1 Free Boundary Problems: Introduction

Speaking roughly, a free boundary problem has two unknowns: a set $\Omega \subseteq \mathbb{R}^n$ and a function $u : \Omega \to \mathbb{R}$. Neither the $\mathbb{R}^n$ for the domain nor the $\mathbb{R}$ for the range are sacred, but let us stick to this formulation for simplicity. The boundary $\partial\Omega$ is the eponymous *free boundary*. The problem itself consists of finding solutions Ω, u which satisfy certain conditions:

- u solves an equation (typically a PDE) on Ω.
- u satisfies a boundary condition with respect to which this PDE is well-posed on $\partial\Omega$.
- u satisfies an additional condition on $\partial\Omega$ which renders the problem overdetermined. This is known as the *free boundary condition.*

A common variation we will also consider is the *two-phase problem*, in which (usually) u also satisfies an equation on $\mathbb{R}^n \setminus \Omega$, as well as a boundary condition for that equation from the side of $\mathbb{R}^n \setminus \Omega$. In this case the free boundary condition will likely involve the values and derivatives of the function from each side.

It will be important to distinguish the global formulation of free boundary problems, which often has additional challenges and subtleties, from the local nature of the problem. To this end, let us assume that we are working over $U \subseteq \mathbb{R}^n$, and that over ∂U some suitable, fixed, and irrelevant to our purposes boundary condition for u is given. The specifics of this will vary from problem to problem, but the general principle is to focus on the free boundary.

Some examples are in order:

1. Here $u > 0$ on Ω, and satisfies $\triangle u = 0$ there. The boundary condition is $u = 0$ on $\partial\Omega$, while the free boundary condition is $|\nabla u| = 1$ (this may also be written as $u_\nu = -1$, where ν is the outward unit normal vector to $\partial\Omega$). This is known as the *Bernoulli problem*, and is one of the oldest and most common archetypes for free boundary problems. The term *one-phase problem*, if no further context is given, often refers to this equation. The variational formulation of this problem is sometimes called the *Alt-Caffarelli problem* after a well-known and popular paper on the subject from 1981.

2. Let $u > 0$ on Ω again, but now $\triangle u = 1$ on Ω. The boundary condition is again $u = 0$. The free boundary condition can be stated in various ways, but one is that $\nabla u = 0$ (as we will discuss, though, this is not really the best way to understand it). This is known as the *obstacle problem*, or is at any rate a very simplified formulation of it. This is the second standard archetype of free boundary problems.

3. Here is a version of the Bernoulli problem with two phases: let $u > 0$ on Ω and $u < 0$ on $\bar{\Omega}^c$, with $u = 0$ on $\partial\Omega$. We assume that $\triangle u = 0$ on both Ω and $\bar{\Omega}^c$ (though not at points on $\partial\Omega$ itself). The free boundary condition is $|\nabla u|_\Omega|^2 - |\nabla u|_{\Omega^c}|^2 = 1$ (here these are meant to indicate the derivatives from either side of the boundary). Note that if $u \geq 0$, this reduces to the one-phase variant.

4. Problems of the following type are often called *Stefan problems*: thinking of $|R^n$ as denoting $n-1$ space variables (x) and one time variable (t), let $u > 0$ satisfy $\partial_t u - \triangle u = 0$ in Ω and $u = 0$ on $\partial\Omega$. The free boundary condition may be written as follows: if V denotes the velocity of $\partial\Omega$ in the direction of the outward normal to it, then $V = |\nabla u|$. Note that to enforce that $u = 0$ on $\partial\Omega$, we have automatically that $V = \partial_t u/|\nabla u|$ (and $V = 0$ if $|\nabla u| = 0$), so this may be expressed as $\partial_t u = |\nabla u|^2$. Other variations on this give rise to different but related free boundary problems.

5. Another common evolution problem is the Hele-Shaw flow, which is like the Stefan problem but with the heat equation replaced with $\triangle u = 0$ (the Laplacian taken only in the space directions, so along slices). This is a blend of elliptic and parabolic equations (the parabolic effect being in the free boundary condition itself).

6. Here is a problem with a different flavor: on Ω, $\triangle u = 0$, with the Neumann boundary condition $u_\nu = 0$ on $\partial\Omega$ (where ν is a unit normal). The free boundary condition is $|\nabla u|_{\text{forward along } \nu}|^2 -$

$|\nabla u|_{\text{backward along } \nu}|^2 = H$, where H is the mean curvature of $\partial\Omega$. This is actually not enough information to identify the problem fully, but one example of this type is the Mumford-Shah energy minimizers from image segmentation.

