limit exists. Our task is to compute this limit.

Note that it suffices to compute $g(x)$ on $\partial^*\{u > 0\}$, the remaining points being $\mathcal{H}^{n-1}$ negligible. At these points, we have from standard measure theory (see [14]) that

$$\lim_r \frac{\mathcal{H}^{n-1}(B_r(x) \cap \partial\{u > 0\}}{r^{n-1}\mathcal{H}^{n-1}(B_1(x) \cap \{x_n = 0\}} = 1.$$

On the other hand,

$$\frac{\triangle u(B_r(x))}{r^{n-1}} = \triangle u_r(B_1) \to \triangle u_0(B_1),$$

where u_0 is as in Corollary 7.2. This can be computed explicitly to be $\sqrt{\Lambda}\mathcal{H}^{n-1}(B_1(x) \cap \{x_n = 0\}$, so

$$g(x) = \sqrt{\Lambda}.$$

□

The following is a different weak way of understanding (7.6), usually referred to as the *viscosity* sense. The idea is that at points where $\{u > 0\}$ is regular from one side, the free boundary condition gives one-sided control on the derivative of u. This is somewhat awkward to state, but the form below will be convenient to work with. Notice that the existence of the ϕ below implies one-sided regularity of $\partial\{u > 0\}$, at least if $\nabla\phi(0) \neq 0$.

Theorem 7.4. *Let u be a local minimizer on B_1, and $0 \in \partial\{u > 0\}$. Let ϕ be a smooth function with $\phi(0) = 0$.*

1. *If $u \geq \phi_+$ on B_1, then $|\nabla\phi(0)| \leq \sqrt{\Lambda}$.*
2. *If $u \leq \phi_+$ on B_1, then $|\nabla\phi(0)| \geq \sqrt{\Lambda}$.*

Proof. Let $u_r(x) = \frac{u(rx)}{r}$ and $\phi_r(x) = \frac{\phi(rx)}{r}$, and consider the blow-up limits $u_r \to u_0$ (along subsequences, in the sense of Theorem 6.2) and $\phi_r \to \phi_0$. As ϕ is smooth, we easily see that $\phi_0(x) = \nabla\phi(0) \cdot x$. Choose coordinates so that $\nabla\phi(0) = |\nabla\phi(0)|e_n$.

Applying Lemma 7.5 below, we have that $u_0(x) = \alpha x_n + o(|x|)$ for some α on $\{x_n > 0\}$; clearly in case (1) $\alpha \geq |\nabla\phi(0)|$ while in case (2) $\alpha \leq |\nabla\phi(0)|$. Now take another blow-up of u_0, obtaining $v = \lim_{r\searrow 0} u_0(\cdot r)/r$, along a subsequence. From Theorem 6.2, this is an entire local minimizer, while from the asymptotics for u_0, we have that

$$v(x) = \alpha x_n$$

on $\{x_n > 0\}$, and in particular vanishes on the hyperplane $P = \{x_n = 0\}$.

If C is the Lipschitz constant for v, then $v \leq -Cx_n$ on $\{x_n < 0\}$. We may then apply Lemma 7.5 to $v1_{\{x_n<0\}}$ on B_1 to obtain that $v = -\beta x_n + o(|x|)$ on $\{x_n < 0\}$. Note that if $\beta \neq 0$, we have that $|\{v = 0\} \cap B_r| = 0$, which is a contradiction. We then perform a blow-up at the origin once again, for v, obtaining an entire local minimizer w with

$$w(x) = \alpha(x_n)_+.$$

It follows from Theorem 7.1 that $\alpha = \sqrt{\Lambda}$, which implies the claimed estimates. □

The repeated blow-ups were just to simplify the proof, and may be easily avoided. The key point here was the following lemma, which essentially states that a (Lipschitz) harmonic function on a domain has linear asymptotic behavior at points on the boundary which have either interior or exterior tangents.

Lemma 7.5. *Let u be a nonnegative continuous function with $|\nabla u| \leq C$ on B_1, and the property that u is harmonic where positive. Assume that $u(0) = 0$. If $u \leq (x_n)_+$ or $u \geq (x_n)_+$, then*

$$u(x) = \alpha x_n + o(|x|)$$

on $\{x_n > 0\}$, for some $\alpha \in [0, \infty)$.

We sketch the proof below, as it is not so easy to find in the literature. It follows Section 11.6 of [3], but the statements there are not quite precise.

Proof. First assume $u \geq (x_n)_+$. Let

$$\alpha(R) = \sup\{\alpha > 0 : u \geq \alpha(x_n)_+ \text{ on } B_R\}.$$

This quantity is decreasing in R, and is always in $[1, C]$ from our assumptions. Set $\alpha = \sup \alpha(R) = \lim_{R \to 0} \alpha(R)$, noting that

$$u \geq \alpha(|x|)(x_n)_+ \geq \alpha(x_n)_+ - [\alpha(|x|) - \alpha]|x| = \alpha(x_n)_+ + o(|x|).$$

Fix $\beta > 0$. Assume that there is a $\delta > 0$ and a sequence of points $x_k \to 0$ such that $(x_k)_n \geq \beta|x_k'|$ and

$$u(x_k) > (\alpha + \delta)((x_k)_n)_+.$$

We have that for each τ,

$$u(z) \geq (\alpha - \tau\delta)z_+$$

for every sufficiently small z (depending on τ). Set

$$v(z) = u(z) - (\alpha - \tau\delta)z_+ > 0$$

on $\{x_n > 0\}$, and notice that this is a superharmonic (indeed, harmonic) function. We have that

$$v \geq v(x_k) - C\kappa|x_k| \geq c\delta(x_k)_n$$

on $B_{\kappa(x_k)_n}(x_k)$ if $\kappa = \kappa(\delta)$ is small, using that $|\nabla v| \leq C$ [as a remark, we could have used the Harnack inequality instead]. In particular, $u \geq c\delta|x_k|$ on a positive-measure subset of $\partial B_{|x_k|}$. Letting h be the harmonic function on $B_{|x_k|}$ with boundary data on $\partial B_{|x_k|}$ given by

$$h(x) = \begin{cases} u(x) & x_n > 0 \\ u(-x) & x_n < 0 \end{cases}$$

and applying the Poisson kernel representation to h easily shows that

$$\partial_n h(y) = |\nabla h(0)| - |D^2 h| O(|x|^2) \geq c(\delta)$$

for $|x| \leq C(\delta, k)$, as $|D^2 h| \leq C/|x_k|$ on this set by standard estimates.

