LOCALIZATION FOR THE NORM-SQUARE OF THE MOMENT MAP AND THE TWO-DIMENSIONAL YANG-MILLS INTEGRAL

Chris T. Woodward

Contents

1. Introduction

The first seven sections of the paper contain a version of localization for the norm-square of the moment map in equivariant de Rham theory. P.-E. Paradan has told me that he obtained the same result but did not entirely publish it; most of the ingredients as well as similar results appear in his papers [**35**], [**36**], and [**37**]. This paper gives a reasonably self-contained proof, different from Paradan's.

The existence of a localization formula, but not the precise form of the contributions, was first suggested by Witten in his study [**47**] of two-dimensional Yang-Mills theory. Later Jeffrey and Kirwan [**22**] gave a formula which had the same purpose of studying the cohomology of the quotient but was expressed in terms of rather different fixed point data. A K-theory version was given by Vergne [**45**] and Paradan [**37**]. A similar result for sheaf cohomology that I learned from C. Teleman is explained in the eighth section.

The formula expresses the integral of an equivariant cohomology class on a Hamiltonian K-manifold with proper moment map as a sum of integrals over fixed point components of one-parameter subgroups corresponding to the critical values of the norm-square of the moment map. If the critical set of the norm-square is non-degenerate in a sense explained below, there is an improved formula 6.0.1 as an integral over the critical set itself. The new observation in this paper, lemma 5.2.1, is that the singularities of the moment map are essentially unchanged under Witten's deformation. The localization formula then follows by arguments introduced by Duistermaat-Heckman [**13**].

The ninth section contains a definition and computation of the Yang-Mills path integral in two dimensions. The idea is to reverse the logic in Witten's paper [**47**], and take the "stationary phase approximation" (that is, the localization formula) as the definition of the path integral. In order to compute it we apply a symmetry argument from Teleman-Woodward [**44**] which reduces the computation to an integration over Jacobians. The result is what the physicists call the Migdal formula for the path integral; its large coupling (topological) limit is the Witten volume formula. More general formulas for intersection pairings and indices on the moduli space are given in [**44**], Meinrenken [**32**], and Jeffrey-Kirwan [**23**].

A different "combinatorial" definition and computation of the 2d Yang-Mills measure, including observables, is given by Levy [**29**]. Putting these results together shows that the "stationary phase" and "combinatorial" definitions of the two-dimensional Yang-Mills integral are equal. This might be seen as the two-dimensional analog of the much harder conjecture regarding the three-dimensional Chern-Simons path integral, that the "combinatorial" definition via Reshetikhin-Turaev [**40**] agrees with the "stationary phase" definition of Axelrod-Singer [**4**] (with leading order term given as in [**14**]).

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2. Localization for one-parameter subgroups

Let K be a compact connected Lie group with Lie algebra $\mathfrak{k}$ and M a K-manifold. The equivariant de Rham cohomology $H_K(M)$ of M (complex coefficients) can be computed in the *Cartan model*,

$$\Omega_K(M) := (S(\mathfrak{k}^*) \otimes \Omega(M))^K$$

where $\Omega(M)$ denotes the space of smooth forms on M and $S(\mathfrak{k}^*)$ the symmetric algebra on $\mathfrak{k}^*$, see [**18**]. The equivariant differential d_K can be written

$$d_K : \ \Omega_K(M) \to \Omega_K(M), \quad (d_K\eta)(\zeta) = (d + 2\pi i \iota(\zeta_M))(\eta(\zeta)), \quad \zeta \in \mathfrak{k}$$

where $\zeta_M \in \mathrm{Vect}(M)$ denotes the generating vector field $\zeta_M(m) = [\exp(-t\zeta)m]$ and $\iota(\zeta_M)$ contraction with ζ_M.

Suppose K acts locally freely on M; then the equivariant cohomology $H_K(M)$ is isomorphic to $H(K\backslash M)$ via pullback p^* by the projection p :

$M \to K\backslash M$. Cartan's homotopy inverse to p^* is constructed as follows. Let

$$\alpha \in \Omega^1(M, \mathfrak{k})^K, \quad \iota(\xi_M)\alpha = -\xi, \quad \forall \xi \in \mathfrak{k}$$

be a connection 1-form on M and

$$\mathrm{curv}(\alpha) \in \Omega^2(K\backslash M, M(\mathfrak{k})), \quad p^* \mathrm{curv}(\alpha) = \mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha]$$

denote its curvature. Let

$$\pi_\alpha : \ \Omega(M) \to p^*\Omega(K\backslash M)$$

be the horizontal projection defined by α. The map

$$\Omega_K(M) \to \Omega(M)^K, \quad \eta \otimes h \mapsto \left((\pi_\alpha \eta) \otimes h \left(\frac{p^* \mathrm{curv}(\alpha)}{2\pi i} \right) \right) \tag{1}$$

has image contained in the space of basic forms and descends to a map $\Omega_K(M) \to \Omega(K\backslash M)$ which is a homotopy inverse to p^*.

Suppose that M is compact and oriented. Integration over M vanishes on equivariant exact forms and defines a push-forward

$$I_{M,K} : \ H_K(M) \to S(\mathfrak{k}^*)^K.$$

We also denote by $I_{M,K}(\eta)$ the push-forward of the cohomology class of an equivariant form η.

For any K-equivariant real oriented vector bundle E of even dimension $2n$, let $\mathrm{Eul}(E) \in H^{2n}_K(M)$ denote its equivariant Euler class, defined as follows. Equip E with a Euclidean metric, let $\mathcal{F}(E)$ denote the orthogonal frame bundle of E, and let $\alpha \in \Omega^1(\mathcal{F}(E), \mathfrak{so}(2n))^K$ be a K-invariant connection 1-form for E. For any $\zeta \in \mathfrak{k}$, the pairing $\alpha(\zeta_{\mathcal{F}(E)})$ is K-invariant and descends to a map

$$\phi : \ M \to \mathrm{Hom}(\mathfrak{k}, \mathfrak{so}(E)).$$

The form $\mathrm{curv}_{\mathfrak{k}}(E) \in \Omega^2_K(M, \mathfrak{so}(E))$ defined by

$$\mathrm{curv}_{\mathfrak{k}}(E) := \mathrm{curv}(E) + 2\pi i \phi$$

is the *equivariant curvature* of E. The Euler class of E is

$$\mathrm{Eul}(E) := \mathrm{Pf}\left(\frac{\mathrm{curv}_{\mathfrak{k}}(E)}{2\pi} \right) \in \Omega^{2n}_K(M)$$

where Pf is the Pfaffian, and the right-hand side denotes the Chern-Weil characteristic form.

Let K be a torus. If E is a complex K-representation then E splits into a sum of weight spaces E_μ for $\mu \in \mathfrak{k}^*$, so that $\exp(\xi)v = e^{2\pi i \mu(\xi)}v$ for $v \in E_\mu$ and $\xi \in \mathfrak{k}$. If E is a real even-dimensional representation of K, then E admits an invariant complex structure and the weights $\mu_1, \ldots, \mu_n$ are independent of the choice of complex structure up to sign. If E is oriented then the product of the complex weights is determined by the orientation on E and

$$(\mathrm{Eul}(E))(\xi) = \prod_{j=1}^n -2\pi i \mu_j(\xi).$$

We will also need cohomology with smooth and distributional coefficients. An equivariant form with smooth coefficients is a smooth equivariant map $\mathfrak{k} \to \Omega(M)$. The equivariant differential extends to equivariant forms with smooth coefficients and its cohomology is the *equivariant cohomology of M with smooth coefficients*. Let $C_0^\infty(\mathfrak{k}^*)$ denote the space of compactly supported smooth functions on $\mathfrak{k}^*$, and $D'(\mathfrak{k}^*)$ the space of distributions on $\mathfrak{k}^*$, that is, the space of continuous linear forms on $C_0^\infty(\mathfrak{k}^*)$. Let $\mathcal{S}(\mathfrak{k}^*)$ denote the space of *Schwartz functions* on $\mathfrak{k}^*$, the space of smooth functions f such that for any polynomial differential operator P, the function Pf is bounded. Its dual $\mathcal{S}'(\mathfrak{k}^*)$ is the space of *tempered distributions* on $\mathfrak{k}^*$. The inclusion $C_0^\infty(\mathfrak{k}^*) \subset \mathcal{S}(\mathfrak{k}^*)$ dualizes to an injection $S'(\mathfrak{k}^*) \to \mathcal{D}'(\mathfrak{k}^*)$. The symmetric algebra $S(\mathfrak{k}^*)$ embeds in $\mathcal{S}'(\mathfrak{k}^*)$ via Fourier transform as the space of distributions supported at the identity. An equivariant differential form with distributional coefficients is an equivariant continuous linear map from $C_0^\infty(\mathfrak{k}^*)$ to $\Omega(M)$. Let $\mathcal{C}_K(M)$ denote the complex of such forms; the equivariant differential extends to $\mathcal{C}_K(M)$ and its cohomology $\mathcal{H}_K(M)$ is the equivariant cohomology with distributional coefficients. The basic results on equivariant cohomology with distributional coefficients are discussed in detail in Kumar-Vergne [**27**]. If M is compact and connected, then the map $I_{M,K}$ extends to $I_{M,K} : \mathcal{H}_K(M) \to \mathcal{D}'(\mathfrak{k}^*)^K$.

In order to state the localization formula, we need to discuss inversion of the Euler class. Suppose that E is an oriented real vector bundle of even dimension, and that a circle subgroup $U(1)_\zeta \subset G$ generated by $\zeta \in \mathfrak{k}$ acts trivially on M fixing only the zero section in E. According to Atiyah-Bott [**2**], the Euler class is invertible after suitably modifying the coefficient ring. For distributional coefficients, the construction is carried out in Paradan [**35**, Section 4]. A definition equivalent to Paradan's goes as follows. Since ζ acts with non-zero weights $\mathrm{Pf}(\phi(m,\xi))$ is a *hyperbolic polynomial*, that is,

$$\mathrm{Pf}(\phi(m, \xi + i\tau\zeta)) \neq 0, \quad \forall \xi \in \mathfrak{k}, \tau < 0, m \in M.$$

By a standard result in distribution theory [**21**, Theorem 12.5.1] $\mathrm{Pf}(\phi(m,\cdot))^{-k}$ has a unique distributional extension with support on $(\zeta,\cdot) \geq 0$ for any $k > 0$. Let $\mathrm{Pf}(\mathrm{curv}_{\mathfrak{k}}(E)/2\pi)_+$ denote the terms containing forms on M of positive degree, so that

$$\mathrm{Pf}(\mathrm{curv}_{\mathfrak{k}}(E)/2\pi) = \mathrm{Pf}(i\phi) + \mathrm{Pf}(\mathrm{curv}_{\mathfrak{k}}(E)/2\pi)_+.$$

Define

$$\mathrm{Eul}(E)_\zeta^{-1} := \mathrm{Pf}(i\phi)^{-1} \left(1 + \frac{\mathrm{Pf}(\mathrm{curv}_{\mathfrak{k}}(E)/2\pi)_+}{\mathrm{Pf}(i\phi)} \right)^{-1}$$

interpreted via its power series expansion, which is finite since $\mathrm{Pf}(\mathrm{curv}_{\mathfrak{k}}(E))_+$ is nilpotent.

For any $\zeta \in \mathfrak{k}$, let M^ζ denote the fixed point set of the one-parameter subgroup $U(1)_\zeta$ generated by ζ. Fix orientations on M^ζ and $T_{M^\zeta}M$ which induce the given orientation on $TM|M^\zeta$. If E is a K-equivariant vector bundle and $K' \subset K$ is a subgroup stabilizing a submanifold $M' \subset M$ then

$\operatorname{Res}^{M,K}_{M',K'} E$ denotes the restriction of E to a K'-equivariant bundle on M'. Similarly, if η is a K-equivariant cohomology form or class, we denote by $\operatorname{Res}^{M,K}_{M',K'} \eta$ the restriction of η to a K'-equivariant form or class on M'.

Theorem 2.0.1. *(Localization for one-parameter subgroups) For any compact oriented K-manifold M, η an equivariant cohomology class with smooth coefficients, and $\zeta \in \mathfrak{k}$,*

$$I_{M,K_\zeta}(\operatorname{Res}^K_{K_\zeta} \eta) = I_{M^\zeta,K_\zeta}(\operatorname{Res}^{M,K}_{M^\zeta,K_\zeta} \eta \wedge \operatorname{Eul}(T_{M^\zeta} M)^{-1}_\zeta).$$

For smooth values of $\operatorname{Eul}^{-1}_\zeta(T_{M^\zeta} M)$, Theorem 2.0.1 is proved in Atiyah-Bott [**3**] and Berline-Vergne [**6**]. For the distributional version, see Guillemin-Lerman-Sternberg [**16**], Canas-Guillemin [**9**], and Paradan [**35**, Section 5].

3. Hamiltonian K-manifolds

Let $T \subset K$ be a maximal torus, $\mathfrak{t}$ its Lie algebra, and $\mathfrak{t}^*$ its dual. Let $\mathfrak{t}^* \to \mathfrak{k}^*$ be the injection whose image is the fixed point set of T. Choose a closed positive Weyl chamber $\mathfrak{t}_+$. Using an invariant metric on $\mathfrak{k}$ to identify $\mathfrak{t}^*$ with $\mathfrak{t}$, let $\mathfrak{t}^*_+$ denote the image of $\mathfrak{t}_+$ in $\mathfrak{t}^*$; this is independent of the choice of metric.

If N is a right K-manifold, then by $N \times_K M$ we mean the quotient of $N \times M$ by the K-action $k(n,m) = (nk^{-1}, km)$.