7. Let, this time, $(u, P) : U \subseteq \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3 \times \mathbb{R}$ satisfy $\partial_t u + u \cdot \nabla u + \nabla P = 0$ and $\operatorname{div} u = 0$ in Ω (the incompressible Euler equations). On $\partial\Omega$, they satisfy the (typical) boundary condition $V = u \cdot \nu$, where ν is the outward unit normal and V is the velocity of Ω in that direction; this condition essentially means that $\partial\Omega$ remains the boundary of the fluid as the fluid evolves. The free boundary condition here is $P = 0$, which renders the problem overdetermined (this is perhaps not entirely obvious). This is the *water waves* problem; it has quite different behavior than any of the previous examples, but is heavily studied in various related formulations in the dispersive equations community.

Free boundary problems may be thought of as PDE for boundaries of sets coupled with PDE for functions on that set. Like with any PDE, it is helpful to ask what *type* of PDE this boundary is satisfying. In all of the examples other than the last two, the answer is either elliptic or parabolic (in the second-to-last, the answer is also elliptic in some weaker and less useful sense, while in the final example, the answer is dispersive). The reason for this answer will only become clear later in the course. However, as a disclaimer, we will focus exclusively on elliptic (and maybe, time-permitting, parabolic) free boundaries; this is not because they are somehow more interesting or popular, but rather reflects my personal ignorance of the dispersive and hyperbolic literature, where the methods used are quite different.

It is best to view all of these examples as *nonlinear* partial differential equations. This, at a first glance, is a confusing remark, as the first two examples (on which we will focus very heavily) are obviously linear: a linear PDE coupled with two linear boundary conditions rendering it overdetermined. Note, though, that it is not just u which we are solving for, but Ω itself. And the fact of the matter is that any problem whose solution is a set is inherently nonlinear; we cannot perform linear algebra on sets, or at least not in any way under which our problems are invariant.

We are therefore faced with the typical challenges of nonlinear PDE: the potential behavior of solutions is limitlessly complicated, the existance theory is not trivial, uniqueness is often just not true, and the tools available to understand regularity are, to speak frankly, primitive. We also have the additional challenge that as our solutions are sets, we have no idea how to even deploy PDE methods: while it may be the case, philosophically, that as argued above the boundary $\partial\Omega$ is solving a differential equation, this will never rise above the level of a deep-seated belief and into the realm of mathematical rigor. We will have to tackle the boundary as-is, as a geometric object, and translate what PDE techniques we wish to bring to the table into a form compatible with that fact.

A historical remark: the theory of free boundary problems saw a leap in level of understanding in the late 1970s and early 1980s similar to what nonlinear elliptic equations saw in the late 50s and early 60s. Prior to these developments, there were some known approaches to existance (especially local-in-time existance to evolution problems and existance for whichever problems could be formulated variationally). The regularity of the solution was understood in some cases (notably for the obstacle problem). A more general understanding was available for evolution problems in one space variable (and possibly stationary problems in two variables, though I'm not as familiar with the literature there). The turning point required a synthesis of techniques in nonlinear PDE with rather different methods developed for a differnet geometric problem: minimal surfaces (we will discuss this problem to some extent as well). With these tools, it is now possible to understand the finer properties of solutions as well as their boundaries.

2 Harmonic Functions

A core aspect of free boundary theory is an extremely clear understanding of the PDE being solved in the domain. To that end, let us start with some review:

Definition 2.1. *A function $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is* harmonic *$u \in C^2(U)$ and $\triangle u = 0$.*

A useful way of thinking about harmonic functions is the following *mean value property*:

Theorem 2.1. *Let u be harmonic on U, and $B_r(x) \subset\subset U$. Then*

$$\fint_{B_r(x)} u = u(x) \tag{2.1}$$

and

$$\fint_{\partial B_r(x)} u = u(x). \tag{2.2}$$

Proof. Let us set

$$f(r) = \fint_{\partial B_r(x)} u = \fint_{\partial B_1} u(x+ry)dy.$$

Set $v(y) = u(x+ry)$, noting that v is harmonic on B_1. Then $f'(r)$ is given by

$$f'(r) = \fint_{\partial B_1} \nabla v(y) \cdot y dy = \fint_{B_1} \operatorname{div}(\nabla v) = 0.$$

Hence f is constant. From the continuity of u at x it follows that $\lim_{r \searrow 0} f(r) = u(x)$, giving (2.2). Then (2.1) follows by integrating (2.2) in r. □

Lemma 2.2. *Let $u \in L^1_{loc}(U)$ satisfy (2.1) for every $B_r(x) \subset\subset U$ (for some representative of u). Then u is a smooth function on U, and is also harmonic.*

Proof. Fix some $B_{3r}(x) \subset\subset U$. We have that for any $y \in B_{2r}(x)$,

$$|u(y)| \leq \frac{1}{|B_r|} \int_{B_r(y)} |u| \leq C(r) \|u\|_{L^1} < \infty.$$

We also have that u is a continuous function, from the continuity property for the Lebesgue integral

$$\lim_{y \to x} \int_{B_r(y)} u = \int_{B_r(x)} u.$$

One may then check that for any continuous u, the function (for fixed r)

$$g(x) = \int_{B_r(x)} u$$

is continuously differentiable, with derivative

$$\nabla g(x) = \int_{\partial B_r(x)} u(z)(z-x)dz$$

(by checking this on smooth functions and approximating, for example). It follows that u is continuously differentiable on B_r, with derivative bounded by a quantity depending only on r and the L^1 norm of u. Now, the components of ∇u also satisfy (2.1), and so we may continue in this manner to obtain that u is smooth on U (in fact, the careful reader may use the explicit form of the estimate to deduce that u is real analytic).

