Worksheet 5
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Entire Class

1. Let's return to citrus fruits but in a different context.
    a. How many ways are there to distribute five citrus fruits of different types (a pomelo, a blood orange, a satsuma mandarin, a tangelo, and a calamansi) into three different-colored baskets (cerulean, lavender, and magenta) without any restrictions on number of fruits per basket?

    b. What about 7 different-colored baskets?

    c. What about 7 different-colored baskets such that each basket has at most one fruit in it?

2. Let's generalize: suppose we have k *labelled* fruits that we want to put in n *labelled* baskets.
   (We use *labelled* to mean we can tell them apart, whereas *unlabelled* will mean the same as *identical.* Some people might call these *distinguishable* vs *indistinguishable.)*

    a. How many ways are there to do this? What if we want each basket to have at most one fruit in it?

    b. Let's formalize: rewrite your conclusions about fruits in baskets as statements about sets and functions. What type of function corresponds to each basket having at most one fruit? If we wanted each basket to have at *least* one fruit, what types of functions are these?

3. Now let's consider a slightly different situation:
    a. How many ways are there to distribute the same five citrus fruits as in Problem 1 into two *identical* baskets, all colored chartreuse? (it may help to list out some examples)
    b. What about five citrus fruits into three identical baskets?

Let X be a set with n elements. A ***partition*** of X is a distribution of its elements into nonempty subsets. Formally, a partition is a set $\{A_1, A_2, ..., A_m\}$of nonempty(!) subsets of X that are disjoint (meaning $A_i \cap A_j = \emptyset$) and such that their union $\bigcup_{i=1}^{m} A_i$ is all of X.

Note that the order of a partition does not matter: {{a}, {b, c}, {d}} is the same partition as {{b, c}, {a}, {d}} of the set X = {a, b, c, d}. So we can think of the parts as being unlabelled.

Let **S(r, t)** (where S stands for “citruS numbers”) denote the number of partitions of an r-element set into t parts. The real name for S(r, t) is “Stirling numbers of the second kind.”

4.
   a. Rewrite 3(a) and 3(b) as questions about partitions of a set.

   b. Now generalize: how many ways are there to distribute k labelled fruits into n identical/unlabelled baskets? (your answer should be in terms of citruS numbers)

   c. What if each basket gets at most 1 fruit? At least 1 fruit?

5. Let’s consider yet another fruit distribution problem.
   a. How many ways are there to distribute five identical kiwi fruits into three different-colored baskets?

   b. What about four identical kiwi fruits into three different-colored baskets?

6. Let's leave behind the fruits and baskets for now. Suppose you have five quarreling sheep and you want to use fences to create three separate pastures. One such fencing might look like this:

   🐑|🐑🐑|🐑🐑 or 🐑🐑|🐑🐑|🐑 <- these two fencings are different.
   You could also do this:
   |🐑🐑🐑🐑|🐑 <- The fence on the left side indicates that there is an empty pasture to the left - maybe you want to save some grass for a garden.
   Or even:
   🐑||🐑🐑🐑🐑 <- Now your empty pasture is in the middle.
   Note that this is allowed:
   ||🐑🐑🐑🐑🐑 <- You do have three pastures. Two are empty. This doesn't address the sheep quarreling, but hey, you're the farmer.

   a. How many ways can you fence your sheep? (at least five apparently…)

   b. Now suppose you have k sheep and you want to separate them into n different pastures. How many ways are there to do this?

      _ _ _ _ _ _ _ _ -> 🐑 | 🐑 🐑 | 🐑 🐑 *...mysterious picture....*

7. Are Problems 5 and 6 related? Come up with a relationship and explain why it works.

8. Using the previous problem, how many ways are there to distribute k identical fruits into n labelled baskets such that each basket has at most one fruit? What about at least one fruit? What about exactly one fruit? Give the corresponding sheep/pasture scenarios for each question. (Which one corresponds to the sheep practicing social distancing?)

Stars and Bars Method:

9. Let’s summarize by making the connection to functions. Let X be a k-element set and Y be an n-element set. Fill out the following chart with the number of $f: X \to Y$. The column says what type of function f is, and the row says whether X and Y are distinguishable or indistinguishable.

	functions	injections	surjections	bijections
X and Y labelled				
X labelled, Y unlabelled				
X unlabelled, Y labelled				

Note that there is one scenario we still haven’t considered: both X and Y unlabelled. This type of problem is beyond what we will cover, but some of the optional reading will discuss it if you’re curious.

This chart is commonly called “The Twelvefold Way” where the 12 usual entries don’t include the bijections column but do include a row for X, Y both unlabelled.