From comparison

$$v \geq c(\delta)(x_n)_+$$

there, so

$$u \geq (\alpha + \delta(c(\delta) - \tau))(x_n)_+.$$

Choose $\tau \ll c(\delta)$, and then k so the above holds, and then $|x|$ so that

$$u \geq (\alpha + \delta\frac{c(\delta)}{2})(x_n)_+$$

on $|x| < C(\delta, k)$. This, however, implies that $\alpha(R) \geq \alpha + c\delta/2$ for small R, which is a contradiction.

To summarize, we have shown that for every β and δ, there is an $r(\beta, \delta)$ such that on $B_{r(\beta,\delta)} \cap \{x_n \geq \beta|x'|\}$ we have

$$u(x) \leq (\alpha + \delta)(x_n)_+.$$

So for every integer k, there is an $r(k)$ such that on $B_{r(k)} \cap \{kx_n \geq |x'|\}$,

$$u(x) \leq (\alpha + \frac{1}{k})(x_n)_+.$$

Along the conical boundary $T = B_R \cap \{kx_n = |x'|\}$, we have that $x_n \le 1/k$, so

$$u(x) \le (\alpha + 1)\frac{R}{k}.$$

On the tangential region $B_R \cap \{0 < kx_n < |x'|\}$, then, the gradient bound on u tells us that

$$u(x) \le u(y) + \frac{CR}{k} \le \frac{CR}{k},$$

where y is the closest point on T to x, which is a distance of at most $\frac{1}{k}$. Hence

$$u(x) \le \alpha(x_n)_+ + C\frac{|x|}{k} = \alpha(x_n)_+ + o(|x|),$$

as promised.

Now consider the opposite case, where $u \le (x_n)_+$. The proof is essentially the same; we just wish to highlight one important point. In this case set

$$\alpha(R) = \inf\{\alpha > 0 : u \le \alpha(x_n)_+ \text{ on } B_R\},$$

which are similarly increasing, and $\alpha = \alpha(0+)$. Then

$$u \le \alpha(x_n)_+ + o(|x|)$$

like before. For the other direction, we similarly take a sequence of points x_k in the same region for which

$$u(x_k) \le (\alpha - \delta)((x_k)_n)_+,$$

and then set

$$v = (\alpha + \tau)(x_n)_+ - u.$$

The main difference is that here, we only know that u is subharmonic on $\{x_n > 0\}$, so v is superharmonic. However, the argument above only used that v is superhamonic, and this is the only place the equation was used at all, so the proof goes through as before (with appropriate modifications to the signs). □

A couple of remarks: the first part, where $u \ge (x_n)_+$, did not require the Lipschitz continuity of u if the Harnack inequality was used where noted, but the last part concerning tangential regions would then not follow. The second part does require the Lipschitz continuity, and one may construct counterexamples otherwise, at least if the $o(|x|)$ is intended in the usual L^∞ sense as here. Readers familiar with the Alt-Caffarelli-Friedman monotonicity formula (or even the related simpler formula for harmonic functions on a half-space with homogeneous Dirichlet boundary conditions) should be able to give a simpler proof of the second part, though that argument is harder to generalize.

8 Flat Implies Smooth: The Setup

Over the next several sections, we will prove that $\partial^*\{u > 9\}$ is composed of a union of $C^{1,\alpha}$ graphs. More precisely, we deal with the following objects:

Definition 8.1. *Let $u \in C^{0,1}_{loc}(B_1)$ be a nonnegative function. We say that u is a* viscosity solution *if it satisfies the following, where ϕ is any smooth function with $\phi(x) = u(x)$:*

1. *$\triangle u = 0$ on $\{u > 0\}$.*
2. *If $u \ge \phi_+$ on B_1, then $|\nabla\phi(x)| \le 1$.*
3. *If $u \le \phi_+$ on B_1, then $|\nabla\phi(x)| \ge 1$.*

From Theorem 7.4, we know that if u is a local minimizer on B_1, then $u/\sqrt{\Lambda}$ is a viscosity solution. In fact, this is an extremely weak notion compared to minimality. The reader is welcome to assume u is a local minimizer instead below, but the proof will only use the viscosity solution properties of u.

It is worth describing which properties of local minimizers are still valid for viscosity solutions. one may check, without much difficulty, that in fact the assumption that $|\nabla u|$ is bounded is superfluous, and that this follows from the other properties (if u is continuous, say). On the other hand, almost none of the other properties we have established are true, including the lower bound of Theorem 4.1 or the density estimates in Theorem 5.1. For some specific subtypes of viscosity solutions these may be recovered, but this is not trivial and generally not possible. Hence our result should be thought of as consisting of two parts. The first is the following theorem about viscosity solutions:

Theorem 8.1. *Let u be a viscosity solution on B_1. Assume that $|u - x_n| \leq \varepsilon_F$ on $B_1 \cap \{u > 0\}$ for some $\varepsilon_F = \varepsilon_F(n)$ small. Then $\partial\{u > 0\} \cap B_{1/2}$ coincides with a graph $\{(x', g(x'))\}$ for a function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\|g\|_{C^{1,\alpha}(B_{1/2})} \leq C(n)$.*

The proof of this theorem will occupy us for the next three sections. If u is a minimizer, it was first proved in [1]. For viscosity solutions, an alternative proof may be found in the sequence of papers [4, 6, 5]. The proof we present here, however, is more recent and due to Daniela De Silva [10].

At this point, one may also prove higher regularity for the free boundary:

Theorem 8.2. *Let u and g be as in Theorem 8.1. Then g is analytic.*

This theorem is due to Kinderlehrer, Nirenberg, and Spruck [13], and uses a change of variables known as the partial hodograph transform.

At least for minimizers, we may combine these results with Corollary 7.2 to see that it applies to every point of $\partial^*\{u > 0\}$, so in particular at $\mathcal{H}^{n-1}$-a.e. point, at sufficiently small scales after a rotation. We will discuss the remainder $\Sigma = \partial\{u > 0\} \setminus \partial^*\{u > 0\}$ more at the end of these notes.