Finally, arguing as in the proof of Theorem 2.1 we have that (2.2) holds for u as well, and that

$$\int_{B_r(x)} \triangle u = 0$$

for every $B_r(x) \subset\subset U$; this latter fact implies that u is harmonic. □

The following definitions will become more relevant soon, but let us save them for now:

Definition 2.2. *We say that u is* subharmonic *on U if $u \in C^2(U)$ $\triangle u \geq 0$ (resp.* superharmonic*, ≤ 0).*

Definition 2.3. *We say that u is* mean value subharmonic *on U if $u \in L^1_{loc}(U)$, is upper semicontinuous, and for every $B_r(x) \subset\subset U$ (and some representative of u),*

$$u(x) \leq \fint_{B_r(x)} u.$$

(resp. mean value superharmonic*, lower, $\geq$).*

To clarify, I use the term *upper semicontinuous* here and below to refer to functions $\Omega \to [0, 8)$, i.e. they are finite from above. Requiring upper semicontinuity is not necessary in all definitions of subharmonic which follow; in some, it will follow from the remainder of the definition. However, this is not always clear, and in some of the definitions is not true.

It is possible to show, with some effort, that an upper semicontinuous function satisfying the mean value inequality is actually locally integrable so long as it is not $-\infty$, though we do not pursue this point further. It will, in fact, then also satisfy the remaining natural mean value properties, such as the monotonicity of both interior and boundary mean value integrals. On the other hand, one may show that if the function is assumed to be locally integrable and has monotone mean value integrals, then it admits an upper semicontinuous representative. I do not know whether merely the mean value inequality as stated above, without any further assumptions, implies the existence of an upper semicontinuous representative.

Proposition 2.3. *A subharmonic function is mean value subharmonic. A mean value subharmonic function with a C^2 representative is subharmonic.*

There are three basic, essential ways to think about harmonic functions besides the mean value property, which we examine below.

2.1 Maximum Principle

Theorem 2.4. *Let u be a mean value subharmonic function on a connected domain U. Assume that there is an $x \in U$ such that*

$$u(x) = \sup_U u.$$

Then u is constant.

Proof. We have that

$$u(x) \leq \fint_{B_r} u \leq \sup_U u (< \infty)$$

for any r with $B_r(x) \subset\subset U$, which means that $u = \sup_U u$ almost everywhere on every such B_r. Given any other point y in U with the Lebesgue density of $\{u < \sup_U u\}$ at y greater than zero, we may connect x and y by a finite path of balls which contain each of their neighbor's centers, and repeating this argument on these balls will give a contradiction. □

While the property below, known as the Harnack inequality, is not exactly a consequence of the maximum principle, it may be thought of as a kind of quantitative version, and belongs in the same category of tools.

Theorem 2.5. *Let u be a nonnegative harmonic function on B_2. Then there is a constant $C = C(n)$ such that*

$$\sup_{B_1} u \leq C u(0).$$

Proof. We have that

$$u(x) = \fint_{B_{1/2}(x)} u = C(n) \int_{B_{1/2}(x)} u \leq C(n) \int_{B_{3/2}} u = C(n)u(0),$$

where the mean value property was used twice. □

An alternative, equivalent, form replaces $u(0)$ by $\inf_{B_1} u$ and can be proved similarly.

The maximum principle suggests some other reasonable notions of subharmonic functions:

Definition 2.4. *An upper semicontinuous function u is said to be* comparison subharmonic *if for all $B_r(x) \subset\subset U$ and all superharmonic functions v on $B_r(x)$ with $v \geq u$ on $\partial B_r(x)$, we have $v \geq u$ on $B_r(x)$.*

This is the weakest type of subharmonic function worth working with, and is a commonly used definition.