3.1. Basic definitions and results. A *Hamiltonian K-manifold* consists of a smooth K-manifold M, a symplectic form ω and an equivariant *moment map* $\Phi : \; M \to \mathfrak{k}^*$ satisfying

$$\iota(\xi_M)\omega = -d(\Phi, \xi), \; \forall \xi \in \mathfrak{k}. \tag{2}$$

If ω is closed but degenerate the data are called a *degenerate Hamiltonian K-manifold*. We denote by $K_M \subset K$ the *principal stabilizer*, that is, the stabilizer of a generic element in $\Phi^{-1}(\mathfrak{t}^*_+)$. The *principal orbit-type stratum* for K resp. $\mathfrak{k}$ is the set of points $m \in M$ with K_m conjugate to K_M resp. $\mathfrak{k}_m$ conjugate to $\mathfrak{k}_M$. For references on the following, see [**26**],[**28**].

Theorem 3.1.1. *Let M be a connected Hamiltonian K-manifold with proper moment map.*

a) *(Kirwan Convexity) The intersection $\Delta(M) := \Phi(M) \cap \mathfrak{t}^*_+$ is a convex polyhedron called the* moment polyhedron *of M.*
b) *(Principal cross-section) The open face $\sigma(M)$ of $\mathfrak{t}^*_+$ containing $\Delta(M)$ in its closure is the* principal face *for M. The inverse image $\Phi^{-1}(K\sigma(M))$ is an open subset of M whose complement is codimension at least two, and the map $K \times_{K_\sigma} \Phi^{-1}(\sigma(M)) \to M, \;\; [k,m] \mapsto km$ is a diffeomorphism onto its image.*

The *rank* of M is the dimension of $\Delta(M)$. M is *maximal rank* if and only if $\mathfrak{k}_M$ is trivial.

The moment map condition (2) is equivalent to the condition that the *equivariant symplectic form*

$$\omega_K(\xi) = \omega + 2\pi i(\Phi, \xi) \in \Omega_K(M).$$

is equivariantly closed. The *equivariant Liouville form* is

$$\mathcal{L} := \exp(\omega_K) = \exp(\omega)\exp(2\pi i(\Phi, \xi)).$$

Let M be a Hamiltonian K-manifold with proper moment map. The *Duistermaat-Heckman measure* $\mu_{M,K}$ is the push-forward of the measure defined by the top degree component of $\exp(\omega)$ under Φ,

$$\mu_{M,K} := \Phi_*(\exp(\omega)) \in \mathcal{D}'(\mathfrak{k}^*)^K.$$

More generally, suppose that $\eta \in \Omega_K(M)$ is closed. If M is compact, we define the *twisted Duistermaat-Heckman distribution* as the Fourier transform of the push-forward of $\mathcal{L} \wedge \eta$:

$$\mu_{M,K}(\eta) = \mathcal{F}_{\mathfrak{k}}(I_{M,K}(\mathcal{L} \wedge \eta)).$$

If M is a Hamiltonian K-manifold M with proper moment map Φ, suppose that $\xi_1, \ldots, \xi_{\dim(\mathfrak{k})}$ are coordinates on $\mathfrak{k}$ and that $\eta = \sum_I \eta_I \xi_I$ where I ranges over multisets with elements $1, \ldots, \dim(\mathfrak{k})$ and $\xi_I := \prod_{i \in I} \xi_i$. Define

$$\mu_{M,K}(\eta) := \sum_I \partial_I \Phi_*(\exp(\omega) \wedge \eta_I) \in \mathcal{D}'(\mathfrak{k}^*)^K \tag{3}$$

where $\partial_I = \mathcal{F}_{\mathfrak{k}}(\xi_I)$, the Fourier transform of ξ_I. $\mu_{M,K}(\eta)$ is supported on the image of Φ and depends only on the cohomology classes of ω_K and η.

Later we will need a variation of this construction when M is a compact Hamiltonian K-manifold with boundary. Let $\eta \in \Omega_K(M)$ be closed and $\alpha \in \Omega^1_K(M) = \Omega^1(M)^K$ an invariant one-form. Consider the family of (possibly degenerate) symplectic forms and moment maps

$$\omega_s = \omega + s\mathrm{d}\alpha, \quad (\Phi_s(m), \xi) = (\Phi(m), \xi) + s\alpha(\xi_M).$$

Let $\mu_{M,K,s}(\eta)$ denote the corresponding twisted Duistermaat-Heckman distribution.

Proposition 3.1.2. *Let $h \in C_0^\infty(\mathfrak{k}^*)^K$ be such that* $\operatorname{supp}(h) \cap \Phi_s(\partial M)$ *is empty for $s \in [0,1]$. $(\mu_{M,K,s}(\eta), h)$ is independent of $s \in [0,1]$.*

This follows from the same argument as in the case without boundary, since the relevant integrals are supported on the interior.

3.2. Coadjoint orbits. The following material is mostly covered in Berline-Getzler-Vergne [**5**, Section 7.5]. We parametrize coadjoint orbits by their intersection with the positive chamber:

$$\mathfrak{t}^*_+ \to K\backslash\mathfrak{k}^*, \quad \lambda \mapsto K\lambda.$$

A symplectic form on $K\lambda$ is defined by the Kirillov-Kostant-Souriau formula

$$\omega_m(\xi_M(m), \zeta_M(m)) = (m, [\xi, \zeta]). \tag{4}$$

The action of K on $K\lambda$ is Hamiltonian with moment map given by the inclusion into $\mathfrak{k}^*$. The weights on the tangent space at a T-fixed point $w\lambda$ are the roots α with $(\alpha, w\lambda) > 0$. By localization (2.0.1)

$$(I_{K\cdot\lambda,T}(\mathcal{L}))(\zeta) = \sum_{[w]\in W/W_\sigma} \frac{\exp(2\pi i(w\lambda,\zeta))}{\prod_{(\alpha,w\lambda)<0} 2\pi i(\alpha,\zeta)} \tag{5}$$

Let ρ denote the half-sum of positive roots of $\mathfrak{k}$. The symplectic volume of $K\cdot\lambda$ is the equal to

$$\begin{aligned} \mathrm{Vol}(K\cdot\lambda) &= (I_{K\cdot\lambda,T}(\mathcal{L}))(0) && (6)\\ &= \lim_{t\to 0}(I_{K\cdot\lambda,T}(\mathcal{L}))(t\rho) && (7)\\ &= \frac{\prod_{(\alpha,\lambda)<0}(\alpha,\lambda)}{\prod_{(\alpha,\lambda)<0}(\alpha,\rho)}. && (8)\end{aligned}$$

A computation at the tangent space at λ shows that the symplectic and Riemannian volumes are related by

$$\mathrm{Vol}(K\cdot\lambda) = (2\pi)^{\dim(K/K_\lambda)/2} \prod_{(\alpha,\lambda)>0} (\alpha,\lambda)\,\mathrm{Vol}(K/K_\lambda) \tag{9}$$

which implies

$$\mathrm{Vol}(K/K_\lambda)^{-1} = (2\pi)^{\dim(K/K_\lambda)/2} \prod_{(\alpha,\lambda)>0} (\alpha,\rho). \tag{10}$$

Similarly, let $\mathrm{RVol}(K\cdot\lambda)$ denote the volume of $K\cdot\lambda$ with respect to the Riemannian metric induced by the embedding $K\cdot\lambda \to \mathfrak{k}^*$. We have

$$\mathrm{RVol}(K\cdot\lambda) = (2\pi)^{\dim(K/K_\lambda)} \prod_{(\alpha,\lambda)>0} (\alpha,\lambda)^2\,\mathrm{Vol}(K/K_\lambda) \tag{11}$$

If $K_\lambda = T$ then

$$\mathrm{RVol}(K\cdot\lambda) = \Pi(\lambda)^2\,\mathrm{Vol}(K/K_\lambda)$$

where

$$\Pi(\xi) = \prod_{\alpha>0} 2\pi(\alpha,\xi) = i^{-\dim(K/T)/2}\,\mathrm{Eul}(\mathfrak{k}/\mathfrak{t}).$$

3.3. Symplectic quotients. The *symplectic quotient* of M at $\lambda\in\mathfrak{k}^*$ is

$$M_{(\lambda)} := K_\lambda\backslash\Phi^{-1}(\lambda).$$

If $\Phi^{-1}(\lambda)$ is contained in the principal orbit-type stratum for K (resp. $\mathfrak{k}$) then $M_{(\lambda)}$ has the structure of a symplectic manifold (resp. orbifold), with symplectic form $\omega_{(\lambda)}$ the unique two-form that pulls back to the restriction of ω to $\Phi^{-1}(\lambda)$. For λ such that $\Phi^{-1}(\lambda)$ is contained in the principal orbit-type stratum, we denote by $\mathcal{L}_{(\lambda)} = \exp(\omega_{(\lambda)})$ the Liouville form on $M_{(\lambda)}$.

The relation between the cohomology of M and the cohomology of the quotient was studied by Kirwan [**25**]. Let λ be a regular value of Φ and κ_λ the composition of restriction $H_K(M) \to H_K(\Phi^{-1}(\lambda))$ with

$$H_K(\Phi^{-1}(\lambda)) \to H(M_{(\lambda)}).$$

κ_λ extends to cohomology with smooth coefficients. By [**25**], if λ is central and M is compact then κ_λ is surjective.

The cohomological pairings on the symplectic quotient are encoded in the twisted Duistermaat-Heckman distributions. Let $\mu_{\mathfrak{k}^*}, \mu_{\mathfrak{t}^*}$ denote the Lebesgue measures induced by the metrics on $\mathfrak{k}^*, \mathfrak{t}^*$ respectively.

Proposition 3.3.1. *Let M be a (possibly degenerate) Hamiltonian K-manifold with proper moment map Φ. Let $\eta \in \Omega_K(M)$ be a closed equivariant form.*

a)
$$\mu_{M,K}(\eta) = \frac{\mathrm{Vol}(K/K_M)}{\mathrm{Vol}(K \cdot \lambda)} I_{M_{(\lambda)}}(\mathcal{L}_{(\lambda)} \wedge \kappa_\lambda(\eta))\mu_{\mathfrak{k}^*}$$
for λ in a neighborhood of any regular value of Φ.

b) *$I_{M_{(\lambda)}}(\mathcal{L}_{(\lambda)} \wedge \kappa_\lambda(\eta))$ is a polynomial in λ on any subset of regular values of Φ on which K_λ is constant.*

c) *Let $\lambda \in \mathfrak{t}^*_+$ be a regular value of Φ. For $\nu \in \mathfrak{t}^*_{\mathrm{reg}} \cap \mathfrak{t}^*_+$,*
$$\lim_{\nu \to \lambda} (\# W_\lambda)^{-1} \mathcal{I}_{M_{(\nu)}}(\mathcal{L}_{(\nu)} \wedge \kappa_\nu(\eta \wedge \mathrm{Eul}((\mathfrak{k}_\lambda/\mathfrak{t})^*))) = I_{M_{(\lambda)}}(\mathcal{L}_{(\lambda)} \wedge \kappa_\lambda(\eta)).$$

Proof. (a),(b) are basic results of Duistermaat-Heckman, see [**12**],[**22**] . (c) By the coisotropic embedding theorem that $M_{(\nu)}$ is a K_λ/T-fiber bundle over $M_{(\lambda)}$. The class $\kappa_\lambda(\mathrm{Eul}((\mathfrak{k}_\lambda/\mathfrak{t})^*))$ restricts to the Euler class of the tangent bundle on the fiber K_λ/T, which has Euler characteristic $\#(K_\lambda/T)^T = \# W_\lambda$. The result follows from fiber integration. □

We will need a more general "reduction in stages" version of this result. Let $K_1 \subset K$ be a normal subgroup, and $K_2 = K/K_1$ the quotient. Let $\Phi = (\Phi_1, \Phi_2)$ be the decomposition of Φ according to the splitting $\mathfrak{k} \cong \mathfrak{k}_1 \oplus \mathfrak{k}_2$.

Proposition 3.3.2. *Let M be a (possibly degenerate) Hamiltonian K-manifold with proper moment map Φ. Let $\eta \in \Omega_K(M)$ be closed and μ_1 a regular value of Φ_1.*

$$\mu_{M,K}(\eta) = \frac{\mathrm{Vol}(K_1/K_{1,M})}{\mathrm{Vol}(K_1 \cdot \lambda_1)} \int_{\mathfrak{k}_1^*} (\mu_{M_{(\lambda_1)},K_2}(\kappa_{\lambda_1}(\eta)) \otimes \delta_{\lambda_1}) d\lambda_1$$

in a neighborhood of μ_1. Furthermore, $\mu_{M_{(\lambda_1)},K_2}(\kappa_{\lambda_1}(\eta))$ is a polynomial in λ_1 as λ_1 varies in any subset of regular values for Φ_1 for which K_{1,λ_1} is constant.

Here the twisted Duistermaat-Heckman distribution $\mu_{M_{(\lambda_1)},K_2}(\eta_{(\lambda_1)})$ on $M_{(\lambda_1)}$ is a distribution on $\mathfrak{k}_2^*$. Its tensor product with $\delta_{(\lambda_1)}$ is a distribution on $\mathfrak{k}^*$ depending on λ_1.

We apply this to prove a polynomiality result for non-regular values. Let $\lambda_0, \lambda_1 \in \mathfrak{k}^*$ and

$$R_{\lambda_0,\lambda_1} = \lambda_0 + \mathbb{R}_{\geq 0}(\lambda_1 - \lambda_0)$$

the ray starting from λ_0 to λ_1. Let $\mathfrak{k}_1$ denote the span of λ_1, and $\mathfrak{k}_2$ the quotient $\mathfrak{k}/\mathfrak{k}_1$. We say that a distribution $\mu \in \mathcal{D}'(\mathfrak{k}^*)$ is *polynomial along the ray* R_{λ_0,λ_1} if the restriction of μ to some neighborhood of $R_{\lambda_0,\lambda_1} - \{\lambda_0\}$ is equal to the restriction of

$$\int_{\mathfrak{k}_1^*} (\mu_2(\lambda_1) \otimes \delta(\lambda_1))\mathrm{d}\lambda_1$$

for some distribution $\mu_2(\lambda_1) \in \mathcal{D}'(\mathfrak{k}_2^*)$ depending polynomially on $\lambda_1 \in \mathfrak{k}_1^*$. We say μ is *polynomial near* $\lambda_2 \in R_{\lambda_0,\lambda_1}$ if this holds in a neighborhood of λ_2.

