Quantum K-theory of geometric invariant theory quotients

AMS Summer Institute in Algebraic Geometry, 2015

University of Utah, Salt Lake City

Chris Woodward, Rutgers University New Brunswick, work in progress with Eduardo Gonzalez, U. Massachusetts Boston.

Outline

I. Review of quantum K-theory and existing computations

II. Description of quantum Kirwan map in quantum K-theory.

III. Applications to quantum K-theory of toric DM stacks, abelianization.

Outline

I. Review of quantum K-theory and existing computations

II. Description of quantum Kirwan map in quantum K-theory.

III. Applications to quantum K-theory of toric DM stacks, abelianization.

Scrabble points: Q 10 points, J, X 8 points, K 5 points, ψ 10 points.

MUMFORD'S GIT

G complex reductive group

X smooth polarized projective (or quasiprojective) G-variety

X^{ss} semistable locus

$X /\!\!/ G = X^{\mathrm{ss}}/G$ assume locally free, stack-theoretic git quotient: proper smooth Deligne-Mumford stack with projective coarse moduli space

Example: $X = \mathbb{C}^n, G = \mathbb{C}^\times, X^{\mathrm{ss}} = (\mathbb{C}^n) - \{0\}, X /\!\!/ G = \mathbb{P}^{n-1}$.

Kirwan's map

$K^G(X)$ *equivariant K-homology group*: Grothendieck group of coherent G-sheaves on X.

$K_G(X)$ *equivariant K-cohomology ring* Grothendieck group of G-equivariant vector bundles on X, isomorphic via duality to $K^G(X)$

Kirwan map: $\kappa_X^G : K_G(X) \to K(X/\!\!/G), \quad [E] \mapsto [(E|X^{ss})/G]$

Can use either algebraic or topological K-theory.

Kirwan surjectivity: Kirwan, [22], Harada-Landweber [17, Theorem 3.1], Halpern-Leistner [16, Corollary 1.2.3]:

EXAMPLE: K-THEORY OF TORIC VARIETIES

Suppose that X is a vector space, G a torus, $X/\!\!/G$ a smooth toric DM stack with projective coarse moduli space

$K_G(X) = \mathrm{Rep}(G)$ representation ring

$[X_i] \in K_G(X)$ weight space of i-th factor $X_i \subset X$

$\mathcal{O}_{D_i}$ structure sheaf for i-th prime invariant divisor $D_i \subset X/\!\!/G$

Then $\kappa_X^G(1 - [X_i]^{-1}) = [\mathcal{O}_{D_i}]$

If $D_{i_1} \cap \ldots \cap D_{i_k} = \emptyset$ then

$$\begin{aligned} \prod_{j=1}^{k}(1 - [X_{i_j}]^{-1}) &= \prod_{j=1}^{k}[\mathcal{O}_{D_{i_j}}] \\ &= [\mathcal{O}_{D_{i_1}\cap\ldots\cap D_{i_k}}] = 0. \end{aligned}$$

These are all the relations in the K-theory by Vezzosi-Vistoli [30, Section 6.2].

K-THEORY OF TORIC VARIETIES CONTINUED

Relations: if $D_{i_1} \cap \ldots \cap D_{i_k} = \emptyset$ then $\prod_{j=1}^{k}(1 - [X_{i_j}]^{-1}) = 0$.

For the projective line $\mathbb{P}^1 = \mathbb{C}^2 /\!/ \mathbb{C}^\times$ both weights are one and the prime divisors are the points at 0 and ∞.

$\{0\} \cap \{\infty\} = \emptyset$ implies

$$K(\mathbb{P}^1) = K(\mathbb{C}^2 /\!/ \mathbb{C}^\times) = \mathbb{Z}[L, L^{-1}]/(1 - L^{-1})^2.$$

Extension to stacks by Borisov-Horja [2]

Quantum K-theory

$\overline{\mathcal{M}}_{g,n}(X, d)$ moduli of stable n-pointed genus g maps of class d

$[\mathcal{O}^{\text{vir}}]$ class of the virtual structure sheaf on $\overline{\mathcal{M}}_{g,n}(X, d)$. (Y.P. Lee [23])

K-theoretic Gromov-Witten invariants

$$\begin{aligned} \tau_{g,n,d} : K_G(X)^{\otimes n} &\to K_G(pt), \\ (\alpha_1, \ldots, \alpha_n) &\mapsto \chi_G(\text{ev}_1^* \alpha_1 \otimes \ldots \otimes \text{ev}_n^* \alpha_n \otimes [\mathcal{O}^{\text{vir}}]) \end{aligned} \quad (1)$$

Do not satisfy Behrend-Manin axioms: splitting axiom has corrections, no divisor axiom.

equivariant quantum K-theory $QK_G(X) := K_G(X) \otimes \Lambda$ where Λ is some Novikov ring (various choices).