Definition 2.5. *An upper semicontinuous function u is said to be* viscosity subharmonic *if for all $\phi \in C^2(U)$ such that phi$(x) = u(x)$ for some $x \in U$ and $\phi > u$ otherwise, we have $\triangle$phi ≥ 0.*

This, on the other hand, is a perhaps odd but very powerful way of thinking about subsolutions, and while not at all necessary here we present it while the going is easier so the reader might internalize it before it becomes truly necessary. Heuristically, if you can touch your subharmonic function from above by a smooth function, then that smooth function better not have $\triangle u < 0$ at that point: if it does, then your subharmonic function has an even smaller Laplacian, as $D^2\phi \geq D^2u$ at this local maximum for the difference, and so the trace is also smaller. The reader may rightly object that D^2u makes no sense and that we are probably testing only at a small subset of the relevant points, but the fact remains:

Proposition 2.6. *A function is viscosity subharmonic if and only if it is comparison subharmonic. A function which is mean value subharmonic is comparison subharmonic.*

It is not true that a comparison subharmonic function must be mean value subharmonic (e.g. $-\infty$ is a counterexample). That turns out to be the only counterexample, though, as we discuss briefly at the end of this section. It is important, especially in the viscosity definition, to assume that u is upper semicontinuous.

Proof. Let u be viscosity subharmonic, and assume that there is a ball $B_r(x) \subset\subset \Omega$ and a superharmonic function h on it such that $h \geq u$ on $\partial B_r(x)$ and $h(y) < u(y)$ for some $y \in B_r(x)$. Observe that for a sufficiently small ε, the function $v(z) = h(z) + \varepsilon(r^2 - |x - z|^2)$ will have that $\sup_{B_r(x)}(u - v) = u(y') - v(y') > 0$ for some $y' \in B_r(x)$ (the supremum is attained, as $u - v$ is upper semicontinuous and is less than or equal to 0 on $\partial B_r(x)$). Let $c = \sup_{B_r(x)}(u - v) > 0$, and define $w(z) = v(z) + c + \varepsilon/2|y' - z|^2$. We have that w is C^2 on $B_s(y')$ for a small s, that $w > u$ away from y', and that $w(y') = u(y')$. From the definition of viscosity subsolution, it follows that $\triangle w(y') \geq 0$, which contradicts that $\triangle w \leq -2n \cdot \varepsilon/2$.

Conversely, let u be a comparison subharmonic function on Ω, and let ϕ be a test function as in the definition of viscosity subharmonic. Assume that $\triangle\phi(x) < 0$; we may then also assume that $\triangle\phi(x) < 0$ on $B_r(x)$. Then by assumption, we may find a $c > 0$ such that $\phi - c \geq u$ on $\partial B_r(x)$, while $\phi(x) - c < u(x)$. This is a contradiction.

That mean value subharmonic functions are comparison subharmonic follows directly from Theorem 2.4. □

Proposition 2.7. *Let Ω be bounded, u be viscosity subharmonic, and $u \leq 0$ on $\partial\Omega$. Then $u \leq 0$ on Ω.*

Proof. Assume that $\sup_\Omega u = M > 0$. Let $w_c(x) = M + c - \varepsilon|x|^2$, where ε is chosen in such a way that $w_0 \geq 0$ on Ω. Let $c(\geq 0)$ be the smallest value such that $\inf_\Omega w_c - u \geq 0$, and let x be the point where the infimum is attained (in Ω). Then $v(y) = w_c(y) + |x - y|^4$ has the following properties: $v(y) = u(y)$; $v > u$ away from y; and $\triangle v(y) = \triangle w_c(y) = -2n\varepsilon < 0$. This contradicts the viscosity subharmonicity of u. □

The maximum principle allows us to construct harmonic functions. This is typically known as Perron's Method, and is outlined below. While the use of viscosity subharmonic functions is entirely unnecessary here, we will use this as an opportunity to see how to work with them.

Lemma 2.8. *Let $\mathcal{F}$ be a family of viscosity subharmonic functions. Let $u(x) = \limsup_{y\to x} \sup_{v\in\mathcal{F}} v(y)$ be the upper semicontinuous envelope of the supremum of them, and assume that $u < \infty$ (at every point). Then u is viscosity subharmonic.*

The upper semicontinuous envelope in this lemma is largely to resolve a technical point: the maximum of two upper semicontinuous functions need not be upper semicontinuous. The minimum, however, is upper semicontinuous, and the reader may verify that u is the smallest upper semicontinuous function to lie above all $v \in \mathcal{F}$.

Proof. Take any point $x \in \Omega$ and function $\phi \in C^2(\Omega)$ such that $\phi(x) = u(x)$ and $\phi > u$ away from x. Fix $\varepsilon > 0$. Then there is a $\delta > 0$ such that $\phi - u > \delta$ outside of $B_\varepsilon(x)$, and a $\tau \in (0, \varepsilon)$ such that $|\phi(\cdot) - \phi(x)| < \delta/4$ on $B_\tau(x)$. In particular, that means that for every $v \in \mathcal{F}$, $\phi - v > \delta$ outside of $B_\varepsilon(x)$. On the other hand, we may find a $y \in B_\tau$ and a $v \in \mathcal{F}$ such that $u(x) - v(y) < \delta/4$ (by the definition of u); it follows that $\phi(y) - v(y) < \delta/2$.