Proposition 3.3.3. *Let M be a (possibly degenerate) Hamiltonian K-manifold with proper moment map Φ. Let $\eta \in \Omega_K(M)$ be closed. Let $\lambda_0 \in \mathfrak{k}^*$ and λ_1 in $\mathfrak{k}_{\lambda_0}^*$ such that $(\lambda_1, \lambda_1 - \lambda_0)$ is a regular value of $(\Phi, \lambda_1 - \lambda_0)$. The distribution $\mu_{M,K}(\eta)$ is polynomial along R_{λ_0,λ_1} near λ_1.*

3.4. Induction. First we define induction for distributions. Let τ be any face of $\mathfrak{t}_+^*$. Let $\mathrm{Vol}_{K_\tau}^K : \mathfrak{t}^* \to \mathbb{R}$ denote the function

$$\mathrm{Vol}_{K_\tau}^K(\lambda) := \frac{\mathrm{Vol}(K \cdot \lambda)}{\mathrm{Vol}(K_\tau \cdot \lambda)}, \quad \lambda \in \mathfrak{t}_+^*.$$

Using (6) $\mathrm{Vol}_{K_\tau}^K$ has a polynomial extension to $\mathfrak{t}^*$ that is invariant under W_τ. We denote by the same name its extension qto $\mathfrak{k}_\tau^*$. Define

$$\mathrm{Ind}_{K_\tau}^K :\ \mathcal{D}'(\mathfrak{k}_\tau^*) \to \mathcal{D}'(\mathfrak{k}^*)^K, \quad (\mathrm{Ind}_{K_\tau}^K \mu, h) = (\mu, \mathrm{Vol}_{K_\tau}^K \mathrm{Res}_{K_\tau}^K h). \tag{12}$$

Restriction to tempered distributions defines a map $\mathcal{S}'(\mathfrak{k}_\tau^*) \to \mathcal{S}'(\mathfrak{k}^*)^K$. The same notation will be used for the Fourier transform $\mathcal{S}'(\mathfrak{k}_\tau) \to \mathcal{S}'(\mathfrak{k})^K$. The reader may note that $\mathrm{Ind}_{K_\tau}^K$ is the "semiclassical limit" of holomorphic induction of representation rings $R(K_\tau) \to R(K)$.

Next, we define induction for Hamiltonian actions. If M is a Hamiltonian K_τ-manifold, one can define a Hamiltonian K-manifold by

$$\mathrm{Ind}_{K_\tau}^K M := K \times_{K_\tau} M$$

with the unique closed equivariant two-form $\mathrm{Ind}_{K_\tau}^K \omega_K$ restricting to ω_K on M. The two-form $\mathrm{Ind}_{K_\tau}^K \omega$ is degenerate if and only if $\Delta(M)$ lies in the union of open faces of $\mathfrak{t}_+^*$ whose closure contains τ.

Finally, we define induction for equivariant forms. The inclusion

$$M \to \mathrm{Ind}_{K_\tau}^K M, \quad m \mapsto [1, m]$$

induces a map $\Omega_K(\mathrm{Ind}_{K_\tau}^K M) \to \Omega_{K_\tau}(M)$. A homotopy inverse is provided by the composition $\mathrm{Ind}_{K_\tau}^K$ of the maps

$$\Omega_{K_\tau}(M) \to \Omega_{K \times K_\tau}(K \times M) \to \Omega_K(\mathrm{Ind}_{K_\tau}^K M) \tag{13}$$

where the last map is the Cartan map (1).

The following proposition shows that taking Duistermaat-Heckman distributions commutes with induction, see Paradan [**36**, 3.13]. For completeness we include a proof.

Proposition 3.4.1. *Let τ be a face of the positive Weyl chamber, M a Hamiltonian K_τ-manifold with proper moment map, and η a closed polynomial K_τ-equivariant form on M.* $\mu_{\mathrm{Ind}_{K_\tau}^K M,K}(\mathrm{Ind}_{K_\tau}^K \eta) = \mathrm{Ind}_{K_\tau}^K \mu_{M,K_\tau}(\eta)$.

Proof. If the closure of τ contains the principal face σ of M, then $\mathrm{Ind}_{K_\sigma}^K = \mathrm{Ind}_{K_\tau}^K \mathrm{Ind}_{K_\sigma}^{K_\tau}$. Therefore, it suffices to prove the proposition for $\tau = \sigma$. Let $\mathrm{conn}_{K_\sigma}^K \in \Omega^1(K, \mathfrak{k}_\sigma)$ be the connection on $K \to K/K_\sigma$ defined using the metric on $\mathfrak{k}$, $\widetilde{\mathrm{curv}}_{K_\sigma}^K \in \Omega_K^2(K/K_\sigma, K(\mathfrak{k}_\sigma))$ its equivariant curvature and $\mathrm{curv}_{K_\sigma}^K \in \Omega^2(K/K_\sigma(\mathfrak{k}_\sigma)$ its ordinary curvature. For each $\lambda \in \sigma$ the pairing with the curvature defines an equivariant two-form $(\widetilde{\mathrm{curv}}_{K_\sigma}^K, \lambda) \in \Omega_K^2(K/K_\sigma)$. Let

$$p_i \in S(\mathfrak{k}_\sigma)^{K_\sigma}, i = 1, 2, \ldots$$

be a basis for the invariant polynomials on $\mathfrak{k}_\sigma$. For each i we have a characteristic form defined via the Chern-Weil homomorphism

$$\widetilde{\mathrm{curv}}_{K_\sigma,i}^K \in \Omega_K(K/K_\sigma).$$

We write

$$\widetilde{\mathrm{curv}}_{K_\sigma,i}^K = \sum_I \mathrm{curv}_{K_\sigma,i,I}^K \xi_I$$

for some forms $\mathrm{curv}_{K_\sigma,i,I}^K \in \Omega(K/K_\sigma)$. Because $(\mathrm{curv}_{K_\sigma}^K, \lambda)$ is the pull-back of the Kirillov-Kostant-Souriau form (4) under the map $K/K_\sigma \to K \cdot \lambda$, we have for any $h \in \mathcal{S}(\mathfrak{k}^*)^K$

$$\sum_I \int_{K/K_\sigma} \mathrm{curv}_{K_\sigma,i,I}^K \wedge \exp(\mathrm{curv}_{K_\sigma}^K, \lambda)(\partial_I h)(\lambda) = (\partial_i \mathrm{Vol}_{K_\sigma}^K h)(\lambda) \tag{14}$$

where ∂_i is the Fourier transform of p_i. Let $\eta_i \in \Omega(M)$ be forms such that $\eta = \sum_i \eta_i p_i$. $\mathrm{Ind}_{K_\sigma}^K(\eta)$ is the form on $\mathrm{Ind}_{K_\sigma}^K M$ whose pull-back to $K \times M$ is

$$\sum_i \pi_2^* \eta_i \wedge \pi_1^* \phi^* (\mathrm{curv}_{K_\sigma}^K)_i.$$

Let $\beta \in \Omega(K \times M)$ be a form which integrates to 1 on the orbits of K_σ on $K \times M$. Using (14) we have (omitting pull-backs which confuse the notation)

$$\begin{aligned}
(\mu_{\mathrm{Ind}_{K_\sigma}^K M,K}(\mathrm{Ind}_{K_\sigma}^K(\eta)), h) &= \sum_{i,I} \int_{K\times M} (\mathrm{Ind}_{K_\sigma}^K \Phi)^*(\partial_I h) \exp(\omega) \wedge \eta_i \wedge \\
&\quad \mathrm{curv}_{K_\sigma,i,I}^K \wedge \beta \wedge \exp(\mathrm{curv}_{K_\sigma}^K, \mathrm{Ind}_{K_\sigma}^K \Phi) \\
&= \sum_i \int_M \Phi^*(\partial_i \mathrm{Vol}_{K_\sigma}^K \mathrm{Res}_{K_\sigma}^K h) \exp(\omega) \wedge \eta_i \\
&= (\mu_{M,K_\sigma}(\eta), \mathrm{Vol}_{K_\sigma}^K \mathrm{Res}_{K_\sigma}^K h) \\
&= (\mathrm{Ind}_{K_\sigma}^K \mu_{M,K_\sigma}(\eta), h).
\end{aligned}$$

as claimed. □

3.5. Comparison of abelian and non-abelian Duistermaat-Heckman distributions. For the sake of computing examples it will be helpful to have the formula that compares the abelian and non-abelian Duistermaat-Heckman measures. The following result of Harish-Chandra compares Fourier transforms over $\mathfrak{k}$ and $\mathfrak{t}$:

Lemma 3.5.1 (Harish-Chandra). *For any* $h \in \mathcal{S}(\mathfrak{k})^K$,

$$\Pi \, \mathrm{Res}_{\mathfrak{t}^*}^{\mathfrak{k}^*} \mathcal{F}_{\mathfrak{k}}(h) = i^{\dim(K/T)/2} \mathcal{F}_{\mathfrak{t}}(\Pi \, \mathrm{Res}_{\mathfrak{t}}^{\mathfrak{k}} h). \tag{15}$$

Proof. Let $\lambda \in \mathfrak{t}^*$.

$$\begin{aligned}
(\mathcal{F}_{\mathfrak{k}}(h))(\lambda) &= (2\pi)^{-\dim(\mathfrak{k}/2)} \int_{\mathfrak{k}} e^{2\pi i(\lambda,\xi)} h(\xi) \mathrm{d}\xi \\
&= (2\pi)^{-\dim(\mathfrak{k}/2)} \int_{\mathfrak{t}_+ \times K/T} e^{2\pi i(\lambda,k\xi)} h(k\cdot\xi) \exp(\omega_\xi) \frac{\mathrm{RVol}_T^K(\xi)}{\mathrm{Vol}_T^K(\xi)} \mathrm{d}\xi \\
&= (2\pi)^{-\dim(\mathfrak{t})/2} \int_{\mathfrak{t}_+} \sum_{w\in W} (-1)^{l(w)} \frac{h(\xi) e^{2\pi i(\lambda,w\xi)} \Pi(\xi)}{\mathrm{Eul}(\mathfrak{k}/\mathfrak{t})(\lambda)} \mathrm{d}\xi \\
&= (2\pi)^{-\dim(\mathfrak{t})/2} i^{\dim(\mathfrak{k}/\mathfrak{t})/2} \int_{\mathfrak{t}} \frac{h(\xi) e^{2\pi i(\lambda,\xi)} \Pi(\xi)}{\Pi(\lambda)} \mathrm{d}\xi \\
&= i^{\dim(\mathfrak{k}/\mathfrak{t})/2} (\Pi^{-1} \mathcal{F}_{\mathfrak{t}}(\mathrm{Res}_T^K h \cdot \Pi))(\lambda).
\end{aligned}$$

□

As I learned from P.-E. Paradan, Harish-Chandra's result implies the following relation between Duistermaat-Heckman measures. Note that the Euler class $\mathrm{Eul}(\mathfrak{k}/\mathfrak{t})$ considered as a distribution on $\mathfrak{t}^*$ is the product of partial derivatives in the direction of the negative roots of $\mathfrak{k}$.

Theorem 3.5.2. *Let M be a compact Hamiltonian K-manifold and $\eta \in \Omega_K(M)$ closed.* $\mu_{M,K}(\eta) = (\#W)^{-1} \mathrm{Ind}_T^K \mathrm{Eul}(\mathfrak{k}/\mathfrak{t}) \mu_{M,T}(\mathrm{Res}_T^K \eta)$.

Proof. Since $\operatorname{Res}^K_T \mathcal{F}_{\mathfrak{k}}^{-1} \mu_{M,K}(\eta) = \mathcal{F}_{\mathfrak{t}}^{-1}\mu_{M,T}(\eta)$ we have using (15)

$$\begin{aligned}
(\mu_{M,K}(\eta), \mathcal{F}_{\mathfrak{k}}(h)) &= (\mathcal{F}_{\mathfrak{k}}^{-1}(I_{M,K}(\mathcal{L}\wedge\eta)), h) \\
&= (\#W)^{-1}(\mathrm{RVol}^K_T \mathcal{F}_{\mathfrak{t}}^{-1}(I_{M,T}(\mathcal{L}\wedge\eta)), \operatorname{Res}^K_T h) \\
&= (\#W)^{-1}(I_{M,T}(\mathcal{L}\wedge\eta), \mathcal{F}_{\mathfrak{t}}(\mathrm{RVol}^K_T \operatorname{Res}^K_T h)) \\
&= (\#W)^{-1}(I_{M,T}(\mathcal{L}\wedge\eta), \mathcal{F}_{\mathfrak{t}}(\Pi^2 \operatorname{Vol}(K/T) \operatorname{Res}^K_T h)) \\
&= (\#W)^{-1}(\mu_{M,T}(\eta), \operatorname{Eul}(\mathfrak{k}/\mathfrak{t})\Pi \operatorname{Vol}(K/T) \operatorname{Res}^K_T \mathcal{F}_{\mathfrak{k}}(h)) \\
&= (\#W)^{-1}(\mu_{M,T}(\eta), \operatorname{Eul}(\mathfrak{k}/\mathfrak{t}) \operatorname{Vol}^K_T \operatorname{Res}^K_T \mathcal{F}_{\mathfrak{k}}(h)).
\end{aligned}$$

□

This formula has as a corollary a result of S. Martin [**30**], which compares cohomological pairings on the abelian and non-abelian quotients. We denote by $M_{T,(\lambda)}$ the symplectic quotients for the action of T:

$$M_{T,(\lambda)} = (\operatorname{Res}^K_T \Phi)^{-1}(\lambda)/T.$$

Proposition 3.5.3. *If λ is a regular value of Φ and $\operatorname{Res}^K_T \Phi$ then*

$$I_{M_{(\lambda)}}(\kappa_\lambda(\eta)) = (\#W_\lambda)^{-1} I_{M_{(\lambda),T}}(\kappa_\lambda(\operatorname{Res}^K_T \eta \wedge \operatorname{Eul}(\mathfrak{k}/\mathfrak{t}) \wedge \operatorname{Eul}((\mathfrak{k}/\mathfrak{k}_\lambda)^*))).$$