The proof of Theorem 8.1 consists of first proving a corresponding *improvement of flatness* lemma:

Lemma 8.3. *Fix $\tau > 0$. Let u be a viscosity solution on B_1 with $0 \in \partial\{u > 0\}$ and assume that*

$$|u - x_n| \leq \varepsilon \leq \varepsilon_I \text{ on } \{u > 0\},$$

for some $\varepsilon_I = \varepsilon_I(n, \tau)$. Then there exists a unit vector $e \in S^{n-1}$ with $|e - e_n| \leq C\varepsilon$, for which

$$\sup_{B_\tau \cap \{u>0\}} |u - x\dot{e}| \leq C(n)\tau^2\varepsilon.$$

Note that without allowing $e \neq e_n$ this lemma would be incompatible with the conclusion of the theorem, for when u is smooth the quantity on the right is comparable in size to g, which behaves like τ on B_τ unless $\nabla g(0) = 0$ (and this is not to be expected under the assumptions here).

Remark 8.4. *The statements*

$$(x - \varepsilon)_+ \leq u \leq (x + \varepsilon)_+$$

and

$$|u - x_n| \leq \varepsilon \text{ on } \{u > 0\}$$

are equivalent, and will be used interchangeably below.

The proof of this lemma is a simple argument by contradiction: assume it fails. Then there is a sequence of $\varepsilon_k \to 0$ and u_k viscosity solutions such that

$$\sup_{B_1 \cap \{u_k > 0\}} |u_k - x_n| = \varepsilon_k$$

but

$$\sup_{B_\tau \cap \{u_k>0\}} |u_k - x\cdot e| \geq C(n)\tau^2\varepsilon_k \tag{8.7}$$

for any e in the class permitted. Set

$$v_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k},$$

which we know is bounded by 1 on $\{u_k > 0\}$. Moreover, we have that $\partial\{u_k > 0\} \subseteq \{|x_n| \leq \varepsilon_k\}$, outside of which both u_k and x_n are harmonic (or constant 0). From this and estimates on harmonic functions, we may deduce that

$$v_k \to v$$

locally uniformly on $B_1 \cap \{x_n > 0\}$, for a function v which is harmonic on this domain. What we intend to show is that (from the $\{x_n > 0\}$ side) v extends to a continuous function satisfying a valid boundary condition. This will then imply a contradiction to (8.7), which is essentially asserting that v_k is not close to any C^2 function with C^2 norm bounded by $C(n)$ (details will be given below).

The main parts of the argument, then, are to upgrade the convergence of v_k to v, and to obtain the boundary condition on v. The former we do with an estimate of De Silva in the next section, after which we show the latter using the viscosity free boundary condition on u_k.

To understand the origins of the boundary condition on v, let us derive a simple version of the linearized Bernoulli problem. Indeed, if $u_k = x_n + \varepsilon_k v_k$, assume that $\partial\{u_k > 0\}$ is a smooth graph over $x_n = 0$. Then on $\partial\{u_k > 0\}$, we have

$$1 = |\nabla u_k(x)|^2 = |e_n + \varepsilon_k \nabla v_k(x)|^2 = 1 + 2\varepsilon_k \partial_n v_k(x) + O(\varepsilon_k^2),$$

which gives

$$\partial_n v_k(x) = O(\varepsilon_k).$$

Taking the limit as $k \to \infty$, the free boundary converges to $\{x_n = 0\}$, on which $\partial_n v = 0$. This is, of course, impossible to justify rigorously a priori, but we will recover this Neumann condition in the viscosity sense.

The final step is to show that Lemma 8.3 implies Theorem 8.1. This is completely standard, but we show the details.

9 Flat Implies Smooth: The Estimate

The main estimate is a kind of improvement of oscillation lemma which may be iterated a finite number of times. It should bring to mind Reifenberg's topological disk theorem, but the conclusion will be much weaker.

Lemma 9.1. *There is a constant $\varepsilon_1(n) > 0$ such that if u is a viscosity solution on B_1 with $0 \in \partial\{u > 0\}$ and*

$$(x_n + a)_+ \leq u \leq (x_n + b)_+$$

for some a, b with $b - a = \varepsilon < \varepsilon_1$, then there are a', b' with $a \leq a' \leq b' \leq b$ and $|a' - b'| < (1-\theta)\varepsilon$ for some $\theta = \theta(n) > 0$, such that

$$(x_n + a')_+ \leq u \leq (x_n + b')_+$$

on $B_{1/10}$.

Proof. Note that it suffices to prove the lemma under the slightly stronger assumption that in addition,

$$\partial\{u > 0\} \cap (\{x_n = -a\} \cup \{x_n = -b\}) = \emptyset,$$

by applying it with a, b replaced by $a - \kappa$, $a + \kappa$, and then sending κ to 0. For the same reason, we may assume the strict inequalities

$$(x_n + a)_+ < u(x) < (x_n + b)_+ \text{ if } u(x) > 0.$$

We assume this from here on.

Let $z = (0, \frac{1}{4})$; we have that

$$\frac{1}{4} + a \leq u(z) \leq \frac{1}{4} + b.$$

The proof will have two largely similar cases: either $u(z) \geq \frac{1}{4} + a + \frac{\varepsilon}{2}$, or not. Let us focus on the case when this does hold.

In this case, set

$$w(x) = u(x) - x_n - a,$$

which has $w(z) \geq \frac{\varepsilon}{2}$ and $w > 0$ on $B_1 \cap \{x_n > \frac{1}{20}\}$ (if ε is small; note that $|a|, |b| \leq \varepsilon$ under our assumptions). Noting also that w is harmonic on this set, we learn that

$$\inf_{B_{9/10} \cap \{x_n \geq \frac{1}{10}\}} w > c(n)\varepsilon,$$

so

$$u > x_n + a + c(n)\varepsilon$$

there.

Let $h(x)$ be the harmonic function which is equal to 1 on $\partial B_{1/8}(z)$ and equal to 0 on $\partial B_{1/2}(z)$; extend it by 0 outside this ball. Define

$$\phi_t(x) = (x_n + a + th(x))_+.$$

Let us collect some information about this family of functions. First, from the definition of h, we have that $\phi_t = (x_n + a)_+$ outside of $B_{1/2}(z)$, so certainly on the set where $|x'| = \frac{1}{2}$. On the disk $\{x_n = \frac{1}{10}\} \cap \{|x'| \leq \frac{1}{2}\}$ we have that $h \leq 1$, so $\phi_t \leq x_n + a + t$. We will always assume that $0 \leq t \leq c(n)\varepsilon$, so that $\phi_t < u$ there.

