Worksheet 5
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Entire Class

Summary:

- We learned the number of ways to distribute fruits into baskets in the situations where (1) the fruits and baskets are labelled, (2) the fruits are labelled, the baskets are unlabelled, (3) the fruits are unlabelled, the baskets are labelled. The summary is at the bottom of the worksheet.
- For type (2), we learned about partitions and citruS number S(r, t) aka “Stirling numbers of the second kind.”
- For type (3), we learned about the stars and bars method of counting.

1. Let’s return to citrus fruits but in a different context.
    a. How many ways are there to distribute five citrus fruits of different types (a pomelo, a blood orange, a satsuma mandarin, a tangelo, and a calamansi) into three different-colored baskets (cerulean, lavender, and magenta) without any restrictions on number of fruits per basket?

    b. What about 7 different-colored baskets?

    c. What about 7 different-colored baskets such that each basket has at most one fruit in it?

(a) Each fruit has five choices, so the answer is 3^5.

(b) Similar to part (a), the answer is 7^5

(c) The first fruit has 7 choices, then the second fruit has 6 choices, and so on. So the answer is P(7, 5) = 7(6)(5)(4)(3).

2. Let’s generalize: suppose we have k *labelled* fruits that we want to put in n *labelled* baskets.
(We use *labelled* to mean we can tell them apart, whereas *unlabelled* will mean the same as *identical.* Some people might call these *distinguishable* vs *indistinguishable.)*

    a. How many ways are there to do this? What if we want each basket to have at most one fruit in it?

b. Let's formalize: rewrite your conclusions about fruits in baskets as statements about sets and functions. What type of function corresponds to each basket having at most one fruit? If we wanted each basket to have at *least* one fruit, what types of functions are these?

(a) Each fruit has n choices, so the answer is n^k

(b) Fruits = domain, baskets = codomain
At most one fruit: injective function (every basket has at most one fruit mapped to it)
At least one fruit: surjective function (every basket is mapped to at least once)
We've shown: number of functions from [k] to [n] is n^k
Number of injective functions from [k] to [n] is P(n, k).

3. Now let's consider a slightly different situation:
   a. How many ways are there to distribute the same five citrus fruits as in Problem 1 into two *identical* baskets, all colored chartreuse? (it may help to list out some examples)
   b. What about five citrus fruits into three identical baskets?

(a) Since the baskets are identical, all that matters is the number of fruits per basket and which fruits those are. The different divisions for numbers of fruit per basket are {5, 0}, {4, 1}, {3, 2}. To get {5, 0} there's one possibility: all fruits go into one basket. For {4, 1}, there are $\binom{5}{1}$possibilities since we choose which fruit goes into a basket by itself. For {3, 2}, there are similarly $\binom{5}{2}$ possibilities.

(b) Now the different divisions are {5, 0, 0}, {4, 1, 0}, {2, 2, 1}, {3, 1, 1}, {3, 2, 0}. We similarly need to calculate how many different ways there are to achieve each. E.g. the possible ways to get {2, 2, 1} includes {{a, b}, {c, d}, {e}}, {{a, c}, {b, d}, {e}}, etc.

Let X be a set with n elements. A ***partition*** of X is a distribution of its elements into nonempty subsets. Formally, a partition is a set $\{A_1, A_2, ..., A_m\}$of nonempty(!) subsets of X that are disjoint (meaning $A_i \cap A_j = \emptyset$) and such that their union $\bigcup_{i=1}^{m} A_i$ is all of X.

Note that the order of a partition does not matter: {{a}, {b, c}, {d}} is the same partition as {{b, c}, {a}, {d}} of the set X = {a, b, c, d}. So we can think of the parts as being unlabelled.

Let **S(r, t)** (where S stands for "citruS numbers") denote the number of partitions of an r-element set into t parts. The real name for S(r, t) is "Stirling numbers of the second kind."

4.

a. Rewrite 3(a) and 3(b) as questions about partitions of a set.

b. Now generalize: how many ways are there to distribute k labelled fruits into n identical/unlabelled baskets? (your answer should be in terms of citruS numbers)

c. What if each basket gets at most 1 fruit? At least 1 fruit?

(a) 3a: How many ways are there to partition a 5-element set into at most 2 parts? Answer: S(5, 2) + S(5, 1)
3b: How many ways are there to partition a 5-element set into at most 3 parts? Answer: S(5, 3) + S(5, 2) + S(5, 1)

(b) The number of ways to distribute k fruits into n identical baskets is

$$S(k,\ n)\ +\ S(k, n\ -\ 1)\ +....+\ S(k, 1)\ =\ \sum_{i=1}^{n} S(k, i).$$

(c) If each basket gets at least one fruit, that means none of the baskets are empty so we are dividing the fruits into exactly k parts. The number of ways to do this is $S(k, n)$.
If each basket gets at most one fruit, and $n \geq k$, there is only one possibility: there are k baskets with exactly 1 fruit and n-k with no fruits.
If $n\ <\ k,$ there are 0 ways because it's not possible to have at most one fruit in each basket.