Let $c_\varepsilon = \inf_\Omega(\phi - v)$. From the information gathered above, we see that $0 \leq c_\varepsilon < \delta/2$. As v is upper semicontinuous, and $\phi - v > \delta$ outside of B_ε, the infimum must be attained at some point x_ε in B_ε. Using $\phi - c_\varepsilon$ as a test function in the definition of viscosity subsolution for v, we see that $\triangle\phi(x_\varepsilon) \geq 0$. As $\varepsilon \to 0$, we have that $x_\varepsilon \to 0$, so from the fact that $\phi in C^2(\Omega)$, we have that $\triangle\phi(x) \geq 0$. □

Lemma 2.9. *Let u be a viscosity subsolution on Ω, and assume that $u_*(x) = \liminf_{y\to x} u(y)$ is not a viscosity supersolution. Then there exists a viscosity subsolution v on Ω with $v \geq u$, $v = u$ outside of $B_r(x) \subset\subset \Omega$, and $v > u$ on $B_{\delta r}(x)$ for some small δ.*

Proof. As u_* is not a viscosity supersolution, we may find a ϕ such that $\phi < u_*$ except at some point x where $\phi(x) = u_*(x)$, and $\triangle\phi \geq \varepsilon > 0$ on $B_r(x)$ for some small ε, r. Let $w(y) = \phi(y) + \varepsilon/2 \cdot (r^2 - |y - x|^2)$; then $w = \phi < u_*$ on $\partial B_r(x)$ and is subharmonic on $B_r(x)$. Applying Lemma 2.8, we see that $v(z) = \limsup_{s\to z} \max\{u(s), w(s)\}$ is viscosity subharmonic on $B_r(x)$; as $u \geq u_* > w$ outside of $B_r(x)$, we also have that v is viscosity subharmonic on Ω. From this we see that $v = u$ outside of $B_r(x)$. On the other hand, on $B_t(x)$ for a small t, we have $w > u_*(x) + \frac{\varepsilon}{4}r^2 > u(x') + \frac{\varepsilon}{8}r^2$ for some x' in $B_{t/2}(x)$; as u is upper semicontinuous, this implies that $w > u$ on $B_s(x')$ for an even smaller s, as claimed. □

Theorem 2.10. *Let Ω be a bounded open set, and g_-, g_+ be viscosity sub and superharmonic with $g_- = g_+$ on $\partial\Omega$. Then there is a harmonic u in $C(\bar{\Omega})$ with $u = g_- = g_+$ on $\partial\Omega$.*

Proof. Let $\mathcal{F} = \{v \text{ viscosity subharmonic on } \Omega : g_- \leq v \leq g_+\}$; this is nonempty as it contains g_- ($g_- \leq g_+$ from Proposition 2.7). Applying Lemma 2.8, we have that $u(x) = \limsup_{y\to x} \sup_{v\in\mathcal{F}} v(y)$ is subharmonic. Assume that u_* (in the notation of Lemma 2.9) is not superharmonic; then there exists a subharmonic v with $v \geq u$, $v = u$ on ∂_Ω, and $v > u$ on some ball. This v has $v \leq g_+$ from Proposition 2.7, and so $v \in \mathcal{F}$. This, however, contradicts $v > u$; we have shown that u_* is superharmonic. Finally, applying Proposition 2.7 to $u - u_*$ (using that they are equal on $\partial\Omega$, as $u \in \mathcal{F}$), we have that $u \leq u_*$, giving $u = u_*$ is harmonic. □

This is often most usefully combined with the following type of fact (far more general versions are available, but we do not pursue this here):

Lemma 2.11. *Let $\Omega = B_1$ and g be a continuous function on ∂B_1. Then there exists a viscosity superharmonic u with $u = g$ on ∂B_1.*

Proof. We will write down a family of superhamonic functions whose infimum is continuous at ∂B_1; this will imply the conclusion provided that they have the following two properties: they are all larger than or equal to g on ∂B_1, and for every $\delta > 0$ and any $x \in \partial B_1$, there is a member of the family v with $v(x) < \delta$.

To do so, we include, first, the constant function $\max g$ in our family. We then add in (for all $x \in \partial B_1$ and $\delta \in (0,1]$) $v_{x,\delta}$, which is defined by:

$$v_{x,\delta}(y) = g(x) + \delta + \alpha_{x,\delta} x \cdot (x - y),$$

where $\alpha_{x,\delta}$ is chosen to be the smallest value possible so that $v \geq g$ on ∂B_1. It remains only to verify that the infimum of all of these functions u is continuous on ∂B_1. Note that u is upper semicontinuous and $u \leq g$ on ∂B_1, by construction. On the other hand, by applying the same construction to $-g$, we produce a family of linear functions $\mathcal{F}'$ with the same properties, but from below. Letting w be their supremum, we have that $g \leq w \leq u \leq g$ on the boundary, and $w \leq u$ on the interior; as w is lower semicontinuous, this implies that both are continuous along ∂B_1. $\square$

The functions g_+ and g_- are known as *barriers.* The use of such arguments, where an explictly given subharmonic function is used with the comparison principle to give some information about a harmonic function, will be very common once we start discussing free boundary problems.