Quantum product

K-theoretic genus zero *Gromov-Witten potential*

$$\mathcal{F} : QK_G(X) \to QK_G(pt), \quad \alpha \mapsto \sum_{d \in H_2(X)} \sum_{n \geq 0} \tau_{0,n,d}(\alpha, \ldots, \alpha) q^d / n!.$$

quantum K-theory pairing $B_\alpha(\beta, \gamma) = \partial_1 \partial_\beta \partial_\gamma \mathcal{F}(\alpha)$.

no grading so B_α **is not constant in** α**!**

quantum K-theory product: $B_\alpha(\beta \star_\alpha \gamma, \kappa) = \partial_\beta \partial_\gamma \partial_\kappa \mathcal{F}(\alpha)$.

Quantum connection $\nabla_\alpha = (1 - \zeta)\partial_\alpha + \alpha\star$ is flat (Givental).

$QK_G(X)$ is a Frobenius manifold but the identity is not flat.

Sample computations

Givental-Lee: K-theoretic graph potential for complete flag varieties

Buch-Mihalcea [3]: computation of three-point invariants for projective spaces and Grassmannians (Anders has a computer program)

Givental-Tonita [9] Iritani-Milanov-Tonita [19]: more computations

Taipale [28]: Qde fundamental solution for partial flag varieties

Gorbounov-Korff [15]: Presentation of quantum cohomology of the Grassmannian.

Maulik-Okounkov [25]: Connections with rep'n theory

Givental June 2015 [10]: permutation-equivariant quantum K-theory I-IV *(Google scholar notification)*

MAIN RESULT: QUANTUM KIRWAN MAP

Thm (G-W) There exists a canonical formal *Kirwan map* in quantum K-theory

$$\kappa_X^G : QK_G(X) \to QK(X/\!\!/G)$$

with the property that the linearization $D_\alpha \kappa_X^G$ is a homomorphism:

$$D_\alpha \kappa_X^G(\beta \star \gamma) = D_\alpha \kappa_X^G(\beta) \star D_\alpha \kappa_X^G(\gamma).$$

If $X/\!\!/G$ is a free quotient (conj in general) $D_\alpha \kappa_X^G$ is surjective for generic α

BACKGROUND: MAPS TO QUOTIENT STACKS

One wants a good notion of stable map to a quotient stack X/G.

Problem: There are two natural stability conditions, corresponding to stability on X and stability of maps to $*/G = BG$ as in Narasimhan-Seshadri-Ramanathan.

Idea: build into the moduli space a datum (one-form) which tells you which stability condition to consider. (Mathematical version of "area-dependent field theory" in physics").

The *quasimaps* of Ciocan-Fontanine-Kim-Maulik [4] etc. are a different version, which enforces X-stability at the special points; works in all genera; related to vortices on curves with cylindrical ends by Venugapalan [31].

AFFINE GAUGED MAPS

Defn: A *affine gauged map* to a quotient stack X/G consists of

a projective nodal curve C of genus zero

non-singular, distinct points $z_1, \ldots, z_n \in C$

a map $u : C \to X/G$

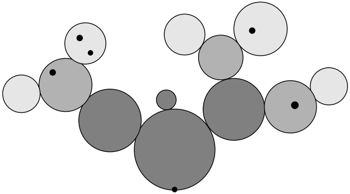
a one-form $\lambda : \; C \to \mathbb{P}(\omega_C \oplus \mathbb{C})$.

Monotonicity and stability conditions

(Monotonicity) λ only has double poles, and is "decreasing" from z_0 to any z_i in the sense that it is $\infty, \ldots, \infty$, finite, $0, \ldots, 0$.