5. Let's consider yet another fruit distribution problem.
   a. How many ways are there to distribute five identical kiwi fruits into three different-colored baskets?

   b. What about four identical kiwi fruits into three different-colored baskets?

What matters here is how many fruits are in each basket e.g. we count the distribution (3, 1, 1) exactly once. But (3, 1, 1) is not the same as (1, 3, 1).

6. Let's leave behind the fruits and baskets for now. Suppose you have five quarreling sheep and you want to use fences to create three separate pastures. One such fencing might look like this:

   🐑|🐑🐑|🐑🐑 or 🐑🐑|🐑🐑|🐑 <- these two fencings are different.
   You could also do this:
   |🐑🐑🐑🐑|🐑 <- The fence on the left side indicates that there is an empty pasture to

the left - maybe you want to save some grass for a garden.
Or even:
🐑||🐑🐑🐑🐑 <- Now your empty pasture is in the middle.
Note that this is allowed:
||🐑🐑🐑🐑🐑 <- You do have three pastures. Two are empty. This doesn't address the sheep quarreling, but hey, you're the farmer.

a. How many ways can you fence your sheep? (at least five apparently…)

b. Now suppose you have k sheep and you want to separate them into n different pastures. How many ways are there to do this?

_ _ _ _ _ _ _ ->  ...mysterious picture....

(a) In order to divide the five sheep into three pastures, we can think about placing 2 fenceposts somewhere among the sheep. If we represent the sheep and posts as 7 blanks, this is the same as choosing 2 spots for the fenceposts to be and the rest are sheep.
So the number of ways to place the fence posts is $\binom{7}{2}$. (alternatively, you can think of choosing 5 spots for the sheep, and we know that $\binom{7}{5}$is the same as $\binom{7}{2}$).

(b) For n pastures and k sheep, you need n-1 fence posts, so there are k+(n-1) spots where you want to place n-1 fence posts, which means there are $\binom{k+(n-1)}{n-1}$ possibilities for fencing the sheep.

7. Are Problems 5 and 6 related? Come up with a relationship and explain why it works.

(Problem 5: How many ways are there to distribute five identical kiwi fruits into three different-colored baskets? What about four identical kiwi fruits into three different-colored baskets?)

You have five kiwis and three baskets, so you can think about the kiwis as sheep and the baskets as pastures. Use fences to separate the kiwis into the baskets: again you have 7 places to put the two separating lines, so the answer is $\binom{7}{2}$.

Similarly, for four kiwis and three baskets, you have four items to divide using 2 lines, so there are $\binom{4+2}{2} = \binom{6}{2}$possibilities.

8. Using the previous problem, how many ways are there to distribute k identical fruits into n labelled baskets such that each basket has at most one fruit? What about at least one fruit? What about exactly one fruit? Give the corresponding sheep/pasture scenarios for each question. (Which one corresponds to the sheep practicing social distancing?)

By the previous problem, there are $\binom{k+(n-1)}{n-1}$ ways to distribute k identical fruit into n labelled baskets.
If there has to be at least one fruit in each basket, we can assume every basket has one fruit so we still have $k - n$ fruits left to distribute into the n baskets. We can view this as a sheep/pasture scenario where we put the remaining k-n sheep into n pastures. The number of ways to do this is $\binom{(k-n)+(n-1)}{n-1}$which is the same as $\binom{k-1}{n-1}$.

If there is at most one fruit in each basket or exactly one fruit in each basket...we will discuss this tomorrow after you've done DP5 :)

Stars and Bars Method: reframing a counting problem as arranging k stars and n bars in a line. We use it when we want to count the number of ways to distribute a bunch of identical objects (stars) into a bunch of distinguishable groups, or when we want to arrange two different types of objects in a line.

9. Let's summarize by making the connection to functions. Let X be a k-element set and Y be an n-element set. Fill out the following chart with the number of $f: X \to Y$. The column says what type of function f is, and the row says whether X and Y are distinguishable or indistinguishable.

	functions	injections	surjections	bijections
X and Y labelled (Problem 1, 2)	n^k	$n(n-1)\cdots(n-k+1)$ if $k \le n$, 0 else	$n!\,S(k,n)$	$n! = k!$ if k=n, 0 else
X labelled, Y unlabelled (Problem 3,4)	$\sum_{i=1}^{n} S(k,i)$	1 if $k \le n$, 0 else	$S(k,n)$	$1\ if\ k = n,$ 0 else
X unlabelled, Y labelled (Problem 5-8)	$\binom{k+(n-1)}{n-1} = \binom{k+n-1}{k}$	$\binom{n}{k}$	$\binom{(k-n)+(n-1)}{n-1} = \binom{k-1}{n-1}$	$1\ if\ k = n,$ 0 else

Note that there is one scenario we still haven't considered: both X and Y unlabelled. This type of problem is beyond what we will cover, but some of the optional reading will discuss it if you're curious.

This chart is commonly called "The Twelvefold Way" where the 12 usual entries don't include the bijections column but do include a row for X, Y both unlabelled.