Note that we used a kind of global argument to deduce that the largest subharmonic function we produced is continuous. This can be avoided for harmonic functions. To see why, first consider the relationship between maximal subharmonicity and mean value subharmonicity again. Let u be a viscosity subharmonic function and $B_r(x)$ be a ball inside of Ω. Then u is upper semicontinuous on $\partial B_r(x)$, and so may be represented as an infimum of a decreasing sequence of continuous functions $\{v_i\}$ on ∂B_r (this is a fact from real analysis). We may, via the method just shown, extend each v_i to a harmonic function on $B_r(x)$. In particular, this means that

$$v_i(x) = ⨍_{\partial B_r(x)} v_i.$$

As $u(x) \leq v_i(x)$ from the maximum principle, and as the integrals converge by the monotone convergence theorem, we have that

$$u(x) \leq ⨍_{\partial B_r(x)} u$$

(it is possible that both sides are $-\infty$ here). One may also use the fact that as $u \leq v_i$ on $B_r(x)$, we have

$$\int_{B_r(x)} u \leq \int_{B_r(x)} v_i = \frac{r}{n} \int_{\partial B_r(x)} v_i \to \frac{r}{n} \int_{\partial B_r(x)} u;$$

the second equality is equivalent to the monotonicity of the volume means $⨍_{B_r(x)} v_i$, and similarly this implies that

$$⨍_{B_r(x)} u$$

is nondecreasing in r, and so controls $u(x)$.

Some consequences of these facts: first, if u fails to be locally integrable near a point, then it must be $-\infty$ on that entire connected component of Ω (notice that this is also true of local intgrability along circles, perhaps surprisingly). Second, a mean value subharmonic function also satisfies the boundary mean value inequality and the monotonicity of the volume averages (this was not immediately obvious from the definition).

Third, consider any viscosity subharmonic u which is constructed as in Perron's method, as the largest subharmonic function lying below a given function g on $\partial\Omega$. We claim that $u(x) = ⨍_{\partial B_r(x)} u$. Indeed, solve for the harmonic functions v_i as above, with data approximating the data of u on ∂B_r. If $v_i(x) > u(x) + \delta$ for all i (recall that $v_i(x)$ converges to the boundary integral), we may find a v_i with $v_i < u + \delta/4$ on $\partial B_r(x)$. Then $\max\{v_i - \delta/2, u\}$ gives a subharmonic function (from Lemma 2.8) which differs from u only on $B_r(x)$, and yet is larger than it at x, contradicting the maximality of u. Hence u is harmonic (and $u = u_*$). Note that we did not require any information about g or Ω to conclude this, other than a guarantee that u exists and is not $-\infty$ (so, say, g is continuous and bounded).

This is a more traditional approach to Perron's method, and is a bit different from our earlier theorem. It sacrifices some generality and, importantly, it required us to already know how to solve the Dirichlet problem on some given domain. On the other hand, it gives a stronger conclusion: it separates out the question of existence of a harmonic function and the manner in which it satisfies the boundary values, allowing for *local* barrier constructions to be used.

2.2 Dirichlet's Principle

We now take a different perspective on harmonic functions. First, some notation and review:

Definition 2.6. *A* distribution *is a continuous linear functional on $C_c^\infty(\Omega)$. The derivative of a distribution f is defined by*

$$\partial_e f(\phi) = -f(\partial_e \phi).$$

Let $H^1(\Omega)$ represent the Sobolev space of functions in $L^2(\Omega)$ whose distributional derivatives admit representations as $L^2(\Omega)$ functions, equipped with the norm

$$\|u\|_{L^2(\Omega)} + \|\nabla u\|_{H^1(\Omega)}.$$

Let $H_0^1(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$.

Proposition 2.12. *For any bounded $\Omega \subseteq \mathbb{R}^n$, we have that*

$$\|u\|_{L^p} \leq C(\operatorname{diam}\Omega)\|\nabla u\|_{L^2}$$

for $1 \leq p \leq 2* := \frac{2n}{n-1}$. If $p < 2*$, then the embedding $H^1 \to L^p$ is compact: any sequence converging weakly in H^1 will converge strongly in L^p.

$$\|\nabla u\|_{L^2(\Omega)}$$

is an equivalent norm for $H_0^1(\Omega)$.