Proof. From (3.5.2) and 3.3.1 (a) we have for generic λ

$$I_{M_{(\lambda)}}(\kappa_\lambda(\eta)) = I_{M_{(\lambda),T}}(\kappa_\lambda(\operatorname{Res}^K_T \eta \wedge \operatorname{Eul}(\mathfrak{k}/\mathfrak{t}))).$$

The result for arbitrary λ follows from 3.3.1 part (c). □

3.6. Symplectic vector bundles. Let M be a compact Hamiltonian K-manifold and $\pi : E \to M$ a K-equivariant symplectic vector bundle, that is, a vector bundle with structure group $Sp(2n,\mathbb{R})$. We recall from [**17**] that the total space of E can be given the structure of closed two-form, equal to ω on the zero section and non-degenerate in a neighborhood of it: Let $\mathrm{Fr}(E)$ denote the frame bundle of E and ω_F the symplectic form on the fiber $F := \mathbb{R}^{2n}$. The action of $Sp(2n,\mathbb{R})$ on F is Hamiltonian; we denote by $\phi : F \to \mathfrak{sp}(2n,\mathbb{R})^*$ the moment map. Let $\alpha \in \Omega^1(\mathrm{Fr}(E), \mathfrak{sp}(2n,\mathbb{R}))$ be a connection one-form. The two-form

$$\pi^*\omega + d(\alpha,\phi) + \omega_F \in \Omega^2(\mathrm{Fr}(E) \times F)$$

(pullbacks from factors are omitted from the notation) is basic and descends to a closed form ω_E on $E \cong \mathrm{Fr}(E) \times_{Sp(2n,\mathbb{R})} F$ with the required properties. By the symplectic embedding theorem, ω_E is the unique form with these properties up to symplectomorphism on a neighborhood of the zero section. The construction also works equivariantly: If M is a Hamiltonian K-manifold and E a K-equivariant symplectic vector bundle let Φ_F denote the moment map for the K-action on the fiber F. The map

$$\xi \mapsto \pi^*(\Phi,\xi) + (\alpha(\xi_{\mathrm{Fr}(E)}), \phi)$$

is $\mathrm{Sp}(2n,\mathbb{R})$-invariant and descends to a moment map $\Phi_E : E \to \mathfrak{k}^*$.

Let $U(1)_\zeta$ be the one-parameter subgroup generated by a central element $\zeta \in \mathfrak{k}$. Suppose $U(1)_\zeta$ acts on E fixing only the zero section with positive weights. In this case the moment map for the action of $U(1)_\zeta$ on the fiber F is a positive-definite quadratic form; it follows that (Φ_E, ζ) is proper, so Φ_E is proper as well. For any closed form $\eta \in \Omega_K(M)$, localization applied to the total space of E gives

$$\mu_{E,K}(\pi^*\eta) = \mu_{M,K}(\eta \wedge \mathrm{Eul}_\zeta^{-1}(E)). \tag{16}$$

Non-compactness of E can be remedied as in [**38**] or [**36**].

Suppose that $U(1)_\zeta$ acts on E with both positive and negative (but not zero) weights. Let $E = E_- \oplus E_+$ be the decomposition into positive and negative weight bundles. Let $\pi : E' \to M$ be the symplectic vector bundle obtained from E by reversing the symplectic structure on the subbundle $E_- \subset E$ on which $U(1)_\zeta$ acts with negative weights, so that the orientation of E' is $(-1)^{\dim(E_-)}$ times the orientation on E. Since

$$\mathrm{Eul}(E)_\zeta^{-1} = (-1)^{\dim(E^-)}\,\mathrm{Eul}(E')_\zeta^{-1}$$

we have

$$\mu_{M,K}(\eta \wedge \mathrm{Eul}(E)_\zeta^{-1}) = (-1)^{\dim(E^-)}\mu_{E',K}(\pi^*\eta). \tag{17}$$

By Proposition 3.3.3 applied to E', for any $\lambda_0 \in \mathfrak{k}^*$ and non-zero $\lambda_1 \in \mathfrak{k}^*_{\lambda_0}$, the distribution $\mu_{M,K}(\eta \wedge \mathrm{Eul}(E)_\zeta^{-1})$ is polynomial along R_{λ_0,λ_1} near λ_0. Hence

Proposition 3.6.1. *The restriction of $\mu_{M,K}(\eta \wedge \mathrm{Eul}(E)_\zeta^{-1})$ to any sufficiently small neighborhood of λ_0 has a unique extension to a tempered distribution on $\mathfrak{k}^*$ that is polynomial along any ray beginning at λ_0.*

4. The Kirwan-Ness stratification

Let (M, ω) be a Hamiltonian K-manifold with proper moment map Φ and f one-half the norm-square of the moment map,

$$f : M \to \mathbb{R},\ f(m) = \frac{1}{2}(\Phi(m), \Phi(m)).$$

In general, f is not a Morse-Bott function. The critical set of f consists of points m fixed by the vector field generated by $\Phi(m)$:

$$\mathrm{crit}(f) = \{m \in M,\ \ (\Phi(m)_M)(m) = 0\}.$$

Hence $\Phi^{-1}(0)$ is a component of $\mathrm{crit}(f)$. For any connected component $C \subset \mathrm{crit}(f)$ the intersection $\Phi(C) \cap \mathfrak{t}_+$ consists of a single point ξ. (See [**25**, 3.15] for the case M compact; the case Φ is proper is similar.) Define

$$\Xi(M) = \{\xi(C),\ \ C \subset \mathrm{crit}(f)\}.$$

For any $\xi \in \Xi(M)$ let

$$C_\xi = \{m \in M,\ \ \Phi(Km) \cap \mathfrak{t}_+ = \xi\}$$

which may be a finite union of connected components. Choose a K-invariant almost complex structure on M, and consider the corresponding K-invariant Riemannian metric. For any $m \in M$ let $\{m_t,\ t \in [0,\infty)\}$ denote the trajectory of $-\operatorname{grad}(f)$. For any $\xi \in \Xi(M)$, let M_ξ denote the stable set of the corresponding critical component C_ξ, that is, the set of m with limit point in C_ξ. The Kirwan-Ness stratification is

$$M = \bigcup_{\xi \in \Xi(M)} M_\xi.$$

For each $\xi \in \Xi(M)$ let $U(1)_\xi$ denote the one-parameter subgroup generated by ξ. Let Z_ξ denote the component of the fixed point set of $U(1)_\xi$ containing $C_\xi \cap \Phi^{-1}(\mathfrak{t}_+)$, Y_ξ the set of points in M which flow to Z_ξ under $-\operatorname{grad}(\Phi,\xi)$, and $\varphi_\xi : Y_\xi \to Z_\xi$ the map given by the limit of the flow. Let G denote the complexification of K, K_ξ, G_ξ the stabilizers of ξ under the adjoint action of K, G and P_ξ the standard parabolic corresponding to ξ. Since (Φ,ξ) is a Morse-Bott function, Y_ξ is a smooth K_ξ-invariant submanifold. Let Z_ξ° denote the set of points in Z_ξ which flow to C_ξ under $-\operatorname{grad}(\operatorname{Res}^M_{Z_\xi} f)$, and $Y_\xi^\circ = \varphi_\xi^{-1}(Z_\xi^\circ)$. By the stable manifold theorem (see e.g. [**41**]) there exists a diffeomorphism

$$Y_\xi^\circ = T_{Z_\xi^\circ} Y_\xi^\circ \tag{18}$$

where $T_{Z_\xi^\circ} Y_\xi^\circ$ is the normal bundle of Z_ξ° in Y_ξ°. The following combines results from Kirwan [**25**], Ness [**34**], and more recent improvements due to Heinzner-Loose [**20**].

Theorem 4.0.2. *Let M be a Hamiltonian K-manifold with proper moment map. For all $\xi \in \Xi(M)$,*

a) $Z_\xi \cap \Phi^{-1}(\xi) = C_\xi \cap \Phi^{-1}(\xi)$;
b) *For a suitable choice of invariant almost complex structure, the stratum M_ξ is a smooth invariant submanifold which is identical in a neighborhood of C_ξ to KY_ξ°;*
c) *Suppose that M is equipped with an invariant Kähler structure. For the metric defined by the structure M_ξ is a G-invariant Kähler submanifolds, Y_ξ° is P_ξ-stable and there exist equivariant diffeomorphisms*

$$K \times_{K_\xi} Y_\xi^\circ \to G \times_{P_\xi} Y_\xi^\circ \to M_\xi. \tag{19}$$

5. Localization for the norm-square of the moment map

In this section we will prove

Theorem 5.0.3. *Let M be a Hamiltonian K-manifold with proper moment map and $\eta \in \Omega_K(M)$ closed.*

$$\mu_{M,K}(\eta) = \sum_{\xi \in \Xi(M)} \mu_{M_\xi,K}(\operatorname{Res}^M_{M_\xi} \eta \wedge \operatorname{Eul}(T_{M_\xi} M)_\xi^{-1}). \tag{20}$$

Unfortunately the expressions on the right-hand side do not yet make sense; the moment map for the K-action on M_ξ is not in general proper. The diffeomorphisms (18) and (19) lead us to expect

$$\mu_{M_\xi,K}(\mathrm{Res}^M_{M_\xi}\eta\wedge\mathrm{Eul}(T_{M_\xi}M)^{-1}_\xi) = \mathrm{Ind}^K_{K_\xi}\mu_{Y^\circ_\xi,K_\xi}(\mathrm{Res}^{M,K}_{Y^\circ_\xi,K_\xi}\eta\wedge\mathrm{Eul}(T_{M_\xi}M)^{-1}_\xi)$$

and

$$\mu_{Y^\circ_\xi,K_\xi}(\mathrm{Res}^{M,K}_{Y^\circ_\xi,K_\xi}\eta\wedge\mathrm{Eul}(T_{M_\xi}M)^{-1}_\xi) = \mu_{Z^\circ_\xi,K_\xi}(\mathrm{Res}^{M,K}_{Z^\circ_\xi,K_\xi}\eta\wedge\mathrm{Eul}(\nu_\xi)^{-1}_\xi) \tag{21}$$

where

$$\nu_\xi := T_{M_\xi}M|_{Z^\circ_\xi}\oplus T_{Z^\circ_\xi}Y^\circ_\xi.$$

By Proposition 3.6.1, we may define

Definition 5.0.4. $\mu_{Z^\circ_\xi,K_\xi}(\mathrm{Res}^{M,K}_{Z^\circ_\xi,K_\xi}\eta\wedge\mathrm{Eul}(\nu_\xi)^{-1}_\xi)$ is the unique extension of the restriction of $\mu_{Z_\xi,K_\xi}(\mathrm{Res}^{M,K}_{Z_\xi,K_\xi}\eta\wedge\mathrm{Eul}(\nu_\xi)^{-1}_\xi)$ to a neighborhood of ξ that is polynomial on any ray starting at ξ. Define

$$\mu_{M_\xi,K}(\mathrm{Res}^M_{M_\xi}\eta\wedge\mathrm{Eul}(T_{M_\xi}M)^{-1}_\xi) := \mathrm{Ind}^K_{K_\xi}\mu_{Z^\circ_\xi,K_\xi}(\mathrm{Res}^{M,K}_{Z^\circ_\xi,K_\xi}\eta\wedge\mathrm{Eul}(\nu_\xi)^{-1}_\xi).$$

Properness of the moment map insures that the sum on the right hand side of (20) is locally finite and so well-defined. Theorem 5.0.3 determines the push-forward from the restrictions to Z^0_ξ and the normal bundle data. In Section 6 we explain how in good cases one can localize the integral further to $\mathrm{crit}(f)$. See Section 7 for examples.

5.1. Witten's deformation. Let M be a Hamiltonian K-manifold with proper moment map Φ, $v_\Phi\in\mathrm{Vect}(M)$ the Hamiltonian vector field for $f=\frac{1}{2}(\Phi,\Phi)$, g an invariant compatible metric on M, J the associated almost complex structure, and α the invariant one-form

$$\alpha(\cdot)=g(v_\Phi,\cdot)=\omega(v_\Phi,J(\cdot)).$$

We write

$$\mathrm{d}_K\alpha(\xi)=d\alpha+2\pi i(\phi,\xi),\qquad(\phi,\xi):=\iota(\xi_M)\alpha.$$

Witten introduced the deformation

$$\omega_s:=\omega+s\mathrm{d}\alpha,\qquad\Phi_s:=\Phi+s\phi,\qquad\tilde{\omega_s}:=\omega_s+2\pi i\Phi_s.$$

Note that

$$(\phi,\Phi)=g(v_\Phi,v_\Phi)\geq 0 \tag{22}$$

and equality holds only on $\mathrm{crit}(f)$.

Lemma 5.1.1. *Φ_s is proper for all $s\geq 0$.*

Proof. We have $\|\Phi_s\|^2=\|\Phi\|^2+2s(\Phi,\phi)+s^2\|\phi\|^2\geq\|\Phi\|^2$ by (22). Hence $\|\Phi_s\|^2\leq C$ implies $\|\Phi\|^2\leq C$ which shows that Φ_s is proper. □

Define $\mu_{M,K,s}(\eta) \in \mathcal{D}'(\mathfrak{k}^*)^K$ by

$$(\mu_{M,K,s}(\eta), h) = \sum_I \int_M \eta_I \wedge \exp(\omega_s) \partial_I h(\Phi(m) + s\phi(m)). \tag{23}$$

Since the cohomology class of ω_s is independent of s, so is $\mu_{M,K,s}(\eta)$. Let

$$U = \bigcup_{\xi \in \Xi(M)} U_\xi$$

be an open neighborhood of $\mathrm{crit}(f) \subset M$, so that

a) α is non-zero on $M - U$,
b) each U_ξ is an open neighborhood of C_ξ,
c) U_ξ are pairwise disjoint, and
d) each U_ξ intersects only orbit-type strata whose closures intersect C_ξ.