(Stability) u is X-stable on $\lambda^{-1}(\infty)$ and G-stable on $\lambda^{-1}(0)$

Any component on which u is trivializable has at least three special points, or two special points and non-zero, finite λ.



Darkly shaded: $\lambda^{-1}(\infty)$ Lightly shaded: $\lambda^{-1}(0)$.

Moduli spaces of affine gauged maps

$\overline{\mathcal{M}}^G(\mathbb{A}, X)$ moduli stack of stable affine gauged maps is a Deligne-Mumford stack with perfect obstruction theory.

The combinatorial type of such an affine map is a *colored tree* Γ in the language of Boardman-Vogt from the 1960's.

$\overline{\mathcal{M}}^G(\mathbb{A}, X) = \cup_\Gamma \mathcal{M}^G_\Gamma(\mathbb{A}, X)$

$\overline{\mathcal{M}}^G(\mathbb{A}, X, d)$ locus of homology class $d \in H^G_2(X, \mathbb{Q})$.

PROPERNESS

Like many similar theories, once one has the right definitions via symplectic geometry one can try to kill the symplectic geometry completely. I will only talk about the algebraic story.

(Originally we wanted the symplectic approach because it is more general. E.g. Wang-Xu arXiv:1505.05945 does compactness for vortices with Lagrangian boundary conditions)

VALUATIVE CRITERION FOR PROPERNESS

$(C, \underline{z}, \lambda, u : C \to X/G) \to S - \{s_0\}$ family over the punctured smooth curve. Want unique extension over central fiber.

Define $\overline{u} : u^{-1}(X^{ss}) \to X /\!\!/ G$. Note $u^{-1}(X^{ss}) \supset \{z_0\} \neq \emptyset$

$\underline{w} = u^{-1}(X - X^{ss})$ the *base points* with some ordering.

Properness of $X /\!\!/ G$: $(\overline{C}, \overline{z} \cup \overline{w}, \overline{u} : \overline{C} \to X /\!\!/ G)$ extends as a stable map (after étale cover) to central fiber.

Canonicity: the one-form λ extends over the central fiber, so that $\lambda(z_i) < \infty, i > 0$.

Extending the bundle

Collapse components of $\overline{C}$ on which $\lambda = 0$.

The bundle $P := \overline{u}^*(X^{ss} \to X /\!\!/ G)$ extends canonically except over the limits of the base points.

Colliot-Thélène and Sansuc [6], see also Ciocan-Fontanine et al [4], Solis [27] implies the bundle extends uniquely over the limits of the base points.

Once one has the bundle extension, u extends as a stable section of $P \times_G X$ by properness of C, X.

Definition of quantum Kirwan

$\bar{I}_{X/\!\!/G}$ rigidified inertia stack of $X/\!\!/G$

$\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1, \ldots, \mathrm{ev}_n) : \overline{\mathcal{M}}_n^G(\mathbb{A}, X) \to \bar{I}_{X/\!\!/G} \times (X/G)^n$
evaluation maps at $z_0, \ldots, z_n$

$M : QK(X/\!\!/G) \to QK(X/\!\!/G)$ Maurer-Cartan map combining all multiplications.

$$\kappa_X^G : QK_G(X) \to QK(X/\!\!/G),$$
$$\alpha \mapsto M^{-1} \sum_{d,n} (q^d/n!)\, \mathrm{ev}_{0,*}^d(\mathrm{ev}_1^* \alpha \otimes \ldots \otimes \mathrm{ev}_n^* \alpha). \quad (2)$$

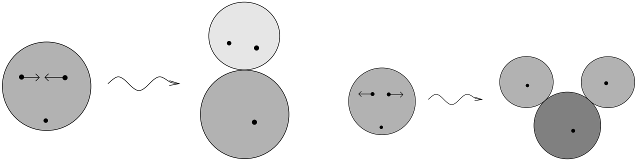
The homomorphism property

The linearized map $D_\alpha \kappa_X^G : QK_G(X) \to QK(X/\!\!/G)$ is a $\star$-homomorphism:

$$D_\alpha \kappa_X^G(\beta \star_\alpha \gamma) = D_\alpha \kappa_X^G(\beta) \star_{\kappa_X^G(\alpha)} D_\alpha \kappa_X^G(\gamma)$$

for any $\alpha \in QK_G(X)$.