For a sufficiently regular Ω (say Lipschitz), there is a well-defined, continuous linear map $H^1(\Omega) \to L^2(\Omega)$, called the *trace*. We let $H^{1/2}(\Omega)$ be the image of H^1 under this trace map; it's straightforward to check that $H^{1/2}(\partial\Omega)$ includes smooth functions.

Now consider the variational problem

$$\min\{\frac{1}{2}\int_\Omega |\nabla u|^2 : u \in H^1(\Omega),\ \operatorname{trace}(u) = g\}, \tag{2.3}$$

where Ω is a sufficiently smooth domain and g is given.

Theorem 2.13. *The problem* (2.3) *admits a unique solution u; moreover, u is harmonic.*

Proof. There are several things to check. Let us start with a general way of showing that a minimizer exists. Let

$$\alpha = \inf\{E[u] := \frac{1}{2}\int_\Omega |\nabla u|^2 : u \in H^1(\Omega),\ \operatorname{trace}(u) = g\}$$

and u_k be a sequence of functions in H^1 with

$$\lim_k E[u_k] = \alpha.$$

Let $v \in H^1$ be any function with trace g. Then $w_k = v - u_k \in H_0^1$, and

$$\frac{1}{2}\int_\Omega |\nabla w_k|^2 = E[u_k] + E[v] - \int_\Omega \nabla v \cdot \nabla u_k \leq C.$$

It follows that w_k admits a weakly convergent subsequence, with $\nabla w_k \to \nabla w$ weakly in L^2. This gives

$$E[w] \leq \liminf E[w_k],$$

so setting $u = v - w$, $E[u] = \alpha$ (we used the weak convergence). Moreover, $w \in H^1_0$, so $v \in H^1$ and has trace g; we have obtained a minimizer.

For the uniqueness, let u and v be two minimizers. Then

$$E[\frac{u+v}{2}] = \alpha/2 + \frac{1}{2}\int_\Omega \nabla u \cdot \nabla v \leq \alpha,$$

with equality if and only if $\nabla u = c\nabla v$. This, together with the trace condition, implies $u = v$.

Now we must check that u is harmonic. We will check that $\triangle u = 0$ in the sense of distributions; it is then not difficult to check that u satisfies the mean value property and is hence harmonic (further details and alternative approaches will be discussed later). To do so, take any $\phi \in C^\infty_c(\Omega)$. Then

$$E[u + t\phi] \geq E[u]$$

implies that

$$t\int_\Omega \nabla u \cdot \nabla phi \geq -t^2 E[\phi].$$

Sending t to 0 from either side gives

$$0 = \int_\Omega \nabla u \cdot \nabla phi = \int_\Omega u \triangle phi,$$

which implies the conclusion. □

This suggests the following definitions:

Definition 2.7. *A function in $H^1(\Omega)$ is called* variationally subharmonic *if $E[u] \leq E[\phi]$ for any ϕ with $u - \phi = 0$ on $\partial\Omega$ (in the sense of traces) and $\phi \leq u$.*

Definition 2.8. *A function in $H^1(\Omega)$ is called* weakly subharmonic *if $\int_\Omega \nabla u \cdot \nabla \phi \leq 0$ for any $\phi \in H^1_0(\Omega)$.*

Definition 2.9. *A function in $L^1(\Omega)$ is called* distributionally subharmonic *if $\triangle u \geq 0$ in the sense of distributions.*

An important point to make is that the Laplacian of a function which is distributionally subharmonic defines a nonnegative functional on the space of continuous compactly supported functions, and so $\triangle u$ is given by a positive Borel measure by the Reisz representation theorem. In particular, all steps of the proof of Theorem 2.1 still work in this case (using also the L^1 assumption), and we may deduce that u is mean value subharmonic (the converse is also true). We are deliberately ignoring the upper semicontinuity issue, but the clever reader should be able to show that a distributionally subharmonic function in fact admits an upper semicontinuous representative.

Variational and weak subharmonicity is equivalent, by a variant of the uniqueness argument used above, and both imply distributional subharmonicity. These notions are not, however, equivalent (due to the extra H^1 assumption).

2.3 Green's Functions

Our final perspective on harmonic functions will be through Green's functions and the Newtonian potential. The key premise is the following one: the function

$$\Phi(x) = \begin{cases} c(n)|x|^{n-2} & n > 2 \\ -c(n)\log|x| & n = 2, \end{cases}$$

known as the *fundamental solution* of the Laplace equation, or the Newtonian potential, satisfies $-\triangle\Phi = \delta_0$ (if $c(n)$ is chosen appropriately).

Lemma 2.14. *For any $v \in C_c^\infty(\mathbb{R}^n)$,*

$$\int \Phi \triangle v = -v(0).$$

in particular, v is (distributionally) superharmonic.