Any union of sufficiently small neighborhoods U_ξ of C_ξ has these properties. Since v_Φ is tangent to the K-orbits, ϕ is non-zero on $M - U$. By (22), for any $R > 0$ there exists an $s(R)$ such that $\|\Phi(m) + s\phi(m)\| > R$ for all $m \in M - U$ and $s > s(R)$. Since h has support in a ball of some radius $R(h)$,

$$\int_{M-U} \sum_I \eta_I \wedge \exp(\omega + s\mathrm{d}\alpha) \partial_I h(\Phi(m) + s\phi(m)) = 0 \tag{24}$$

for $s > s(R(h))$. Let $\mu_{\xi,s} := \mu_{U_\xi,K,s}(\eta)$ denote the distribution defined (23) except that integration is over U_ξ. By (24), for s sufficiently large

$$(\mu_{M,K}(\eta), h) = \sum_{\xi \in \Xi(M)} (\mu_{\xi,s}, h). \tag{25}$$

5.2. The limit distributions. We will show that $\mu_{\xi,s}$ has a distributional limit $\mu_{\xi,\infty}$ as $s \to \infty$. In most of this section we will assume that ξ is central, that is, fixed by the coadjoint action of K. An equivariant map $\varphi : M \to \mathfrak{k}^*$ will be called *annihilating* if

$$\mathrm{Im}(D_m \varphi) = \mathrm{ann}(\mathfrak{k}_m) := \{\lambda \in \mathfrak{k}^*,\ \lambda(\mathfrak{k}_m) = \{0\}\} \quad \forall m \in M.$$

This is part of the definition of an *abstract moment map* in [**24**],[**15**]. We say that $\xi \in \Xi(M)$ is *minimal* if (ξ, ξ) is the minimum value of (Φ, Φ).

Lemma 5.2.1. a) *$\phi | U_\xi$ is annihilating;*
b) *For s sufficiently large, $\Phi_s | U_\xi$ is annihilating;*
c) *If ξ is minimal, then $\Phi_s(\partial U_\xi)$ does not contain ξ for any $s \in [0, \infty)$.*

Proof. (a) We have

$$d(\phi, \zeta) = dg(v_\Phi, \zeta_M) = d\omega(\zeta_M, Jv_\Phi) = dL_{Jv_\Phi}(\Phi, \zeta) = L_{Jv_\Phi} d(\Phi, \zeta). \tag{26}$$

By a result of Duistermaat, the flow φ_t of $-\mathrm{grad}(f) = -J\Phi_M$ gives a deformation retract of M_ξ onto C_ξ, see [**48**, Appendix]. Since ξ is central, Φ is constant on C_ξ. It follows that $\varphi_t^* \mathrm{d}(\Phi, \zeta) \to 0$ as $t \to \infty$ for any $\zeta \in \mathfrak{k}_\xi$. If $\mathrm{d}(\phi, \zeta) = 0$ then (26) implies $\varphi_t^* \mathrm{d}(\Phi, \zeta) = d(\Phi, \zeta)$ for all t. Taking the limit $t \to \infty$ gives $\mathrm{d}(\Phi, \zeta) = 0$. Conversely, $\mathrm{d}(\Phi, \zeta)(m) = 0$

implies $\zeta_M(m) = 0$ and so $\mathrm{d}(\phi, \zeta)(m) = -\iota(\zeta_M(m))\mathrm{d}\alpha = 0$. (b) Since $\operatorname{Im} D_m\Phi = \operatorname{Im} D_m\phi$, we have $\operatorname{Im}(D_m\Phi + sD_m\phi) = \operatorname{Im}(D_m\phi)$ for s sufficiently large. (c) Suppose that ξ is minimal. Then (ξ, ξ) is the minimum of (Φ, Φ) and $C_\xi = \Phi^{-1}(K\xi)$, so that $(\Phi, \Phi) > (\xi, \xi)$ on ∂U_ξ. Hence $(\Phi_s, \Phi_s) = (\Phi, \Phi) + 2s(\Phi, \phi) + s^2(\phi, \phi) \geq (\Phi, \Phi) > (\xi, \xi)$ on ∂U_ξ. □

Lemma 5.2.2. *(compare Paradan [**36**, 3.8])*

a) *Let $[s_1, s_2]$ and $\lambda \in \mathfrak{k}^*$ be such that for all $s \in [s_1, s_2]$, $\Phi_s(\partial U_\xi)$ does not contain λ. Then the restriction of $\mu_{\xi,s}$ to a neighborhood of λ is independent of $s \in [s_1, s_2]$.*
b) *$\mu_{\xi,s}$ converges to a limit $\mu_{\xi,\infty}$ as $s \to \infty$.*
c) *For any $\lambda \in \mathfrak{k}_\xi^*$ and any s sufficiently large, $\mu_{\xi,s}$ is polynomial on the ray $R_{\xi,\lambda}$ near λ.*
d) *If ξ is minimal then $\mu_{\xi,\infty}$ is equal to $\mu_{\xi,0}$ in a neighborhood of ξ, and is polynomial on any ray beginning at ξ.*

Proof. (a) follows from 3.1.2. (b) By 5.2.1 (a), for any $\xi \in \Xi(M)$ and any $\lambda \in \mathfrak{k}^*$, there exists a $s > 0$ such that $\Phi_{s_1}(\partial U_\xi)$ does not contain λ for $s_1 > s$. Therefore, the claim follows from (a). (c) By 3.3.3, it suffices to show that $(\lambda, \lambda - \xi)$ is a regular value for $(\Phi_s, \lambda - \xi)$ for s sufficiently large. The fixed point set $U_\xi^{\lambda-\xi}$ of $\lambda - \xi$ satisfies $(\Phi(U_\xi^{\lambda-\xi}), \lambda - \xi) = (\xi, \lambda - \xi)$. Since $(\lambda, \lambda - \xi) \neq (\xi, \lambda - \xi)$, $\lambda - \xi$ acts infinitesimally freely on $(\Phi_s, \lambda - \xi)^{-1}((\lambda, \lambda - \xi))$. By 5.2.1 (e) Φ_s is a non-degenerate for s sufficiently large, hence $(\lambda, \lambda - \xi)$ is a regular value of $(\Phi_s, \lambda - \xi)$. (d) Assume ξ is the minimum. By 5.2.1 (b), $0 \notin \Phi_s(\partial U_\xi), \forall s \geq 0$. The claim follows from (a) and (c). □

Let ν_ξ' denote the bundle obtained from flipping the negative weights, as in (17), and U_ξ' the induced Hamiltonian K-structure on U_ξ with ω', $\Phi'|U_\xi'$ the new two-form and moment map. Define

$$\Phi_s^u|U_\xi = (1-u)\Phi_s|U_\xi + u\Phi_s'|U_\xi.$$

and $\mu_{\xi,s}^u(\eta)$ the twisted Duistermaat-Heckman distribution for ω_s^u, as above. By Proposition 3.1.2,

Proposition 5.2.3. *For any $\lambda \in \mathfrak{k}^*$, for s sufficiently large, $\mu_{\xi,s}^u$ is independent of $u \in [0, 1]$ in a neighborhood of λ.*

Corollary 5.2.4. $\mu_{\xi,\infty} = \operatorname{Ind}_{K_\xi}^K \mu_{Z_\xi^\circ, K_\xi}(\operatorname{Res}_{Z_\xi^\circ, K_\xi}^{M,K} \eta \wedge \operatorname{Eul}(\nu_\xi)_\xi^{-1})$.

Proof. By (17), $\mu_{\nu_\xi', K_\xi}(\pi^*\eta) = \mu_{Z_\xi, K_\xi}(\eta \wedge \operatorname{Eul}(\nu_\xi)_\xi^{-1})$. In the case ξ is central, 5.2.3 and 5.2.2 (d) imply $\mu_{\xi,\infty} = \mu_{\nu_\xi', K_\xi}(\pi^*\eta)$ in a neighborhood of ξ. Combining the two equalities proves the claim. The general case follows from 3.4.1. □

Theorem 5.0.3 follows from Lemma 5.2.2(b), Corollary 5.2.4, and (25).

Corollary 5.2.5. *If M is a Hamiltonian K-manifold with proper moment map and finite number of orbit-type strata, and $\eta \in \Omega_K(M)$ closed, then $\mu_{M,K}(\eta)$ is a tempered distribution.*

Proof. If M has a finite number of orbit-type strata, then the sum in Theorem 5.0.3 is finite. By Corollary 3.6.1 $\mu_{\xi,\infty}$ is tempered for all $\xi \in \Xi(M)$, hence $\mu_{M,K}(\eta)$ is a finite sum of tempered distributions. $\square$

5.3. Further comments.

a) One-parameter localization 2.0.1 for central, generic one-parameter subgroups is a special case of localization via the norm-square 5.0.3. Indeed, Let $\mathfrak{z} \subset \mathfrak{k}$ denote the center of $\mathfrak{k}$, and suppose that $\zeta \in \mathfrak{z}$. We can use ζ to shift the moment map $\Phi_s = \Phi + s\zeta$. For sufficiently large s, an element $m \in M$ is fixed by $\Phi_s(m)$ if and only if it is fixed by ζ. The subsets Z_ξ are components of M^ζ, and 5.0.3 reduces to 2.0.1.

b) The statement and proof of (5.0.3) are the same in the case that M is a Hamiltonian K-orbifold with proper moment map.

6. Pairing with invariant functions

Let M be a Hamiltonian K-manifold with proper moment map, and $\eta \in \Omega_K(M)$ closed. By (12) and (15), for each $\xi \in \Xi(M)$, the contribution from ξ to $(\mu_{M,K}(\eta), h)$ is

$$(\mu_{Z_\xi^\circ, K_\xi}(\operatorname{Res}^{M,K}_{Z_\xi^\circ, K_\xi} \eta \wedge \operatorname{Eul}(\nu_\xi)_\xi^{-1}), \operatorname{Vol}^K_{K_\xi} \operatorname{Res}^K_{K_\xi} h). \tag{27}$$

Hence

$$(\mu_{M,K}(\eta), h) = \sum_{\xi \in \Xi(M)} (\mu_{Z_\xi^\circ, K_\xi}(\eta \wedge \operatorname{Eul}_\xi^{-1}(\nu_\xi)), \operatorname{Vol}^K_{K_\xi} \operatorname{Res}^K_{K_\xi} h). \tag{28}$$

Suppose that $\Phi^{-1}(\xi)$ is contained in the principal orbit-type stratum for the action of K_ξ on Z_ξ. In this case the contribution from ξ can be expressed as an integral over the symplectic quotient

$$Z_{(\xi)} := (Z_\xi)_{(\xi)} = C_\xi / K.$$

This observation is essentially in Duistermaat-Heckman [**13**] and has nothing to do with the norm-square of the moment map; that is, Z_ξ can be any Hamiltonian action. In this section, we work out the relevant formulas.

Let K'_ξ denote the identity component of the generic stabilizer of K_ξ on $\Phi^{-1}(\xi) \cap Z_\xi$. The assumption that $\Phi^{-1}(\xi) \cap Z_\xi$ is contained in the principal orbit-type stratum of Z_ξ implies that K'_ξ acts trivially on the annihilator of $\mathfrak{k}'_\xi$. It follows that K'_ξ is normal and the quotient $K''_\xi := K_\xi / K'_\xi$ is a compact connected Lie group. Let K''_{ξ, Z_ξ} denote the (finite) generic stabilizer of K''_ξ on Z_ξ. Let κ_ξ denote the composition of the restriction $H_K(M) \to H_{K_\xi}(\Phi^{-1}(\xi))$ with the isomorphism

$$H_{K_\xi}(\Phi^{-1}(\xi) \cap Z_\xi) \to H_{K'_\xi}(Z_{(\xi)}) = H(Z_{(\xi)}) \otimes S(\mathfrak{k}_\xi^{\prime,*})^{K'_\xi}.$$

This extends to forms with smooth coefficients. In particular, any $h \in \mathcal{S}(\mathfrak{k}_\xi^*)^{K_\xi}$ defines a characteristic class $\kappa_\xi(h) \in H_{K'_\xi}(Z_{(\xi)})$. We denote by

$$\nu_{(\xi)} := K'_\xi \backslash (\nu_\xi|_{\Phi^{-1}(\xi)}) \to Z_{(\xi)}$$

the quotient bundle.

Theorem 6.0.1. *If $\Phi^{-1}(\xi) \cap Z_\xi$ is contained in the principal orbit-type stratum in Z_ξ, then* (27) *is equal to*

$$\mathrm{Vol}(K''_\xi / K''_{\xi, Z_\xi}) \int_{Z_{(\xi)} \times \mathfrak{k}'_\xi} \mathcal{L}_{(\xi)} \wedge \mathrm{Eul}(\nu_{(\xi)})_\xi^{-1} \wedge \kappa_\xi(\eta \wedge \mathrm{Vol}_{K_\xi}^K \mathrm{Res}_{K_\xi}^K h).$$

Note that the inverted Euler class has tempered-distributional coefficients, while the remaining factor has coefficients in the ring of Schwartz functions. The integral over $\mathfrak{k}'_\xi$ refers to the pairing of these coefficient rings.