We claim that ϕ_t is subharmonic on $G := \{|x'| \leq \frac{1}{2}\} \times \{|x_n| \leq \frac{1}{10}\}$. Indeed, this is obvious, as th is subharmonic, so $x_n + a + th$ is subharmonic, and the positive part of a subharmonic function is subharmonic. Furthermore, we have that at a point x of $\partial\{\phi_t > 0\}$ where $h > 0$, we have that

$$|\nabla \phi_t(x)|^2 = |e_n + t\nabla h|^2 \geq 1 + 2te_n \cdot \nabla h.$$

on this set, $\nabla h \cdot e > 0$, as h is radial and centered at z, while $x_n < z_n$ here.

Let

$$t_* = \max\{t \in [0, c(n)\varepsilon] : \phi_t \leq u \text{ on } G\}.$$

We claim that $t_* = c(n)\varepsilon$. If this is not the case, then there must be a point $x \in \bar{G} \cap \overline{\{u > 0\}}$ at which $u(x) = \phi_{t_*}(x)$; we also still have that $\phi_{t_*} \leq u$ on G. This immediately implies that $t_* > 0$, for there are no such points for ϕ_0 by our starting assumptions.

Now, x cannot be in the boundary of G. Indeed, on the part of the boundary where $x_n = \frac{1}{10}$, we have already checked that $\phi_t < u$. On the rest of it, $\phi_t = \phi_0$, so if x is located there $t_* = 0$, a contradiction.

We also cannot have $\phi_t(x) > 0$: if this is the case, then as ϕ_t is subharmonic, from the strong maximum principle we have that on $\{\phi_t > 0\} \cap G$, $u = \phi_t$; this contradicts the fact that $u < \phi_t$ on a part of the boundary of G where this is the case.

This leaves only the case that $u(x) = \phi_t(x) = 0$. Note that at such a point, we must have that $h > 0$ (otherwise $t_* = 0$, which is a contradiction), and so ϕ_t is locally the positive part of a smooth function. Applying the definition of viscosity solution, this tells us that

$$|\nabla \phi_t(x)| \leq 1,$$

directly contradicting our earlier computation of $|\nabla \phi_t|$.

We have shown that $\phi_{c(n)\varepsilon} \leq u$. On $B_{1/10} \subset\subset B_{1/2}(z)$ we have that $h \geq c'(n) > 0$, so

$$u \geq (x_n + a + \varepsilon c(n)c'(n))_+.$$

Set $b' = b$, $a' = a + \varepsilon(n)c'(n)$, and $\theta = 1 - c(n)c'(n)$ to conclude.

The remaining case proceeds similarly, using

$$\phi_t = (x_n + b - th)_+$$

instead; the only thing to note is that in this case when computing $|\nabla \phi_t|^2$ one must reabsorb the t^2 term into the t term, possibly choosing $c(n)$ smaller there to make that work. □

When iterated, this gives the following estimate:

Lemma 9.2. *Let u be viscosity solution on B_1 with $0 \in \partial\{u > 0\}$. Then there is an $\varepsilon_2 > 0$ and constants C and $\alpha < 1$, such that if*

$$|u - (x_n)_+| \leq \varepsilon < \varepsilon_2,$$

and

$$v(x) = \frac{u(x) - x_n}{\varepsilon},$$

then

$$|v(x) - v(y)| \leq C|x - y|^\alpha$$

for all $x, y \in B_{1/2} \cap \{u \bar{>} 0\}$ with $|x - y| \geq C\frac{\varepsilon}{\varepsilon_1}$.

Proof. We first show this for a point $x \in \partial\{u > 0\}$ and a point $y \in B_{1/2}$ with $u(y) > 0$.

Notice that we may apply Lemma 9.1 repeatedly on $B_{1/2 \cdot 10^{-k}}(x)$, so long as $4 \cdot 10^k \cdot \varepsilon \leq \varepsilon_1$. At each stage, we obtain a pair a_k, b_k with $b_k - a_k \leq 2\varepsilon(1-\theta)^k$, such that

$$(z_n - x_n + a_k)_+ \leq u(z) \leq (z_n - x_n + b_k)_+$$

on $B_{1/2 \cdot 10^{-k-1}}(x)$. As x is in the free boundary $\partial\{u > 0\}$, $|a_k|, |b_k| \leq 2\varepsilon(1-\theta)^k$. and so this implies that

$$|u(z) - (z_n - x_n)_+| \leq 2\varepsilon(1-\theta)^k.$$

Using this with $z = y \in B_{10^{-k-1}/2}(x) \setminus B_{10^{-k-2}/2}(x)$, we have that $|(z-y)_n - ((z-y)_n)_+| \leq 2\varepsilon(1-\theta)^k$, so

$$|u(y) - y_n + x_n - u(x)| \leq 4\varepsilon(1-\theta)^k \leq 4C\varepsilon|x - y|^\alpha$$

for some small α. This gives

$$|v(x) - v(y)| \leq C|x - y|^\alpha.$$

We succeeded in doing this for as long as

$$\varepsilon_1 \geq \varepsilon 4 \cdot 10^k \geq \frac{\varepsilon}{50}|x - y|^{-1},$$

or $|x - y| \geq C\frac{\varepsilon}{\varepsilon_1}$.

Now take any x and y in $B_{1/2}$ with $|x - y| \geq 10C\frac{\varepsilon}{\varepsilon_1}$, and let $d(x) = d(x, \partial\{u > 0\})$. Let x_* be the point in $\partial\{u > 0\}$ where $d(x)$ is attained. If $|x - y| \leq d(x)/8$, we have from above that

$$|v(z) - v(x')| \leq Cd(x)^\alpha$$

for all $z \in B_{d(x)/4}(x)$. The function v is harmonic on $B_{d(x)/4}$, so

$$\sup_{B_{d(x)/8}(x)} |\nabla v| \leq C\frac{\operatorname{osc}_{B_{d(x)/4}(x)} v}{d(x)} \leq Cd^{\alpha-1}(x).$$

Thus

$$|v(x) - v(y)| \leq Cd(x)^{\alpha-1}|x - y| \leq C|x - y|^\alpha.$$

In the alternative case where $d(x) \leq 8|x - y|$, we may simply do

$$\begin{aligned}|v(x) - v(y)| &\leq |v(x) - v(x_*)| + |v(x_*) - v(y)| \\ &\leq Cd^\alpha(x) + C(|x - y| + d(x))^\alpha \\ &\leq C|x - y|^\alpha,\end{aligned}$$

applying the main estimate twice. □

10 Flat Implies Smooth: The Linearization

The purpose of this section is to give a proof of Lemma 8.3, armed with the estimate in Lemma 9.2. We first review some facts about viscosity harmonic functions and the Neumann problem.