Proof. Take any such v. Then we have

$$\int \Phi \triangle v = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} \Phi \triangle v.$$

We compute the derivatives of Φ:

$$\nabla \Phi = c(n) \frac{x}{|x|^n}$$

and

$$\triangle \Phi = 0$$

away from the origin. Using the divergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_\varepsilon} \Phi \triangle v &= \int_{\mathbb{R}^n \setminus B_\varepsilon} \triangle \Phi v + \int_{\partial B_\varepsilon} \frac{x}{|x|} \cdot \nabla \Phi v - \frac{x}{|x|} \cdot \nabla v \Phi \\ &= c(n) \varepsilon^{1-n} \int_{\partial B_\varepsilon} v + O(\varepsilon). \end{aligned}$$

Sending $\varepsilon \to 0$ concludes the argument. □

When solving boundary value problems, it helps to localize the fundamental solution to a function on the given domain; this is called the Green's function:

Definition 2.10. *Let Ω be a domain, and let $G(x, y)$ be a solution to the boundary value problem*

$$-\triangle_x G(x, y) = \delta_y \qquad x \in \Omega G(x, y) = 0 \qquad x \in \partial\Omega.$$

Such a G is called the Green's function *of Ω.*

The regularity assumption in the below is not strictly necessary.

Lemma 2.15. *Let Ω be a bounded open set with Lipschitz boundary. Then there is a unique Green's function G for Ω.*

Proof. The uniqueness follows from the maximum principle applied to the difference.

To prove the existence, one only needs to find (for every y) a function $h(x)$ which is harmonic on Ω and agrees with $\Phi(x - y)$ on $\partial\Omega$. As the latter is clearly in $H^{1/2}(\partial\Omega)$, we may do so via the variational method (note, though, that to show that the resulting function is continuous would require further arguments which we do not pursue here). Alternatively, we may use Perron's method here after showing that Lipschitz domains admit the upper and lower barriers required (this is true, and may be verified as an exersize). □

Let $K(x, y) = \partial_{\nu,x} G(x, y) : \partial\Omega \times \Omega \to \mathbb{R}$; this is usually known as the *Poisson kernel.* At least if the domain is sufficiently smooth, this is a well-defined quantity, and can be used to solve the Dirichlet problem. We are not going to explore the smoothness requirements in depth here; the main point is that the divergence theorem must apply. A careful reader may in fact verify the identity below for Lipschitz (or weaker) domains.

Lemma 2.16. *Let Ω be a bounded, open, smooth domain, and g be a continuous function on $\partial\Omega$. Then*

$$u(x) = \int_{\partial\Omega} K(x, y) g(y) dy$$

solves

$$\begin{cases} \triangle u = 0 \qquad \text{in } \Omega u = g \qquad \text{in } \partial\Omega \end{cases}$$

The boundary condition may be interpreted in $H^{1/2}$ sense.

The boundary condition may also be interpreted in the sense of continuous functions, with a bit more work (which we delay until later).

Proof. Let ϕ be any smooth function on $\bar{\Omega}$. Then we have that

$$\int_{\Omega} \triangle_x G(x,y)\phi(x) - G(x,y)\triangle\phi(x)dx = \int_{\partial\Omega} \partial_\nu G(x,y)\phi(x) - \partial_\nu\phi(x)G(x,y),$$

which gives

$$\phi(x) - \int_{\Omega} G(x,y)\triangle\phi(x)dx = \int_{\partial\Omega} \partial_{\nu,x}G(x,y)\phi(x).$$

Let v be the solution to the boundary value problem in question. Replacing v by v_ε, which is a sequence of smooth functions converging to v in $H^1(\Omega) \cap C^2_{\text{loc}}(\Omega)$ (this may be arranged by a standard density theorem on a small region around $\partial\Omega$ and by mollifying directly near $\partial\Omega$), we have that

$$v(x) = \int_{\partial\Omega} K(x,y)g(y)dy.$$

This implies the conclusion. □

A similar argument shows that $G(x,y) = G(y,x)$.

We close here to mention the following construction: given a smooth Ω and a continuous function g on $\partial\Omega$, we may define a funcional from $C(\partial\Omega)$ to $\mathbb{R}$, ω_x, via

$$\omega_x(g) = h_g(x),$$

where h_g is the solution to the Laplace equation on Ω with boundary values g. Then so long as $g \geq 0$, so is h, meaning that this is a positive functional. By the Reisz representation theorem, this means that Ω_x is given by a Borel measure supported on $\partial\Omega$. This measure is called the *harmonic measure*. The above lemma shows that the Radon-Nikodym derivative of ω_x with respect to surface measure on $\partial\Omega$ is given by the Poisson kernel $K(\cdot, x)$. This notion is easy to generalize to all situations where one can solve the Dirichlet problem (and, in fact, to arbitrary domains, via either limiting arguments or by using Perron's method directly).