Proof by divisor equivalence: $\overline{\mathcal{M}}_2(\mathbb{A}) \cong \mathbb{P}^1$ via the "distance" between z_1 and z_2.



Application: quantum K-theory of toric varieties, toric DM stacks

G torus

X vector space with weight spaces X_j with weights $\mu_j, j = 1, \ldots, k$. Define

$$\zeta_+(d) = \prod_{\mu_j(d) \geq 0} (1 - [X_j]^{-1})^{\mu_j(d)}$$

$$\zeta_-(d) = q^d \prod_{\mu_j(d) \leq 0} (1 - [X_j]^{-1})^{-\mu_j(d)}$$

Thm: $(QK(X /\!\!/ G), \star_{\kappa_X^G(0)})$ is generated by $[X_1]^{\pm}, \ldots, [X_k]^{\pm}$ with K-theoretic Batyrev relations $\zeta_+(d) - \zeta_-(d), d \in H_2^G(X)$.

(N.B. there is a formal completion I am not discussing.)

USEFUL OR USELESS?

The indeterminacy $\kappa_X^G(0)$ is the *canonical bulk deformation* depending on the presentation of $X /\!\!/ G$ as git quotient.

In the cohomology case, it vanishes for all Fano varieties, so one gets a presentation of small quantum cohomology.

In K-theory, $\kappa_X^G(0)$ might not vanish even for Fano varieties (no grading.) $\kappa_X^G(0)$ does vanish for weighted projective spaces, so one at least gets an explicit result in that case.

Proof is an Euler class computation

The proof does not use any hypergeometric functions, any symplectic geometry, or any formulas for fundamental qde solutions.

Let $E_+ := \bigoplus_{\mu_j(d)\geq 0} \mathrm{ev}_1^* X_j^{\oplus \mu_j(d)}$ vector bundle over $\mathcal{M}_1^G(\mathbb{A}, X)$.

Section $\sigma : \mathcal{M}_1^G(\mathbb{A}, X) \to E_+$ given by $u \mapsto$ derivatives of u_j at z_1 up to order $\mu_j(d)$.

Derivatives makes sense because of λ! The zeroes of the section are isomorphic moduli spaces of maps of lower degree:

$$\begin{array}{ccc} \sigma^{-1}(0) & \longrightarrow & \overline{\mathcal{M}}_1^G(\mathbb{A}, X, d') \\ \big\downarrow & & \\ \overline{\mathcal{M}}_1^G(\mathbb{A}, X, d'-d) & & \end{array}$$

PROOF IS AN EULER CLASS COMPUTATION

Let $E_+ := \bigoplus_{\mu_j(d)\geq 0} \mathrm{ev}_1^* X_j^{\oplus \mu_j(d)}$ vector bundle over $\mathcal{M}_1^G(\mathbb{A}, X)$.

Section $\sigma : \mathcal{M}_1^G(\mathbb{A}, X) \to E_+$ given by $u \mapsto$ derivatives of u_j at z_1 up to order $\mu_j(d)$.

Derivatives makes sense because of λ! The zeroes of the section are isomorphic to moduli spaces of maps of lower degree:

poly's of degree d′ vanishing to order d $\longrightarrow$ poly's of degree d′

$\downarrow$

poly's of degree d′ - d

Euler class computation continued

poly's of degree d′ vanishing to order d ⟶ poly's of degree d′

↓

poly's of degree d′ - d

.

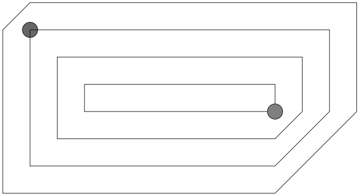
The class $\zeta_+(d)$ is the Euler class of E_+ whose zero set is on upper left so

$$\begin{aligned}\chi(\mathcal{M}_1^G(\mathbb{A}, X, d'), \mathrm{ev}_1^* \zeta_+(d) \otimes \mathcal{O}_{d'}^{\mathrm{vir}}) &= \chi(\sigma^{-1}(0), \mathcal{O}_{d'}^{\mathrm{vir}}) \\ &= \chi(\mathcal{M}_1^G(\mathbb{A}, X, d'-d), \mathrm{ev}_1^* \zeta_-(d) \otimes \mathcal{O}_{d'-d}^{\mathrm{vir}}) \quad (3)\end{aligned}$$

for any degree d'.