Definition 10.1. *We say that $u \in C^0(U)$ is viscosity harmonic on U if:*

1. *for every $\phi \geq u$ on $V \subset\subset U$ with $\phi \in C^\infty$ and $\phi(x) = u(x)$, we have $\triangle\phi(x) \geq 0$.*
2. *for every $\phi \leq u$ on $V \subset\subset U$ with $\phi \in C^\infty$ and $\phi(x) = u(x)$, we have $\triangle\phi(x) \leq 0$.*

Proposition 10.1. *u is viscosity harmonic on U if and only if:*

1. *for every $\phi > u$ on $V \setminus \{x\} \subset\subset U$ with ϕ a quadratic polynomial and $\phi(x) = u(x)$, we have $\triangle\phi(x) \geq 0$.*
2. *for every $\phi < u$ on $V \setminus \{x\} \subset\subset U$ with ϕ a quadratic polynomial and $\phi(x) = u(x)$, we have $\triangle\phi(x) \leq 0$.*

Proof. The only if part is immediate. Otherwise, given any $\phi \geq u$ on V, write

$$\phi(y) = \phi_2(y) + O(|y-x|^3),$$

where ϕ_2 is a quadratic polynomial. Then for every $\varepsilon > 0$, there exists a $\tau > 0$ such that

$$v_\varepsilon = \phi_2(y) + \varepsilon|y-x|^2$$

is strictly larger than u on $B_\tau(x) \setminus \{x\}$. Applying the assumption gives that $\triangle v_\varepsilon(x) \geq 0$, so

$$\triangle\phi(x) = \triangle\phi_2(x) \geq -\varepsilon.$$

Send $\varepsilon \to 0$ to conclude, and argue similarly for functions touching from below. □

Proposition 10.2. *A function $u \in C^0(U)$ is viscosity harmonic if and only if it is harmonic (meaning, C^2 and has $\triangle u = 0$ pointwise).*

Proof. That a harmonic function is viscosity harmonic is clear: if ϕ touches u from above, then $D^2\phi \geq D^2 u$, and we can take traces.

For the opposite direction, take any $x \in U$ and let h be the harmonic replacement of u on $B_r(x) \subseteq U$. If $\min(h-u) < 0$, then also $m = \min(h - u + \varepsilon(r^2 - |x - \cdot|^2)) < 0$ for small ε. Then $h(z) - m + \varepsilon(r^2 - |x-z|^2)$ touches u from above at a point $y \in B_r(x)$. As u is viscosity harmonic, this gives $\triangle h(y) > 2n\varepsilon > 0$, which is a contradiction. Thus $h \geq u$. Likewise $h \leq u$, using the other inequality, which implies that u is harmonic on $B_r(x)$. □

Definition 10.2. *We say that $u \in C^0(B_1 \cap \{x_n \geq 0\})$ satisfies the Neumann condition in the viscosity sense if:*

1. *for every $\phi \geq u$ with $\phi \in C^\infty$ and $\phi(x) = u(x)$ at x with $x_n = 0$, we have $\partial_n\phi(x) \geq 0$.*
2. *for every $\phi \leq u$ with $\phi \in C^\infty$ and $\phi(x) = u(x)$ at x with $x_n = 0$, we have $\partial_n\phi(x) \leq 0$.*

Lemma 10.3. *Let u be harmonic on $B_1 \cap \{x_n > 0\}$ and satisfy the Neumann condition in the viscosity sense. Let $\bar{u}$ be the even extension of u to B_1:*

$$\bar{u}(x) = \begin{cases} u(x) & x_n \geq 0 \\ u(x', -x_n) & x_n < 0. \end{cases}$$

Then $\bar{u}$ is harmonic on B_1.

Proof. Let ϕ be such that $\phi > \bar{u}$ on $B_1 \setminus \{x\}$ and $\phi(x) = u(x)$, and assume for contradiction that $\triangle u(x) < 0$. This implies that $x_n = 0$, as $\bar{u}$ is harmonic away from $\{x_n = 0\}$. Let $\psi(x) = \frac{\phi(x', x_n) + \phi(x', -x_n)}{2}$; then ψ has the same properties as ϕ and

$$\triangle \psi(x) = \triangle \phi(x) < 0.$$

We may assume that $\triangle \psi < 0$ on $B_\tau(x)$, and $\psi > u + \omega$ on $\partial B_\tau(x)$, for some small τ. Let

$$\psi_\varepsilon(z) = \psi(z) - \varepsilon z_n - \min_{B_\tau(x)} (\psi - \varepsilon z_n - u).$$

Then $\psi_\varepsilon \geq u$, and $\psi_\varepsilon(y) = u(y)$ for some $y \in \bar{B}_\tau(x)$. If $\varepsilon < \omega$, then $x \in B_\tau(x)$, for otherwise $\min B_\tau(x)(\psi - \varepsilon z_n - u) \leq 0$, meaning $\psi_\varepsilon > \psi - \varepsilon\tau > \psi - \omega > u$ on ∂B_τ. We cannot have $y_n \neq 0$, as $\triangle \psi_e(y) = \triangle \psi(y) < 0$ there.

Now apply the viscosity Neumann property to ψ_ε and u, to obtain that

$$\partial_n \psi(y) - \varepsilon = \partial_n \psi_\varepsilon(y) \geq 0.$$

But $\partial_n \psi(y) = 0$ by symmetry, which contradicts the above.

A similar argument works when touching from below. □

Proof of Lemma 8.3. We argue by contradiction, using the notation introduced after the statement of Lemma 8.3, with

$$v_k = \frac{u_k - x_n}{\varepsilon_k}.$$

We have that $U_k := \{u_k > 0\}$ contains $\{x_n > \varepsilon_k\}$, and v_k is harmonic on this set. Extract a subsequence of u_k such that $v_k \to v$ locally uniformly on $U := B_1 \cap \{x_n < 0\}$; this is possible as $|v_k| \leq 1$ by assumption and standard estimates on harmonic functions. The function v retains the estimate $|v| \leq 1$ and is harmonic on U.

