1 Continuity of Solutions (obstacle problem)

Below, let D be a smooth bounded open set, and $\sigma : [0,\infty) \to [0,\infty)$ a continuous, nondecreasing function with $\lim_{t\searrow 0}\sigma(t) = 0$. Let U be a ball (this can be removed with appropriate modifications, but is not relevant here). Let

$$[v]_{\sigma,U} = \sup_{x,y\in U} \frac{|u(x)-u(y)|}{\sigma(|x-y|)}.$$

Let (for C^1 functions v)

$$[v]_{\sigma,1,U} = \sup_{x,y\in U} \frac{|u(x)+(y-x)\nabla u(x)-u(y)|}{|x-y|\sigma(|x-y|)}.$$

Lemma 1.1. *For C^1 functions v, the seminorms $[v]_{\sigma,1;U}$ and $[\nabla v]_{\sigma,U}$ are equivalent, in the sense that*

$$[v]_{\sigma,1,U} \leq C_1[\nabla v]_{\sigma,U} \leq C_2[v]_{\sigma(C_3,\cdot),1,U}$$

where $C_i = C_i(n)$.

Proof. We have

$$|u(x)+(y-x)\nabla u(x)-u(y)| = |\int_0^1 [\nabla u(yt+x(1-t)) - \nabla u(x)]\cdot(y-x)| \leq |y-x|\sigma(|x-y|)[\nabla u]_{\sigma,U}$$

from the fundamental theorem of calculus. For the other direction,

$$|(\nabla u(x)-\nabla u(y))\cdot(x-y)| \leq |u(x)+(y-x)\nabla u(x)-u(y)|+|u(y)+(x-y)\nabla u(y)-u(x)| \leq 2|y-x|\sigma(|x-y|)[\nabla u]_{\sigma,1,U}.$$

Fix a unit vector e orthogonal to $x-y$; we may then find a point $z\in U$ such that $e\in \text{span}(x-y,x-z)$, $|z-y|+|z-x|\leq 4|x-y|$ and $|(z-y)\cdot e| \geq \frac{1}{4}|z-y|$, $|(z-x)\cdot e|\geq \frac{1}{4}|y-x|$. Indeed, such a z may always be found on the intersection of $B_{|x-y|/2}(\frac{x+y}{2})$ and the plane of points eqidistant from x and y; at least half of this disk will be contained in u, and one may check that this contains sufficient points. Then we have

$$|u(x)+(z-x)\nabla u(x)-u(z)-u(y)-(z-y)\nabla u(y)+u(z)+u(y)+(x-y)\nabla u(y)-u(x)| \leq 9|y-x|\sigma(4|x-y|)[\nabla u]_{\sigma,1,U},$$

so

$$|(z-x)(\nabla u(x)-\nabla u(y))| \leq 9|y-x|\sigma(4|x-y|)[\nabla u]_{\sigma,1,U}.$$

Hence, using $x-z = e(x-z)\cdot e + (x-y)\frac{(x-z)\cdot(x-y)}{|x-y|^2}$

$$|(x-z)\cdot e||e\cdot(\nabla u(x)-\nabla u(y))| \leq \frac{|(x-z)\cdot(x-y)||(x-y)\cdot(\nabla u(x)-\nabla u(y))|}{|x-y|^2} + |(x-z)\cdot(\nabla u(x)-\nabla u(y))|,$$

and

$$|e\cdot(\nabla u(x)-\nabla u(y))| \leq 100\sigma(4|x-y|)[\nabla u]_{\sigma,1,U}.$$

□

Lemma 1.2. *Let u be a continuous function on D, smooth on the closed set $F\subseteq D$, with the property that*

$$\triangle u = 0$$

on $D\setminus F$, and

$$|u(x)-u(y)| \leq \sigma(|x-y|)$$

for $x\in F$ and $y\in D$. Then if $U\subseteq D$ is a ball of radius r whose quadruple is also contained in D and U intersects F,

$$|u(x)-u(y)| \leq C_1\sigma(C_2|x-y|) + C_3|x-y|\frac{\sigma(4r)}{r}$$

for any $x,y\in U$.

Note that for $\sigma(t) = Ct^\alpha$, this implies $[u]_{\sigma,U} \leq C_4$. Indeed, this will be true for any σ with $t/s \leq \sigma(t)/\sigma(s)$ for $t \leq s$.

Proof. Take any $x, y \in U$ and set $\min\{d(x, F), d(y, F)\} = d$. Note that $d < 2r$ from the assumption that U intersects F, so B_d is contained in the quadruple of U If $|x - y| \leq \frac{1}{2}d$, then we have

$$|u(x) - u(y)| \leq C(n)\frac{|x-y|}{d} \operatorname*{osc}_{B_d(x)} u$$

by applying elliptic estimates on $B_d(x)$. As $\partial B_d(x)$ intersects F, we have a point $z \in \partial B_d$ such that $|u(y) - u(z)| \leq \sigma(|y - z|)$; this guarantees that $\operatorname{osc}_{B_d(x)} u \leq 2\sigma(2d)$.

If instead $|x - y| \geq \frac{1}{2}d$, let x' and y' be the two points in F closest to x, y respectively. Assume that $d = |x - x'| \leq 2|x - y|$; then $|y - y'| \leq |x - y| + |x - x'| \leq 3|x - y|$ and $|x' - y'| \leq 6|x - y|$. This gives

$$|u(x) - u(y)| \leq |u(x) - u(x')| + |u(x') - u(y')| + |u(y') - u(y)| \leq 3\sigma(6|x - y|).$$

□

Here is a version for the derivatives instead.

Lemma 1.3. *Let u be a continuous function on D, smooth on the closed set $F \subseteq D$, with the property that*

$$\triangle u = 0$$

on $D \setminus F$, and

$$|u(x) + (y - x)\nabla u(x) - u(y)| \leq |x - y|\sigma(|x - y|)$$

for $x \in F$ and $y \in D$. Assume also that there is a $\phi : D \to \mathbb{R}$ with $u = \phi$ on F and $[\phi]_{\sigma,1,D} \leq 1$. Then if $U \subseteq D$ is a ball of radius r whose quadruple is also contained in D and U intersects F,

$$|\nabla u(x) - \nabla u(y)| \leq C_1\sigma(C_2|x - y|) + C_3|x - y|\frac{\sigma(4r)}{r}$$

for any $x, y \in U$.

The extra assumption about ϕ is not really needed, and may be removed by the same argument as was used to prove the harder part of Lemma 1.1.

Proof. Take any $x \in U \setminus F$ and find $y \in F$ such that $|x - y| = d(x, F) < 2r$. Then we have that

$$v(x) = u(x) - \nabla u(y)(x - y) - u(y)$$

is harmonic on $B_{d(x,F)}(x)$, and so

$$|\nabla u(x) - \nabla u(y)| = |\nabla v(x)| \leq C(n)\frac{4|x - y|\sigma(2|x - y|)}{|x - y|} \leq C\sigma(2|x - y|).$$

Combined with the assumption on ϕ, this implies that for any $x \in U$ and $y \in F \cap U$,

$$|\nabla u(x) - \nabla u(y)| \leq C_1\sigma(C_2|x - y|).$$

The conclusion now follows by applying Lemma 1.2 to each component of ∇u. □