Injectivity part of proof

Using the mmp we get an equality of dimension [20],
$\dim(QK_G(X)/\text{ relations}) = \dim(QK(X/\!\!/G))$.



Mmp running for a twice blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

This shows that $QK_G(X)/\text{ relations} = QK(X/\!\!/G)$.

CONSEQUENCES

(i) The quantum K-theory and quantum cohomology of toric DM stacks are isomorphic over $\mathbb{Z}$.

(ii) Mirror description: $QK(X/\!\!/G) = \text{Jac}(W)$ where W is the Givental potential and $\text{Jac}(W)$ its Jacobian ring.

(iii) Toric DM stacks related by crepant transformations have isomorphic QK. (True in general by Givental-Tonita [9]?)

(iv)$QK(X/\!\!/G) = \bigoplus_{Z_i} QK(Z_i)$ where Z_i ranges over centers of mmp transitions (True in general?)

Gauged K-theoretic Gromov-Witten invariants

Goal: "formula" for fundamental qde solution in quantum K-theory

A *gauged map* from C to X consists of

- (Curve) a nodal projective curve $\hat{C}$
- (Markings) $z_0, \ldots, z_n \in \hat{C}$ disjoint from each other and the nodes;
- (Parametrization) a stable map $v : \hat{C} \to C$ of degree $[C]$;
- (Map) a map $\hat{C} \to X/G$ to the quotient stack X/G, corresponding to a bundle $P \to C$ and section $u : \hat{C} \to (v^*P)(X)$

Mundet semistability

Mundet semistability interpolates between bundle stability and target stability: $u : C \to X/G$ morphism.

σ parabolic reduction of $u^*(X \to X/G)$

ξ is a central weight of the corresponding Levi subgroup

Mundet weight sum of the Ramanathan and Hilbert-Mumford weights

$$\mu^M(\sigma, \xi) = \mu^R(\sigma, \xi) + \lambda \mu^{HM}(\sigma, \xi).$$

$u : \hat{C} \to C \times X/G$ is *Mundet semistable* if

$$\mu^M(\sigma, \xi) \leq 0$$

for all pairs (σ, ξ).

$\overline{\mathcal{M}}_n^G(C, X, d)$ stack of Mundet semistable maps.

Gauged graph invariants

For $\alpha_1, \ldots, \alpha_n \in K_G(X)$ and $d \in H_2^G(X)$ define

$$\tau^G_{X,n,d}(C, \alpha_1, \ldots, \alpha_n) = \chi(\overline{\mathcal{M}}^G_n(C, X, d), \mathrm{ev}_0^* \alpha_0 \otimes \mathrm{ev}_1^* \alpha_1 \otimes \ldots \mathrm{ev}_n^* \alpha_n \otimes [\mathcal{O}^{\mathrm{vir}}]) \quad (4)$$

$$\tau^G_X : QK_G(X) \to \Lambda, \quad \alpha \mapsto \sum_{n \geq 0} \sum_{d \in H_2^G(X,\mathbb{Z})} (Q^d / n!) \tau^G_{X,n,d}(C, \alpha, \ldots, \alpha). \quad (5)$$

This is the K-theoretic analog of the "G-function" in Givental's paper.

Localized graph invariants

Localization in the case that $\mathbb{C} = \mathbb{P}^1$ gives what Givental calls I-functions

$$\tau^G_{X,\pm} : QK_{G\times\mathbb{C}^\times}(X) \to QK_{\mathbb{C}^\times}(X /\!\!/ G)$$

Example: In the toric case

$$\tau^G_{X,-}(\alpha,\zeta,q) = \sum_{d\in H_2^G(X)} q^d \exp\left(\frac{\Psi_d\alpha}{1-\zeta^{-1}}\right) \frac{\prod_{j=1}^k \prod_{m=-\infty}^0 (1 - X_j^{-1}\zeta^m)}{\prod_{j=1}^k \prod_{m=-\infty}^{\mu_j(d)} (1 - X_j^{-1}\zeta^m)}. \tag{6}$$

where $\Psi_d\alpha$ is the twist of α by the character d followed by classical Kirwan

ADIABATIC LIMIT THEOREM

The diagram

$$\begin{array}{ccc} QK_G(X) & \xrightarrow{\kappa_X^G} & QK(X/\!\!/G) \\ & {\scriptstyle \tau_X^G} \nwarrow \quad \nearrow {\scriptstyle \tau_{X/G}} & \\ & \Lambda_X^G & \end{array}$$

commutes in the limit $\rho \to \infty$.