Applying Lemma 9.2 to u_k, we deduce that on $B_{1/2} \cap \bar{U}_k$,

$$|v_k(x) - v_k(y)| \leq C(|x - y| + \varepsilon_k)^\alpha$$

(the reader is invited to check that this is equivalent to the conclusion of that lemma). Hence for any pair of points $x, y \in U \cap B_{1/2}$, passing to the limit gives

$$|v(x) - v(y)| \leq C|x - y|^\alpha,$$

and so in particular v extends to a continuous function on $\bar{U} \cap B_{1/2}$. We may also upgrade the convergence to the following more uniform sense: for every $\delta > 0$ there exists a K such that if $k > K$, $x \in \bar{U}_k \cap B_{1/2}$, and $y \in \bar{U} \cap B_{1/2}$, if $|x - y| \leq K^{-1}$, then $|v_k(x) - v(y)| \leq \delta$. Recalling that $v_k(0) = 0$ and $0 \in \bar{U}_k$, this implies that $v(0) = 0$.

We also know that

$$\sup_{B_\tau \cap \{u > 0\}} \frac{|u_k(x) - x \cdot e|}{\varepsilon_k} \geq C(n)\tau^2$$

for all unit vectors e with $|e - e_n| \leq C\varepsilon_k$. Assume for a moment that $v \in C^2(B_{1/4} \cap \bar{U})$, and that $\partial_n v(0) = 0$. Then

$$|v(x) - \nabla v(0) \cdot x| \leq [v]_{C^2} \tau^2$$

on B_τ, and so

$$\sup_{B_\tau \cap \bar{U}_k} |v_k(x) - \nabla v(0) \cdot x| \leq o_k(1) + [v]_{C^2}\tau^2 \leq 2[v]_{C^2}\tau^2$$

for large k. On the left, we may rewrite

$$|v_k(x) - \nabla v(0) \cdot x| = |\frac{u_k(x) - (e_n + \varepsilon_k \nabla v(0)) \cdot x}{\varepsilon_k}|.$$

Set $e_k = \frac{e_n + \varepsilon_k \nabla v(0)}{|e_n + \varepsilon_k \nabla v(0)|}$. We have that

$$|e_n + \varepsilon_k \nabla v(0)|^2 = 1 + 2\varepsilon_k e_n \cdot \nabla v(0) + \varepsilon_k^2 |\nabla v(0)|^2,$$

noting that, importantly, the middle term vanishes from our assumption. This gives

$$|e_n + \varepsilon_k \nabla v(0) - e_k| \leq C|\nabla v(0)|^2 \varepsilon_k^2$$

and

$$|e_n - e_k| \leq C|\nabla v(0)|\varepsilon_k,$$

so

$$|\frac{u_k(x) - e_k \cdot x}{\varepsilon_k}| \leq 3[v]_{C^2} \tau^2$$

for large k. This is a contradiction if $[v]_{C^2}, |\nabla v(0)| \leq C(n)$, which is what we will now attempt to show.

We claim that v has the following property: let ϕ be a smooth function with $\phi \leq v$ on $\bar{U} \cap B_{1/2}$ with equality only at $x \in B_{1/2}$ with $x_n = 0$. Then $\partial_n \phi(x) \leq 0$.

Assume this is not the case. Then we may as well assume that

$$\phi(y) = a + cy_n + \nu' \cdot (y' - x') + (y - x)A(y - x),$$

where $c > 0$ and A is a matrix with a strictly positive trace, on $B_\tau \cap \bar{U}$, while $\phi < u - \omega$ on $\partial B_\tau(x) \cap \bar{U}$. This may be arranged by finding a function of this type below the original ϕ. Up to taking τ smaller, we may also assume that $\partial_n \phi \geq c/2$ on B_τ.

We have that

$$v_k \geq \phi - o_k(1)$$

on $\bar{U} \cap B_\tau(x)$; writing this out,

$$u_k \geq x_n + \varepsilon_k \phi - \varepsilon_k o_k(1).$$

Now, on $\partial B_\tau(x) \cap \bar{U}_k$, we have that

$$u_k \geq x_n + \varepsilon_k \phi + \varepsilon_k \omega - \varepsilon_k o_k(1),$$

while at some point $x_k \in \bar{U}_k \cap B_\tau(x)$ we have

$$u_k \leq x_n + \varepsilon_k \phi + \varepsilon_k o_k(1).$$

Consider, then, $w^t = (x_n + \varepsilon_k \phi + \varepsilon_k t)_+$. For $t < -\omega/2$, we have $w^t \leq u$ on $B_\tau(x)$, and for $t \leq \omega$ $w^t \leq u$ on $\partial B_\tau(x)$. Yet on the other hand. for some $t < w/2$, $w^t > u$ at a point in $B_\tau(x)$; let t_* be the smallest such t, and let z be a point in $B_\tau(x) \cap \bar{U}_k$ where $w_{t_*}(z) = u_k(z)$.

Note that as ϕ has $\triangle \phi > 0$, $\triangle w_t > 0$ where w_t is positive. This gives a contradiction if $u_k(z) > 0$. On the other hand,

$$|\nabla w_t(z)|^2 = 1 + 2t\varepsilon_k \partial_n \phi(z) + t^2 \varepsilon_k^2 |\nabla \phi(z)|^2 > 1,$$

which gives a contradiction to the viscosity solution property of u_k if $u_k(z) = 0$. This establishes the claim.

A similar argument shows that if $\phi > v$ on $B_{1/2}$ except at x, where $\phi(x) = v(x)$, then $\partial_n \phi \geq 0$.

We extend v by even reflection to $B_{1/2}$:

$$v(x', x_n) = v(x', -x_n),$$

giving a continuous function harmonic away from $\{x_n = 0\}$. We have shown that v satisfies the Neumann condition in the viscosity sense on $\bar{U} \cap B_{1/2}$, so v is harmonic on B_1 by Lemma 10.3.

To conclude: we have shown that v is harmonic on $B_{1/2}$ and $|v| \leq 1$, so

$$|\nabla v(0)| + [v]_{C^2(B_{1/4})} \leq C(n).$$

Moreover, v is symmetric across $\{x_n = 0\}$, so $\partial_n v(0) = 0$. This gives the contradiction, as previously checked. □

11 Flat Implies Smooth: The Conclusion

Finally, we use Lemma 8.3 to prove Theorem 8.1. The bulk of the work has already been done, and we just sketch one way to conclude here.