Proof. Suppose that $\xi = 0$, $\eta = 1$, M is maximal rank, and $\Phi^{-1}(0)$ is contained in the principal orbit-type stratum for M. Let $\alpha \in \Omega^1(\Phi^{-1}(0), \mathfrak{k})^K$ denote a connection one-form for the action of K on $\Phi^{-1}(0)$. Let $\pi_0 : \Phi^{-1}(0) \to M_{(0)}$ denote the projection. By the coisotropic embedding theorem, a neighborhood of $\Phi^{-1}(0)$ is K-symplectomorphic to a neighborhood of $\Phi^{-1}(0) \times \{0\}$ in the Hamiltonian K-manifold

$$(\Phi^{-1}(0) \times \mathfrak{k}^*, \quad \pi_0^* \omega_{(0)} + \mathrm{d}(\lambda, \alpha)). \tag{29}$$

Let π_1, π_2 denote the projections

$$\Phi^{-1}(0) \overset{\pi_1}{\to} T \backslash \Phi^{-1}(0) \overset{\pi_2}{\to} M_{(0)}.$$

The two-form $\mathrm{d}\alpha$ is T-basic and descends to a closed $\mathfrak{k}$-valued two-form $\pi_{1,*} \mathrm{d}\alpha$. By (29) and (3.3), the volumes of quotients at generic $\lambda \in \mathfrak{k}^*$ are

$$p(\lambda) := \mathrm{Vol}(M_{(\lambda)}) = I_{T \backslash \Phi^{-1}(0)}(\exp(\pi_2^* \omega_{(0)} + (\lambda, \pi_{1,*} \mathrm{d}\alpha))).$$

Let $h \in \mathcal{S}(\mathfrak{k}^*)^K$, $f = \mathcal{F}_{\mathfrak{t}}^{-1}(\Pi \, \mathrm{Res}_{\mathfrak{t}^*}^{\mathfrak{k}^*} h)$ and $p = \sum_I p_I \lambda_I$. Using (11)

$$\begin{aligned}
\mathrm{Vol}(K/K_M)^{-1}(\mu_{M,K}, h) &= ((\mathrm{Vol}_T^K)^{-1} p, h)_{\mathfrak{k}^*} \\
&= (\# W)^{-1} (\mathrm{Res}_{\mathfrak{t}^*}^{\mathfrak{k}^*} p, \Pi h)_{\mathfrak{t}^*} \\
&= (\# W)^{-1} (\mathcal{F}_{\mathfrak{t}}^{-1}(\mathrm{Res}_{\mathfrak{t}^*}^{\mathfrak{k}^*} p), \mathcal{F}_{\mathfrak{t}}^{-1}(\Pi \, \mathrm{Res}_{\mathfrak{t}^*}^{\mathfrak{k}^*} h))_{\mathfrak{t}} \\
&= (\# W)^{-1} \sum_I p_I (\partial_I f)(0) \\
&= (\# W)^{-1} I_{T \backslash \Phi^{-1}(0)}(\exp(\pi_2^* \omega_{(0)}) f(\pi_{1,*} \frac{\mathrm{d}\alpha}{2\pi i}))
\end{aligned}$$

defined using the formal power series of f at 0. Choose an orthonormal basis $\xi_1, \ldots, \xi_r$ for $\mathfrak{t}$ and $\xi_{r+1}, \ldots, \xi_n$ for $\mathfrak{k}/\mathfrak{t} \cong \mathfrak{t}^\perp$. We can replace the integral over $T \backslash \Phi^{-1}(0)$ with

$$\frac{1}{\mathrm{Vol}(T)} \int_{\Phi^{-1}(0)} (\exp(\pi_2^* \omega_{(0)}) \wedge f(\frac{\mathrm{d}\alpha}{2\pi i}) \wedge \prod_{j=1}^r (\alpha, \xi_j)).$$

It remains to integrate over the fiber K of $\Phi^{-1}(0) \to M_{(0)}$. Writing

$$d\alpha = \pi_0^* \operatorname{curv}(\alpha) - \frac{1}{2}[\alpha, \alpha]$$

we see that the component of $f(\frac{\mathrm{d}\alpha}{2\pi i}) \wedge \prod_{j=1}^r (\alpha, \xi_j)$ that contributes to the fiber integral is

$$(\mathcal{F}_t^{-1} \Pi f) \left(\pi_0^* \frac{\operatorname{curv}(\alpha)}{2\pi i}) \right) \wedge \prod_{i=1}^n (\alpha, \xi_j).$$

Integrating over the fiber changes the factor $\prod_{i=1}^n (\alpha, \xi_j)$ to $\operatorname{Vol}(K)$. We have

$$\begin{aligned} (\#W)^{-1} (\mathcal{F}_t^{-1} (\Pi^2 \operatorname{Res}_{\mathfrak{t}^*}^{\mathfrak{k}^*} h))(\frac{\operatorname{curv}(\alpha)}{2\pi i}) &= (\mathcal{F}_{\mathfrak{k}}^{-1}(h))(\frac{\operatorname{curv}(\alpha)}{2\pi i}) \\ &= \mathcal{L}_0 \wedge \kappa_0(h)). \end{aligned}$$

It follows that

$$(\mu_{M,K}, h) = \operatorname{Vol}(K/K_M) I_{M_{(0)}} (\mathcal{L}_0 \wedge \kappa_0(h)).$$

(This computation is essentially contained in Witten [**47**].) The general case is the same, except that the forms on $Z_{(\xi)}$ are K'_ξ-equivariant and the form η is to be included. □

Corollary 6.0.2. *Let M be a compact Hamiltonian K-manifold, $\eta \in \Omega_K(M)$ closed, and $h \in \mathcal{S}(\mathfrak{k}^*)^K$. The pairing $(I_{M,K}(\eta), \mathcal{F}_{\mathfrak{k}}^{-1}(h))$ is equal to*

$$\sum_{\xi \in \Xi(M)} \operatorname{Vol}(K''_\xi / K''_{\xi, Z_\xi}) \int_{Z_{(\xi)} \times \mathfrak{k}'_\xi} \kappa_\xi (\eta \wedge \operatorname{Vol}_{K_\xi}^K \operatorname{Res}_{K_\xi}^K h) \wedge \operatorname{Eul}(\nu_{(\xi)})_\xi^{-1}.$$

Proof. Consider the family of equivariant symplectic forms $\epsilon \omega_K$ for $\epsilon \in (0,1]$. The corollary follows from taking the limit $\epsilon \to 0$ of (28), using Theorem 6.0.1: The stratification is independent of ϵ, and $\mathcal{L}_{(\xi)} \to 1$ as $\epsilon \to 0$. □

If $\Phi^{-1}(\xi) \cap Z_\xi$ is not contained in the principal orbit-type stratum of Z_ξ, then (27) can be written as a finite sum of integrals over symplectic quotients near ξ, by the gluing rule in Meinrenken [**33**], but I do not know a nice formula for the limit.

7. Examples

In these examples we will compare the one-parameter and norm-square localization formulas.

7.1. $K = U(1)$ **acting on** $M = \mathbb{P}^1$**.** We identify the Lie algebra $i\mathbb{R}$ of $U(1)$ with $\mathbb{R}$ by division by $2\pi i$. If we choose on $\mathbb{R}$ the standard inner product then the weight lattice becomes identified with $\mathbb{Z}$, $\mu_{\mathfrak{t}^*}$ is Lebesgue measure on $\mathbb{R}$, and the volume of K is 1. The action of $U(1)$ on $\mathbb{P}^1$ by $z[w_0, w_1] = [z^{-a} w_0, z^{-b} w_1]$ has moment map

$$\Phi([w_0, w_1]) = \frac{a|w_0|^2 + b|w_1|^2}{|w_0|^2 + |w_1|^2}.$$

There are two fixed points, at $w_0 = 0$, resp. $w_1 = 0$. The tangent weights at the fixed points are $\pm(b-a)$. Let $H_\pm$ denote the Heaviside distributions, equal to $\mu_{\mathfrak{k}^*}$ on the positive (resp. negative) real numbers and zero elsewhere. One-parameter localization with $\xi > 0$ gives

$$\mu_{M,K} = \frac{1}{b-a}(\delta(a) \star H_+ - \delta(b) \star H_+) = \frac{\chi_{[a,b]}\mu_{\mathfrak{k}^*}}{b-a}$$

where $\chi_{[a,b]}$ the characteristic function for the interval $[a,b]$ and $\star$ denotes convolution. For negative action chamber $\xi < 0$ localization gives

$$\mu_{M,K} = \frac{1}{b-a}(-\delta(a) \star H_- + \delta(b) \star H_-) = \frac{\chi_{[a,b]}\mu_{\mathfrak{k}^*}}{b-a}.$$

This is shown graphically in Figure 1.

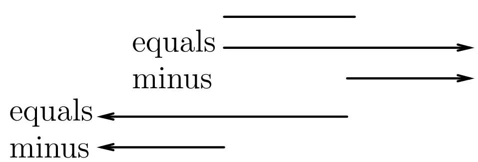

Figure 1. One-parameter localization for $\mathbb{P}^1$

The Kirwan-Ness localization is as follows for $a < 0 < b$ is as follows. The critical components of f are the two T-fixed points and the zero level set:

$$C_a = \{w_0 = 0\} = M_a, \quad C_b = \{w_1 = 0\} = M_b$$

$$C_0 = \Phi^{-1}(0), \; M_0 = \{[w_1, w_2], w_1 w_2 \neq 0\} = M - M_a - M_b.$$

For $\xi = 0$, $Z_\xi^\circ = M_\xi$ is the complement of the K-fixed points. The Duistermaat-Heckman measure for Z_ξ° is $\mu_{\mathfrak{k}^*}\chi_{[a,b]}/(b-a)$. Its unique extension which is piecewise polynomial on any ray beginning at 0 is $\mu_{\mathfrak{k}^*}/(b-a)$. Theorem 5.0.3 gives

$$\begin{aligned}
\mu_{M,K} &= \mu_{M,0} + \mu_{M,b} + \mu_{M,a} \\
&= \frac{1}{b-a}\left(\mu_{\mathfrak{k}^*} - H_- \star \delta(a) - H_+ \star \delta(b)\right) \\
&= \frac{1}{b-a}\mu_{\mathfrak{k}^*}(1 - \chi_{(-\infty,a]} - \chi_{[b,\infty)}).
\end{aligned}$$

The formula is shown graphically in Figure 2.

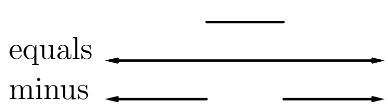

Figure 2. Norm-square localization for $\mathbb{P}^1$

7.2. $SU(3)$ **acting on a** G_2**-coadjoint orbit.** In this example, we apply the localization formulas to the action of $SU(3) \subset G_2$ on a coadjoint orbit of G_2. Let $K = SU(3)$, and ω_1, ω_2 the fundamental weights. Let G_2 denote the connected simple complex group of type G_2. The dual positive Weyl chamber for G_2 is the span of ω_1 and $\omega_1+\omega_2$. Let $P_{\omega_1+\omega_2}$ denote the maximal parabolic of G_2, so that $M = G_2/P_{\omega_1+\omega_2}$ is diffeomorphic to the coadjoint orbit through $\omega_1+\omega_2$. The Weyl group W for $SU(3)$ acts simply transitively on the T-fixed points. First we compute the Duistermaat-Heckman measure using ordinary localization. The contribution to $\mu_{M,T}$ from the fixed point $x(w)$ corresponding to $w \in W$ is

$$\delta_{w\mu} \star \prod_{j=1}^{5} \pm H_{\pm w\beta_j}$$

where β_j are the positive roots of G_2 not vanishing at $\omega_1 + \omega_2$

$$2\omega_1 - \omega_2,\ -\omega_1 + 2\omega_2,\ \omega_1 + \omega_2,\ 3\omega_1,\ 3\omega_2$$

and the signs are determined by the action chamber. The β_j that are not roots of $SU(3)$ are $\beta_5 = 3\omega_1,\ \beta_6 = 3\omega_2$. By (3.5.2)

$$\mu_{M,K} = \frac{1}{6}\operatorname{Ind}_T^K \sum_{w\in W} (-1)^{l(w)} \delta_{w(\omega_1+\omega_2)} \star (\pm H_{\pm 3w\omega_1}) \star (\pm H_{\pm 3w\omega_2}).$$

The contributions are shown in Figure 3. Each contribution is $\pm\mu_{\mathfrak{t}^*}/(9\|\omega_1\|\|\omega_2\|)$ where non-negative. Positive (resp. negative) contributions are shown in light (resp. dark) shading. The moment polytope for M is

$$\Delta = \operatorname{hull}(\omega_1, \omega_2, \omega_1 + \omega_2).$$

Let F_1 be the open face connecting $\omega_2, \omega_1+\omega_2$, F_2 the open face connecting $\omega_1, \omega_1+\omega_2$, and F_3 the open face connecting ω_1, ω_2. Let $F_{ij} = F_1 \cap F_2$.