For the localized graph potentials $\tau_{X/\!\!/G,\pm} \circ \kappa_X^G = \tau_{X,\pm}^G$

Generalizes formulas of Ciocan-Fontanine-Kim [5], Iritani [21] (big I function in QH) and Givental-Tonita [9] (QK of CI of low degree) and Givental [10] 2015 "*toric q-hypergeometric series [$\alpha = 0$] represents a value of the big J-function in the quantum K-theory*"

WALL-CROSSING

Suppose that $L_\pm \to X$ are two polarizations. Difference between K-theoretic Gromov-Witten graph potentials on $X/\!\!/_\pm G$ is measured by an explicit sum of wall-crossing terms:

$$\begin{array}{ccccc}
QK_G(X, L_-) & \longleftarrow & QK_G^\vee(X) & \longrightarrow & QK_G(X, L_+) \\
\Big\downarrow \kappa^G_{X,-} & & & & \Big\downarrow \kappa^G_{X,+} \\
QK(X/\!\!/_- G) & \xrightarrow{\tau_{X/\!\!/_- G}} & \tilde{\Lambda}^G_X & \xleftarrow[\tau_{X/\!\!/_+ G}]{} & QK(X/\!\!/_+ G)
\end{array}$$

Consequence: Invariance of graph potentials under simple flops induced by vgit. (generalizes Y.P. Lee et al. in quantum cohomology case)

Abelianization

Suppose $T \subset G$ is a maximal torus, $\tau_{X/\!\!/T}$ are the $\mathfrak{g}/\mathfrak{t}$-twisted K-theoretic Gromov-Witten graph invariants as in Bertram-Ciocan-Fontanine-Kim [1]. There is a commutative diagram

$$
\begin{array}{ccc}
QK_G(X) & \longrightarrow & QK_T(X) \\
\downarrow{\scriptstyle \kappa_{X,G}} & & \downarrow{\scriptstyle \kappa_{X,T}} \\
QK(X/\!\!/G) & & QK(X/\!\!/T) \\
\downarrow{\scriptstyle \tau_{X/G}} & & \downarrow{\scriptstyle |W|^{-1}\tau_{X/T}} \\
\Lambda_X^G & \longleftarrow & \Lambda_X^T
\end{array}
$$

We didn't prove anything for J-functions yet, this is a result about graph potentials.

Proof of abelianization

First check it in the Mundet chamber $\lambda = 0$ where all bundles are trivial using Halpern-Leistner's [16] virtual localization theorem.

Then show both sides have the same wall-crossing property, hence it holds for $\lambda \to \infty$.

Then use the adiabatic limit theorem.

Open problems: (i) formulas for fundamental qde solutions, presentations for quantum K-theory rings for non-abelian git quotients generalizing Taipale [28] for partial flag case. (ii) Invariance of fundamental qde solutions under weighted flops.

A. Bertram, I. Ciocan-Fontanine, and B. Kim.
Gromov-Witten invariants for abelian and nonabelian quotients.
J. Algebraic Geom., 17(2):275–294, 2008.

Lev A. Borisov and R. Paul Horja.
On the *K*-theory of smooth toric DM stacks.
In *Snowbird lectures on string geometry*, volume 401 of *Contemp. Math.*, pages 21–42. Amer. Math. Soc., Providence, RI, 2006.

Anders S. Buch and Leonardo C. Mihalcea.
Quantum *K*-theory of Grassmannians.
Duke Math. J., 156(3):501–538, 2011.