Proof of Theorem 8.1; sketch. By selecting ε_F sufficiently small, we may guarantee that for every $x \in \partial\{u > 0\} \cap B_{1/2}$, we may apply Lemma 8.3 to $B_{1/2}(x)$, iteratively, with τ so small that $C(n)\tau \leq \frac{1}{2}$. This gives a sequence of unit vectors e^k such that $e^0 = e_n$, $|e^k - e^{k+1}| \leq C\varepsilon_F 2^{-k}$, with

$$\sup_{B_{\tau^k}(x)} |u(z) - (z - x) \cdot e^k| \leq 2^{-k}\tau^k \varepsilon_I \leq \tau^{k(1+\alpha)}\varepsilon_I.$$

The e^k form a Cauchy sequence, so let $e^\infty = \lim e^k$. Summing, this has $|e^k - e^\infty| \leq C2^{-k}\varepsilon_F$, so

$$\sup_{B_{\tau^k}(x)} |u(z) - (z - x) \cdot e^\infty| \leq C\tau^{k(1+\alpha)}\varepsilon_I.$$

A similar argument gives that if e^∞_x is the asymptotic normal at x, then

$$|e^\infty_x - e^\infty_y| \leq C|e^k_x - e^k_y| + C2^{-k}\varepsilon_F \leq C|x - y|^\alpha \varepsilon_F,$$

where $2^{-k-1} \leq |x - y| \leq 2^{-k}$; the second inequality can be verified directly from the fact that if u is close to both $x \cdot a$ and $x \cdot b$, then $|a - b|$ is also small.

This implies that u extends to a $C^{1,\alpha}$ function on $\overline{\{u > 0\}} \cap B_{1/2}$, with $\nabla u(x) = \varepsilon^\infty$ at the free boundary point x. If ε_F is small enough, $e \cdot e^\infty \gg 0$ at every such x, and so $\partial\{u > 0\}$ is a graph over $\{x_n = 0\}$. That it is a $C^{1,\alpha}$ graph follows from the implicit function theorem (or directly estimating its normal vectors e^∞) □

12 The Weiss Formula, Cones, and Singular Points

At this point, we return to the topic of local minimizers and attempt to understand the *singular set* $\sigma = \partial\{u > 0\} \setminus \partial^*\{u > 0\}$ better. We do not give detailed proofs here, but show the general structure of the arguments. For this, it will help to have the following formula (due to Georg Weiss, [15]):

Proposition 12.1. *Let u be a local minimizer on B_1 and $u(0) = 0$. Then for $r < 1$, the quantity*

$$W(u, r) = \frac{1}{r^n}E(u, r) - \frac{1}{r^{n+1}}\int_{\partial B_r} u^2$$

has

$$\partial_r W(u, r) \geq \frac{2}{r^{n+2}}\int (u - x \cdot \nabla u)^2.$$

This may be verified by a direct computation, together with the weak form of the free boundary condition in Theorem 7.1 applied to vector fields approximating $x1_{B_r}$.

Notice that W is invariant under rescaling, in the sense that if $u_r(x) = \frac{u(rx)}{r}$, then $W(u_r, 1) = W(u, r)$. Let $W(u, 0+)$ be the limit of $W(u, r)$ as $r \searrow 0$; we then have that if u_0 is a blow-up limit of u,

$$W(u_0, S) = \lim_{r\searrow 0} W(u_r, S) = \lim_{r\searrow 0} W(u, rS) = W(u, 0+).$$

In particular, $\partial_r W(u_0, r) = 0$, so $u_0 - x \cdot \nabla u_0$ a.e. on $\mathbb{R}^n$. This implies that $u_0(x) = |x|u_0(x/|x|)$, i.e. that u_0 is 1-homogeneous. We have shown that

Proposition 12.2. *Let u be a local minimizer on B_1 and $u(0) = 0$. Then if u_0 is a blow-up limit of u, $u_0(x) = |x|u_0(x/|x|)$. The free boundary $\partial\{u_0 > 0\}$ and the set $\{u > 0\}$ are invariant under dilation.*

The above is only interesting if $0 \in \Sigma$, for otherwise we have already shown a much stronger property (the blow-up is a unique half-linear solution). For singular points, though, this is the starting point of the analysis.

Definition 12.1. *We say that an entire local minimizer u is a* singular cone *if $0 \in \Sigma$ and u is 1-homogeneous. Let n_* be given by*

$$n_* = \min\{n : \exists \text{ a singluar cone in } \mathbb{R}^n\}.$$

First, in light of Theorem 8.1, we may upgrade the convergence of local minimizers slightly to say that limits of singular points are singular points.

Lemma 12.3. *Let $u_k \to u$ be local minimizers on B_1 with the convergence as in Theorem 6.2, and $0 \in \Sigma_{u_k}$. Then $0 \in \Sigma_u$.*

Proof. Assume this is not the case; then $\partial\{u > 0\} \cap B_\tau$ is contained in $\{|x_n| \leq \varepsilon_F \tau/2\}$ for sufficiently small τ and some choice of basis, by Theorem 8.1. It follows that $\partial\{u_k > 0\} \cap B_\tau \subseteq \{|x_n| \leq \varepsilon_f \tau\}$ for large k, from the convergence of the free boundaries. Applying Theorem 8.1 to u_k gives that $0 \in \partial^*\{u_k > 0\}$, a contradiction. □

Next, we have the following observation about cones:

Lemma 12.4. *Let u be a singular cone.*

1. *If there is a point $x \neq 0$, $x \in \Sigma$, then $n_* < n$.*
2. *If $\mathcal{H}^s(\Sigma \cap B_1) > 0$ (with $s \geq 1$), then there exists a singular cone v in $\mathbb{R}^{n-1}$ with $\mathcal{H}^{s-1}(\Sigma_v \cap B_1) > 0$.*

Proof. First, if $x \in \Sigma$, then so is tx for all $t > 0$. Let u_0 be a blow-up of u at x; we claim that if $e = x/|x|$, then $\partial_e u_0 = 0$. Indeed,

$$\nabla u(x + ry) \cdot \frac{x + ry}{|x + ry|} = u(x + ry) \leq C(n)r$$

from the fact that u is 1-homogeneous, so

$$|\nabla u(x + ry) \cdot e| \leq |e - \frac{x + ry}{|x + ry|} + Cr \to 0$$

as $r \to 0$. As this converges to $\nabla u_0(y)$ at almost every point, we have $\partial_e u_0 = 0$. Moreover, u_0 is 1-homogeneous.

Finally, one may check that if u_0 is an entire local minimizer with $\partial_e u_0 = 0$, and $v : \mathbb{R}^{n-1} \to \mathbb{R}$ is such that $u(x) = v(x - (x \cdot e)e)$, then v is an entire local minimizer on $\mathbb{R}^{n-1}$ directly. As $0 \in \Sigma_{u_0}$ by Lemma 12.3, $0 \in \Sigma_v$; this proves (1).