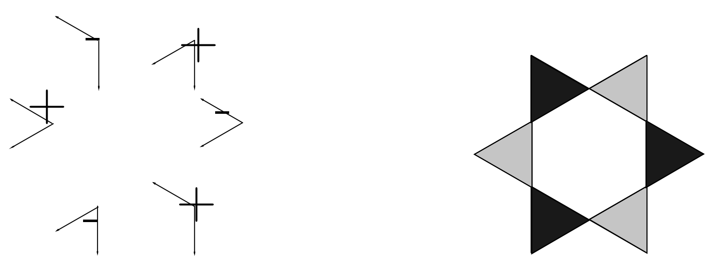

Figure 3. One-parameter localization for G_2/P

We compute the Kirwan-Ness stratification. The inverse image $\Phi^{-1}(F_{12})$ contains a unique point, $x(1) \in M$, which is T-fixed. None of the other T-fixed points map to $\mathfrak{t}^*_+$. Therefore, the remaining points in $\Phi^{-1}(\operatorname{int}(\mathfrak{t}^*_+))$ have one-dimensional stabilizers. Since $\Phi^{-1}(\operatorname{int}(\mathfrak{t}^*_+))$ has dimension $2\dim(T)$, it is a toric manifold, so the inverse image of any face $F \subset \operatorname{int}\mathfrak{t}^*_+$ has infinitesimal stabilizer the annihilator of the tangent space of F. The stabilizers of the faces F_1, F_2, F_3 are

$$\mathfrak{t}_1 = \mathrm{span}(h_1), \ \mathfrak{t}_2 = \mathrm{span}(h_2), \ \mathfrak{t}_3 = \mathrm{span}(h_3)$$

where h_1, h_2, h_3 are the coroots of $SU(3)$. The level set $\Phi^{-1}((\omega_1 + \omega_2)/2)$ is critical with $\xi = (\omega_1 + \omega_2)/2$. The fixed point component Z_ξ has moment

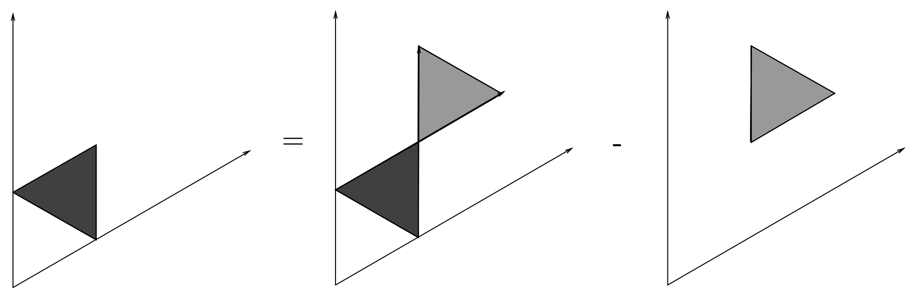

Figure 4. Norm-square localization for G_2/P

image

$$\Phi(Z_\xi) = \mathrm{hull}(2\omega_2 - \omega_1, 2\omega_1 - \omega_2).$$

The unstable manifold Y_ξ has image under the moment map for T

$$\mathrm{proj}^{\mathfrak{k}}_{\mathfrak{t}} \Phi(\overline{Y_\xi}) = \mathrm{hull}(2\omega_2 - \omega_1, 2\omega_1 - \omega_2, \omega_1 + \omega_2).$$

None of the other faces F_j contain points ξ with $\xi \in \mathfrak{t}_j$. Therefore, there are no other critical points in $\Phi^{-1}(\mathrm{int}(\mathfrak{t}^*_+))$. Finally consider the inverse image of the vertices $F_{jk} = F_{13}$ or F_{23}. $\Phi^{-1}(F_{jk})$ does not contain a T-fixed point. $\Phi^{-1}(F_{jk})$ does not contain a point m stabilized by $\mathrm{span}(F_{jk})$. Indeed, since the stabilizer K_m does not contain a maximal torus, K_m cannot intersect the semisimple part $[K_{\Phi(m)}, K_{\Phi(m)}]$. Therefore, K_m is one-dimensional. Let X denote the fixed point component of K_m containing m. Since K_m is one-dimensional, the image $\Phi(X)$ is codimension one, and so meets $\Phi^{-1}(\mathrm{int}(t^*_+))$. This implies that the $\mathfrak{k}_m$ is conjugate to either $\mathfrak{t}_j$ or $\mathfrak{t}_k$, and so $\mathfrak{k}_m$ cannot equal the span of F_{jk}. Therefore,

$$\Xi(M) = \{\omega_1 + \omega_2, \frac{1}{2}(\omega_1 + \omega_2)\}.$$

One can show that the Kirwan-Ness stratification coincides with the orbit stratification for G, just as in the previous example. In particular M is a two-orbit variety, with one open orbit and one of complex codimension two.

The contributions to the norm-square localization formula can be described as follows. For $\xi = (\omega_1 + \omega_2)/2$ we have

$$\mu_{\xi,\infty} = \delta((\omega_1 + \omega_2, \xi) = \frac{1}{2}\|\omega_1 + \omega_2\|) \star H_{\omega_1+\omega_2}/(9\|\omega_1\|\|\omega_2\|).$$

where $H_{\omega_1+\omega_2}$ is the Heaviside distribution for $(\omega_1 + \omega_2, \xi) \geq 0$. Therefore,

$$\mathrm{Ind}^K_T \mu^\infty_\xi = \mathrm{Ind}^K_T (\chi_\Delta - \chi_\Box)\mu_{\mathfrak{t}^*}/(9\|\omega_1\|\|\omega_2\|).$$

where $\chi_\Delta, \chi_\Box$ are the characteristic functions for the polytope Δ, resp. the cone

$$\Box = \mathbb{R}_{\leq 0}(\Delta - (\omega_1 + \omega_2)) + \omega_1 + \omega_2.$$

For $\xi = \omega_1 + \omega_2$ we get

$$\operatorname{Ind}_T^K \mu_\xi^\infty = \operatorname{Ind}_T^K \frac{\chi_\Box \mu_{\mathfrak{t}^*}}{9\|\omega_1\|\|\omega_2\|}.$$

Hence

$$\mu_{M,K} = \operatorname{Ind}_T^K \left(\frac{\chi_\Delta \mu_{\mathfrak{t}^*}}{9\|\omega_1\|\|\omega_2\|} \right).$$

See Figure 4.

8. A remark on sheaf cohomology

In algebraic geometry there is a formula which expresses the index of a sheaf of a stratified variety as a sum over the strata. Let G be a reductive complex group, $R(G)$ the ring of finite linear combinations of irreducible characters, and $\mathcal{R}(G) = \operatorname{Hom}(R(G), \mathbb{Z})$ its dual. Let M be a smooth quasiprojective variety, and $E \to M$ a G-equivariant vector bundle. The *equivariant index* of E is the virtual representation

$$I_{M,K}(E) = \sum_{j=0}^{\dim(M)} (-1)^j H^j(M, E).$$

We will assume that the multiplicity of any irreducible representation is finite, so that $I_{M,K}(E)$ defines an element in $\mathcal{R}(G)$.

Suppose M decomposes into a disjoint union of smooth G-stable subvarieties

$$M = \bigcup_{\xi \in \Xi(M)} M_\xi.$$

Let $T_{M_\xi} M$ is the normal bundle of $M_\xi \to M$, $T^*_{M_\xi} M$ its dual. The Euler class

$$\operatorname{Eul}(T_{M_\xi} M) := \Lambda^{\text{even}}(T_{M_\xi} M) \ominus \Lambda^{\text{odd}}(T_{M_\xi} M)$$

has a formal inverse

$$\operatorname{Eul}(T_{M_\xi} M)^{-1} := (-1)^{\operatorname{codim}(M_\xi)} \det(T^*_{M_\xi} M) \otimes S(T^*_{M_\xi} M)$$

where S resp. Λ denotes the direct sum of symmetric resp. exterior powers and det the top exterior power. Let $\operatorname{Res}^M_{M_\xi}$ denote restriction to M_ξ. The Cousin-Grothendieck spectral sequence (take the Euler characteristic of the local cohomologies) produces a formula in $\mathcal{R}(G)$

$$I_{M,K}(E) = \sum_{\xi \in \Xi(M)} I_{M_\xi,K}(\operatorname{Res}^M_{M_\xi} E \otimes \operatorname{Eul}(T_{M_\xi} M)^{-1}) \tag{30}$$

assuming that the representations on the right hand side have finite multiplicities, see Teleman [**43**] and Hartshorne [**19**, Section 4].

In some sense the localization theorems in equivariant cohomology or K-theory are attempts to extend this result to manifolds with group actions; so far this has been done only in special cases. One parameter localization arises from the stratification defined by the action of a circle subgroup $G = \mathbb{C}^*$. Let $\Xi(M)$ denote the set of connected components of the fixed point set M^G in M. For any $\xi \in \Xi(M)$, let

$$M_\xi = \{m \in M, \quad \lim_{z \to 0} zm \in \xi\}$$

denote the stable manifolds for the flow generated by the action, as in Bialinicki-Birula [**7**]. The formula (30) is a sheaf-theoretic version of 2.0.1. (30) applied to the Kirwan-Ness stratification gives a sheaf-theoretic version of 5.0.3. It remains an open question, at least for me, whether there is a more general formula in equivariant K-theory or in equivariant de Rham theory analogous to (30). For instance, the decomposition of a projective spherical G-variety into G-orbits produces a formula in K-theory not covered by one-parameter localization or localization via the norm-square of the moment map.

9. 2d Yang-Mills

The basic reference for mathematical two dimensional Yang-Mills theory is Atiyah-Bott [**2**]. Let K denote a connected compact Lie group and G the complexification of K. Fix the basic inner product $(\, , \,) : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ and use it to identify $\mathfrak{k}$ with its dual $\mathfrak{k}^*$. Let P be a principal K-bundle and $P(\mathfrak{k}) = P \times_K \mathfrak{k}$ the adjoint bundle. Similarly, let $P(G) = P \times_K G$ the associated principal G-bundle. Let $\Omega^\bullet(X, P(\mathfrak{k}))$ the space of forms with values in $P(\mathfrak{k})$. The inner product on $\mathfrak{k}$ induces a metric on $P(\mathfrak{k})$. Combining this with the wedge product gives map

$$\Omega^k(X, P(\mathfrak{k})) \times \Omega^l(X, P(\mathfrak{k})) \to \Omega^{k+l}(X), \quad (a_1, a_2) \mapsto (a_1 \wedge a_2).$$

Choose a metric on X and let $*$ denote the associated Hodge star operator $\Omega^k(X, P(\mathfrak{k})) \to \Omega^{2-k}(X, P(\mathfrak{k}))$. Let

$$\mathcal{A}(P) = \Omega^1(X, P(\mathfrak{k}))$$

the affine space of connections and

$$K(P) = \mathrm{Aut}_K(P), \; G(P) = \mathrm{Aut}_G(P(G))$$

the group of unitary, resp. complex gauge transformations. For any $A \in \mathcal{A}(P)$, let $F_A \in \Omega^2(X, P(\mathfrak{k}))$ denote its curvature. Yang-Mills theory is the area-dependent quantum field theory with partition function given

$$Z(X) = \sum_P Z(P)$$

where the sum is over isomorphism classes of principal K-bundles P and $Z(P)$ is defined formally by the path integral

$$“Z(P) = \frac{1}{\mathrm{Vol}(K(P))} \int_{\mathcal{A}(P)} \exp(-S(A)) DA”, \quad S(A) = \frac{1}{2\epsilon} \int_X (F_A \wedge *F_A).$$

Formally $Z(P)$ is the pairing of the Duistermaat-Heckman measure for the action of $K(P)$ on $\mathcal{A}(P)$ with a Gaussian on $\Omega^2(X, P(\mathfrak{k}))$.

A definition of the two-dimensional Yang-Mills integral, including observables, is given by Levy [**29**]. Levy's approach is to embed the space of connections mod gauge equivalence into the space of maps of the loop space on Σ to the group mod conjugacy via the holonomy map. Levy constructs a probability measure on this “thickening” of the space of connections and proves that the Yang-Mills integral is given by the Migdal formula.

One can give a different definition of the Yang-Mills integral by assuming that localization for the norm-square (5.0.3) holds. The strategy of defining path integrals by expanding over critical components of the integrand appears in many places, such as perturbative Chern-Simons theory [**4**]. The purpose of this section is to show that with this definition, the Yang-Mills integral is given by the Migdal formula, and hence agrees with Levy's definition. This might be seen as an easy two-dimensional analog of the much harder conjecture regarding the three-dimensional Chern-Simons path integral, that the “exact” definition via Reshetikhin-Turaev agrees with the “perturbative” definition of Axelrod-Singer.

The action of $K(P)$ on $\mathcal{A}(P)$ is Hamiltonian with moment map minus the curvature, and so the Yang-Mills function $S(A)$ is the norm-square of the moment map. The critical points of $S(A)$ are the connections satisfying the *Yang-Mills equation*

$$d_A^* F_A = 0.$$

These are the connections with constant central curvature. Each such connection is gauge equivalent to a connection A' with $F_{A'} = *\xi$; let $\Xi(P)$ denote the set of ξ. In the case $K = U(r)$, P is the principal $U(r)$ bundle of rank r and degree d over a surface X of genus at least one,

$$\Xi(P) = \{(\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \ldots, \mu_r)\} \subset \mathbb{Q}^r$$

the set of non-increasing sequences such that $\mu_j = d_j/r_j$ for some integers d_j and r_j such that $\sum_j d_j = d, \quad \sum_j r_j = r$ and each μ_j appears r_j times. If X is genus zero then only integral μ_j appear.

Minus the gradient flow of $S(A)$ induces a decomposition of $\mathcal{A}(P)$ into stable manifolds

$$\mathcal{A}(P) = \bigcup_{\xi \in \Xi} \mathcal{A}(P)_\xi$$

By results of Donaldson [**11**], Daskalopolous[**10**], Råde [**39**], and Atiyah-Bott [**2**] this is identical to the decomposition by Harder-Narasimhan type of the corresponding holomorphic G-bundle. For $K = U(n)$, $\mathcal{A}(P)_\xi$ consists of connections such that the corresponding holomorphic bundle has Harder-Narasimhan quotients with ranks r_j and degrees d_j.

For each $\xi \in \Xi(P)$, the universal quotient of $\mathcal{A}(P)_\xi$ by $G(P)$ is the moduli space $\mathcal{M}(X, K_\xi; \xi)$ of K_ξ-bundles with constant central curvature ξ. Define a vector bundle $\nu_\xi \to \mathcal{M}(X, K_\xi; s\xi)$ by

$$(\nu_\xi)_{[A]} = (H^1 - H^0)(\overline{\partial}_A, \mathfrak{g}/\mathfrak{g}_\xi) \oplus \mathfrak{p}_\xi/\mathfrak{g}_\xi$$

where $\overline{\partial}_A$ is the corresponding Dolbeault operator. Except for the factor $\mathfrak{p}_\xi/\mathfrak{g}_\xi$, this is the virtual normal bundle for the embedding of moduli stacks induced by $G_\xi \to G$, see [**44**]. The inclusion of $\mathfrak{p}_\xi/\mathfrak{g}_\xi$ in the definition has to do with the fact that the generic complex automorphism group of a bundle of type ξ is the corresponding parabolic P_ξ, which means that $-\mathfrak{p}_\xi/\mathfrak{g}_\xi$ appears in the stacky normal bundle ν_ξ but not in the corresponding formula in Section 5.