I. Ciocan-Fontanine, B. Kim, and D. Maulik.
Stable quasimaps to GIT quotients.
J. Geom. Phys., 75:17–47, 2014.

I. Ciocan-Fontanine and B. Kim.
Big I-functions.
http://www.arxiv.org/abs/1401.7417

J.-L. Colliot-Th el'ene and J.-J. Sansuc.
Fibr es quadratiques et composantes connexes r eelles.
Math. Ann., 244(2):105–134, 1979.

A. Givental.
On the WDVV equation in quantum *K*-theory.
Michigan Math. J., 48:295–304, 2000.
Dedicated to William Fulton on the occasion of his 60th birthday.

A. Givental and Y.-P. Lee.
Quantum *K*-theory on flag manifolds, finite-difference Toda lattices and quantum groups.
Invent. Math., 151(1):193–219, 2003.

A. Givental and V. Tonita.
The Hirzebruch–Riemann–Roch theorem in true genus-0 quantum K-theory.
arXiv:1106.3136.

A. GIvental.
Permutation-equivariant quantum K-theory I-IV.
On his web page, preprints June-July 2015.

R. Goldin, M. Harada, T. S. Holm, and T. Kimura.
The full orbifold *K*-theory of abelian symplectic quotients.
J. K-Theory, 8(2):339–362, 2011.

E. Gonzalez and C. Woodward.
Quantum cohomology and minimal model programs.
arxiv:1207.3253.

E. Gonzalez and C. Woodward.
Quantum Witten localization and abelianization for qde solutions.
arXiv:0811.3358.

E. Gonzalez and C. Woodward.
A wall-crossing formula for Gromov-Witten invariants under variation of git quotient.
arxiv:arXiv:1208.1727.

V. Gorbounov and C. Korff.
Quantum Integrability and Generalised Quantum Schubert Calculus arxiv:1408.4718v1.

D. Halpern-Leistner.
The derived category of a GIT quotient.
arxiv:1203.0276.

Megumi Harada and Gregory D. Landweber.
Surjectivity for Hamiltonian G-spaces in K-theory.
Trans. Amer. Math. Soc., 359(12):6001–6025 (electronic), 2007.

M. Harada, G. D. Landweber, and R. Sjamaar.
Divided differences and the Weyl character formula in equivariant K-theory.
Math. Res. Lett., 17(3):507–527, 2010.

H. Iritani, T. Milanov, and V. Tonita.
Reconstruction and Convergence in Quantum K-Theory via Difference Equations.
arxiv:1309.3750.

H. Iritani.
An integral structure in quantum cohomology and mirror symmetry for toric orbifolds.
Adv. Math., 222(3):1016–1079, 2009.

H. Iritani.
Quantum *D*-modules and generalized mirror transformations.
Topology, 47(4):225–276, 2008.

F. C. Kirwan.
Cohomology of Quotients in Symplectic and Algebraic Geometry, volume 31 of *Mathematical Notes*.
Princeton Univ. Press, Princeton, 1984.

Y.-P. Lee.
Quantum *K*-theory. I. Foundations.
Duke Math. J., 121(3):389–424, 2004.

Y.-P. Lee.
A formula for Euler characteristics of tautological line bundles on the Deligne-Mumford moduli spaces.
Internat. Math. Res. Notices, (8):393–400, 1997.

D. Maulik and A. Okounkov.
Quantum groups and quantum cohomology.
arxiv:1211.1287.

D. Metzler.
Kalkman localization in generalized equivariant cohomology theories.
Preprint.

P. Solis
A complete degeneration of the moduli of G-bundles on a curve.
arXiv:1311.6847.

K. Taipale.
K-theoretic J-functions of type A flag varieties.
Int. Math. Res. Not. IMRN, (16):3647–3677, 2013.

V. Tonita.
A virtual Kawasaki formula.
arxiv:1110.3916.

G. Vezzosi and A. Vistoli.
Higher algebraic K-theory for actions of diagonalizable groups.
Invent. Math., 153(1):1–44, 2003.

S. Venugopalan.
Vortices on surfaces with cylindrical ends.
arxiv:1201.1933.

C. Woodward.
Quantum Kirwan morphism and Gromov-Witten invariants of quotients I.
preprint.

C. Woodward.
Quantum Kirwan morphism and Gromov-Witten invariants of quotients II.
preprint.

C. Woodward.
Quantum Kirwan morphism and Gromov-Witten invariants of quotients III.
arXiv:1408.5864.