(2) can be shown with the aid of a geometric measure theory argument due to Federer, which essentially guarantees that there is a point $x \in B_1$ such that performing a blow-up at x, the limit u_0 has $\mathcal{H}^s(\Sigma_{u_0} \cap B_1) > 0$. The rest follows in the same way. □

For the lemma of Federer; see [14] (in the minimal surface context, but the argument is mostly independent of the problem). The point of the above is the following connection between n_* and the size of Σ.

Theorem 12.5. *Let u be a local minimizer on B_1. Then:*

1. *If $n < n_*$, Σ is empty.*
2. *If $n = n_*$, Σ is a discrete set (i.e. contains no accumulation points).*

3. *If $n > n_*$, then for every $s > 0$, $\mathcal{H}^{n-n_*+s}(\Sigma) = 0$ (i.e. the Hausdorff dimension of Σ is at most $n - n_*$).*

Proof. For (1), assume $x \in Sigma$ and perform a blow-up at x; using Lemma 12.3 the blow-up limit is a singular cone. This is a contradiction to the definition of n_*.

For (2), assume that $x_k \to x$, with $x, x_k \in \Sigma$. Set $r_k = |x_k - x|$ and take a blow-up limit along a subsequence of the r_k at x. If u_0 is the limit, then $0 \in \Sigma_{u_0}$, and also $y = \lim x_k/r_k \in \partial B_1$ (along a subsequence) is in Σ_{u_0}. Applying the previous lemma gives a contradiction.

For (3), we again use the lemma of Federer to perform a blow-up at a point x so that the limit u_0 has $\mathcal{H}^{n-n_*+s}(\Sigma_{u_0} \cap B_1) > 0$. Then apply the lemma above repeatedly, to obtain a singular cone in dimension n_* with $\mathcal{H}^s(\Sigma_v \cap B_1) > 0$. Then apply the first part of the lemma once, to obtain a singular cone in dimension $n_* - 1$, a contradiction. □

13 Classification of Cones

The previous section is not very interesting unless one has some way of estimating n_*. Let us start with the easy cases:

Lemma 13.1. *$n_* > 1$.*

Proof. x_+ is the unique 1-homogeneous entire local minimizer in $\mathbb{R}$, up to reflection: the 1-homogeneous functions in $\mathbb{R}$ are $\alpha x_+ + \beta x_-$; if α, β both are nonzero this is clearly not minimal, while if α or β is not 1 this fails to satisfy the free boundary condition. □

By separating variables, we have that an α-homogeneous harmonic function on a conical set K in $\mathbb{R}^n$ satisfies, on $K \cap \partial B_1$,

$$\triangle_{S^{n-1}} u + (\alpha(n-2) + \alpha^2)u = 0$$

on K. If $u = 0$ on ∂K, then u is a positive Dirichlet eigenfunction on $\partial B_1 \cap K$, meaning $\lambda_1(\partial B_1 \cap K) = \alpha(n-2) + \alpha^2$.

Lemma 13.2. *$n_* > 2$.*

Proof. Let u be a singular cone. Then each connected component of $\partial\{u > 0\} \cap \partial B_1$ has first Dirichlet eigenvalue 1. This means that this set is either a half-circle or the union of two half-circles (the Dirichlet eigenvalues are fully determined by the length of a circular arc), and the latter contradicts density bounds. Thus $\{u > 0\}$ is a half-plane, contradicting that 0 is a singular point. □

We also have, due to Caffarelli, Jerison, and Kenig [7]:

Theorem 13.3. *$n_* > 3$.*

The state of the art, due to Jerison and Savin [12], is:

Theorem 13.4. *$n_* > 4$.*

And finally, this is due to De Silva and Jerison [11]:

Theorem 13.5. *$n_* \leq 7$.*

The proofs of all three of these results are quite involved. The key observation is that local minimizers, in addition to satisfying the stationarity condition (7.6), also have a stability property which can also be obtained by performing domain variations (this time to second order). This takes the form

$$\int_{\partial\{u>0\}} H\phi^2 \le \int_{\{u>0\}} |\nabla\phi|^2$$

for any smooth ϕ compactly supported on $\mathbb{R}^n \setminus \{0\}$. Here H is the mean curvature of $\partial\{u > 0\}$, oriented inwards; it turns out to be nonnegative. The proof of Jerison and Savin proceeds by plugging in certain nonlinear combinations of second derivatives of u into this inequality, after multiplying by a cutoff.

References

[1] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[2] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman. Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc.*, 282(2):431–461, 1984.

[3] Luis Caffarelli and Sandro Salsa. *A geometric approach to free boundary problems*, volume 68 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[4] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$. *Rev. Mat. Iberoamericana*, 3(2):139–162, 1987.

[5] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on X. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 15(4):583–602 (1989), 1988.

[6] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz. *Comm. Pure Appl. Math.*, 42(1):55–78, 1989.

[7] Luis A. Caffarelli, David Jerison, and Carlos E. Kenig. Global energy minimizers for free boundary problems and full regularity in three dimensions. In *Noncompact problems at the intersection of geometry, analysis, and topology*, volume 350 of *Contemp. Math.*, pages 83–97. Amer. Math. Soc., Providence, RI, 2004.

[8] Gui-Qiang Chen, Monica Torres, and William P. Ziemer. Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. *Comm. Pure Appl. Math.*, 62(2):242–304, 2009.

[9] G. David and T. Toro. Regularity of almost minimizers with free boundary. *Calc. Var. Partial Differential Equations*, 54(1):455–524, 2015.

[10] D. De Silva. Free boundary regularity for a problem with right hand side. *Interfaces Free Bound.*, 13(2):223–238, 2011.

[11] Daniela De Silva and David Jerison. A singular energy minimizing free boundary. *J. Reine Angew. Math.*, 635:1–21, 2009.

[12] David Jerison and Ovidiu Savin. Some remarks on stability of cones for the one-phase free boundary problem. *Geom. Funct. Anal.*, 25(4):1240–1257, 2015.

[13] D. Kinderlehrer, L. Nirenberg, and J. Spruck. Regularity in elliptic free boundary problems. *J. Analyse Math.*, 34:86–119 (1979), 1978.

[14] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[15] Georg S. Weiss. Partial regularity for weak solutions of an elliptic free boundary problem. *Comm. Partial Differential Equations*, 23(3-4):439–455, 1998.