Let K'_ξ denote the identity component of the generic automorphism group for $M(X, K_\xi; \xi)$, $K''_\xi = K_\xi/K'_\xi$. Let $K''_{\xi,M}$ denote the (finite) subgroup of K''_ξ contained in the generic automorphism group. Let $\mathcal{M}(X, x, K_\xi; \xi)$ denote the moduli space of bundles with framing at a base point x. If every point in $\mathcal{M}(X, K_\xi; \xi)$ has automorphism group K'_ξ then $M(X, x, K_\xi; \xi)$ is a locally free K''_ξ-space with quotient $\mathcal{M}(X, K_\xi; \xi)$. For any $h \in \mathcal{S}(\mathfrak{k}_\xi)^{K_\xi}$, let $\kappa_\xi(h) \in \mathcal{H}_{K'_\xi}(\mathcal{M}(X, K_\xi; \xi))$ denote the corresponding characteristic class. Let $\mathcal{L}_{(\xi)}$ denote the K'_ξ-equivariant Liouville form on $\mathcal{M}(X, K_\xi; \xi)$, with constant moment map with value ξ. Let $\mu_{\mathcal{A}(X),\xi} \in \mathcal{S}'(\mathfrak{k}^*)^K$ denote the distribution defined by

$$(\mu_{\mathcal{A}(X),\xi}, h) = \int_{\mathcal{M}(X, K_\xi; \xi) \times \mathfrak{k}'_\xi} \mathcal{L}_{(\xi)} \wedge \mathrm{Eul}(\nu_\xi)_\xi^{-1} \wedge \kappa_\xi(\mathrm{Vol}_{K_\xi}^K \mathrm{Res}_{K_\xi}^K h),$$

times $\mathrm{Vol}(K''_\xi/K''_{\xi,M})$, compare with 6.0.1. Let

$$\Xi(X) = \bigcup_P \Xi(P)$$

and define the Yang-Mills partition function by

$$Z(X) := \sum_{\xi \in \Xi(X)} (\mu_{\mathcal{A}(X),\xi}, h)$$

where $h \in \mathcal{S}(\mathfrak{k}^*)^K$ is the Fourier transform of $\hat{h}(\zeta) = \exp\left(-\frac{\epsilon}{2}\|\zeta\|^2\right)$. (There is a slight inconsistency with the previous formal definition of a missing factor of a power of ϵ, which we will ignore.) From our point of view, there is nothing special about the Gaussian and h could be an invariant Schwartz function.

Some care is needed for the definition in the presence of reducible connections. Let $\mathcal{M}(X, K)_\nu$ denote the moduli space of flat K bundles on the once-punctured surface, with holonomy around the puncture conjugate to $\exp(\nu)$. This space admits a holomorphic description in terms of semistable bundles with a *parabolic reduction* at the puncture, described in Mehta-Seshadri [**31**]. Let $Z(K)$ denote the center of K, and $K'' = K/Z(K)$. The function

$$Z(X,\nu) := \#\operatorname{Vol}(Z(K))\operatorname{Vol}(\mathcal{M}(X,K)_\nu)$$

is piecewise polynomial for $\nu \in \mathfrak{k}''$. If every point in $\mathcal{M}(X,K)$ has automorphism group $Z(K)$ then

$$\mu_{\mathcal{A}(X),0} = \operatorname{Vol}(K \cdot \nu)^{-1} Z(X,\nu)\mu_{\mathfrak{k}'',*}$$

for ν in a neighborhood of 0. In case $\mathcal{M}(X,K)$ contains reducibles, this can be taken as the definition of μ_0. There are similar definitions for the other distributions μ_ξ in the presence of reducible connections.

The main result of this section is

Theorem 9.0.1. *("Migdal formula", see* [**46**, 2.51]*) Let K be a compact connected group. The 2-dimensional Yang-Mills partition function is given by*

$$Z(X) = \operatorname{Vol}(K)^{2g}\sum_\nu (\dim V_\nu)^{2-2g}\hat{h}(\nu+\rho)$$

where the sum is over dominant ν in the weight lattice Λ^* plus ρ.

Here ρ is the half-sum of the positive roots which is a weight if $\mathfrak{k}$ is spinnable. Before we give the proof, we note the corollary (as already discussed in [**46**])

Corollary 9.0.2. *Suppose that K is semisimple and $g \geq 2$. The volume of the moduli space $\mathcal{M}(X,K)$ is*

$$\operatorname{Vol}(\mathcal{M}(X,K)) = \#Z(K)\dim(K)^{2g}\sum_\nu(\dim V_\nu)^{2-2g}$$

where $Z(K)$ is the center of K.

Proof. Take the limit $\epsilon \to 0$ in Theorem 9.0.1. By definition of $Z(X)$, the limit

$$\lim_{\epsilon\to 0} Z(X) = \#Z(K)^{-1}\operatorname{Vol}(\mathcal{M}(X,K)).$$

On the other hand, the limit of the right hand side of 9.0.1 is

$$\dim(K)^{2g}\sum_\nu(\dim V_\nu)^{2-2g},$$

which proves the corollary. ∎

The measures $\mu_{\mathcal{A}(X),\xi}$ for ξ generic can be described as follows. The moduli space $\mathcal{M}(X,K_\xi,\xi)$ is the Jacobian of torus bundles with first Chern class ξ, and is diffeomorphic to T^{2g}. The characteristic classes of the bundle ν are computed in [**42**],[**44**, p.8]. One obtains

$$\operatorname{Eul}(\nu_\xi) = (-1)^{2\rho(\xi)}|\operatorname{Eul}(\mathfrak{k}/\mathfrak{t})|^{2g-2}.$$

Integrating over $\mathcal{M}(X, K_\xi, \xi)$ gives

$$\begin{aligned}\mu_{\mathcal{A}(X),\xi} &= i^{(2g-1)\dim(K/T)/2}(-1)^{2\rho(\xi)} \operatorname{Ind}_T^K \int_{T^{2g}} \exp(\omega_{(\xi)})\delta_\xi \operatorname{Eul}(\mathfrak{k}/\mathfrak{t})_\xi^{1-2g} \\ &= i^{(2g-1)\dim(K/T)/2}(-1)^{2\rho(\xi)} \operatorname{Ind}_T^K \operatorname{Vol}(T^{2g})\delta(\xi) \operatorname{Eul}(\mathfrak{k}/\mathfrak{t})_\xi^{1-2g}.\end{aligned}$$

The proof of Theorem 9.0.1 is based on the idea, introduced by C. Teleman [**42**], that the sum over strata is the same as the sum of contributions from the T-bundles. Define

$$\mu_{\mathcal{A}(X)} := \sum_{\xi \in \Xi(X)} \mu_{\mathcal{A}(X),\xi} \in \mathcal{D}'(\mathfrak{k}^*)^K$$

which is a kind of Duistermaat-Heckman measure for $\mathcal{A}(X)$. Let

$$\beta_{\mathcal{A}(X),\xi} := \frac{i^{(g-\frac{1}{2})\dim(K/T)}}{\# W_\xi}(-1)^{2\rho(\xi)} \operatorname{Ind}_T^K \operatorname{Vol}(T^{2g})\delta(\xi) \operatorname{Eul}(\mathfrak{k}/\mathfrak{t})_\zeta^{1-2g}$$

if $\mathcal{M}(X, K_\xi, \xi)$ contains T-bundles, and zero otherwise. Here $\zeta \in \mathfrak{t}_+^*$ is any regular element. We wish to compare $\mu_{\mathcal{A}(X)}$ with

$$\beta_{\mathcal{A}(X)} := \sum_{\xi \in \Xi(X)} \beta_{\mathcal{A}(X),\xi}$$

which is the sum of the "fixed-point contributions" from T-bundles. For any distribution $\mu \in \mathcal{D}'(\mathfrak{k})^K$, define a distribution $\mu_T \in \mathcal{D}'(\mathfrak{t})^{\operatorname{sign}(W)}$ by

$$(\mu_T, \operatorname{Vol}_T^K \operatorname{Res}_T^K h) := (\mu, h).$$

The map $\mu \mapsto \mu_T$ is a right inverse to Ind_T^K. We will need the following lemma. Part (a) is the high level limit of Proposition 3.3 in [**44**]. There is an alternative proof that does not use algebraic geometry, but I will not describe here since it yields weaker results that the argument in [**44**]. Part (b) follows from [**44**].

Lemma 9.0.3. a) *$\mu_{\mathcal{A}(X),T}$ is invariant under translation by the coweight lattice Λ and anti-invariant under W. (In other words, anti-invariant under the action of the affine Weyl group.)*
b) *$(\mu_{\mathcal{A}(X),\xi} - \beta_{\mathcal{A}(X),\xi})_T$ has Fourier transform supported in $\mathfrak{t}_{\mathrm{sing}}$.*

Since $\mu_{\mathcal{A}(X),T}$ is a periodic distribution, its Fourier transform $\mathcal{F}_\mathfrak{t}^{-1}\mu_{\mathcal{A}(X),T}$ is a sum of delta functions at the weights of T:

$$\mathcal{F}_\mathfrak{t}^{-1}\mu_{\mathcal{A}(X),T} = \sum c_\lambda \delta_\lambda.$$

Since $\mu_{\mathcal{A}(X),T}$ is W-anti-invariant, $c_\lambda = 0$ unless λ is regular. By part (b) of Lemma 9.0.3, $\mu_{\mathcal{A}(X),T}$ is equal to $\beta_{\mathcal{A}(X),T}$ plus a distribution whose Fourier transform is supported in $\mathfrak{t}_{\mathrm{sing}}$. We have

$$\beta_{\mathcal{A}(X),T} = \frac{i^{(g-\frac{1}{2})\dim(K/T)}}{\# W} \sum_{\xi \in \Lambda} \delta_\xi \operatorname{Vol}(T^{2g})(-1)^{2\rho(\xi)} \operatorname{Eul}(\mathfrak{k}/\mathfrak{t})^{1-2g}.$$

By the Poisson summation formula

$$\mathcal{F}_t^{-1}\mu_{\mathcal{A}(X),T} = i^{-\dim(K/T)/2}\sum_{\lambda\in\Lambda^*+\rho}(\#W)^{-1}\delta_\lambda \operatorname{Vol}(T^{2g})\prod_{\alpha>0}2\pi(\alpha,\lambda)^{1-2g}$$

where (since $c_\lambda = 0$ for singular λ) the sum is over *regular* λ. Hence

$$\mu_{\mathcal{A}(X)} = i^{-\dim(K/T)/2}\operatorname{Vol}(K)^{2g-1}\operatorname{Vol}(T)\operatorname{Ind}_T^K\sum_\lambda \delta_{\lambda+\rho}\dim(V_\lambda)^{1-2g}$$

where the sum is over λ such that $\lambda+\rho$ is a dominant weight. (If λ is not a weight, V_λ is a representation of the universal cover of K.) Finally pairing with the Gaussian h gives

$$\begin{aligned}(\mu_{\mathcal{A}(X)}, h) &= \frac{\operatorname{Vol}(K)^{2g-1}\operatorname{Vol}(T)}{i^{\dim(K/T)/2}}\left(\sum_\lambda \delta_{\lambda+\rho}\dim(V_\lambda)^{1-2g}, \operatorname{Vol}_T^K\operatorname{Res}_T^K\hat{h}\right)\\ &= \left(\operatorname{Vol}(K)^{2g}\sum_\lambda \delta_{\lambda+\rho}\dim(V_\lambda)^{2-2g}, \operatorname{Res}_T^K\hat{h}\right)\\ &= \operatorname{Vol}(K)^{2g}\sum_\lambda \dim(V_\lambda)^{2-2g}\hat{h}(\lambda+\rho)\end{aligned}$$

which completes the proof of 9.0.1. This computation is done on a physics level of rigor by Blau-Thompson [**8**]. The main point is that the contribution of the semistable stratum is not affine Weyl-invariant, but only becomes so after adding the contributions from the higher strata. As in [**44**], the additional symmetry removes the necessity of doing any hard computations, that is, any integrals other than integrals over Jacobians.

Example 9.0.4. Let $K = SU(2)$ and identify $\mathfrak{t}\to\mathbb{R}$ so that the weight lattice is $\mathbb{Z}/2$ and coweight lattice $\mathbb{Z}$. If X has genus $g=1$, an explicit computation shows

$$Z(X,\nu) = \frac{1}{4}\operatorname{Vol}(T^2)(1-2\nu).$$

For ξ a positive integer, $K_\xi = K$ and $M(X,T;\xi) = T^2$. The normal bundle $\nu_\xi = (\mathfrak{k}/\mathfrak{t})^2$, hence

$$\begin{aligned}\mu_{\mathcal{A}(X),\xi} &= \frac{1}{2}\operatorname{Ind}_T^K\operatorname{Vol}(T^2)\delta(\xi)\operatorname{Eul}(\mathfrak{k}/\mathfrak{t})_+^{-1} - \delta(-\xi)\operatorname{Eul}(\mathfrak{k}/\mathfrak{t}))_-^{-1}\\ &= \frac{1}{2}\operatorname{Vol}(T^2)(\delta(\xi)H_+ - \delta(-x)H_-)\end{aligned}$$

where $H_\pm$ are the Heaviside distributions. The sum is the sawtooth distribution shown below in solid lines in Figure 5. The dotted line is the contribution from $\xi=0$.

It seems an interesting question whether a similar definition could be used for other path integrals, for instance, holomorphic Yang-Mills theory in four dimensions. On the other hand, other path integrals such as full four-dimensional Yang-Mills or two-dimensional Yang-Mills with observables

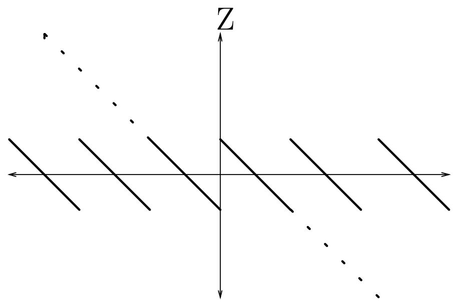

Figure 5. One-point function for genus one, $K = SU(2)$

do not seem to admit heuristic interpretations as pairings in equivariant cohomology, and it appears unlikely that the techniques described here would apply.

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Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.
E-mail address: ctw@math.rutgers.edu

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