Geometry and measurement in middle-school mathematics
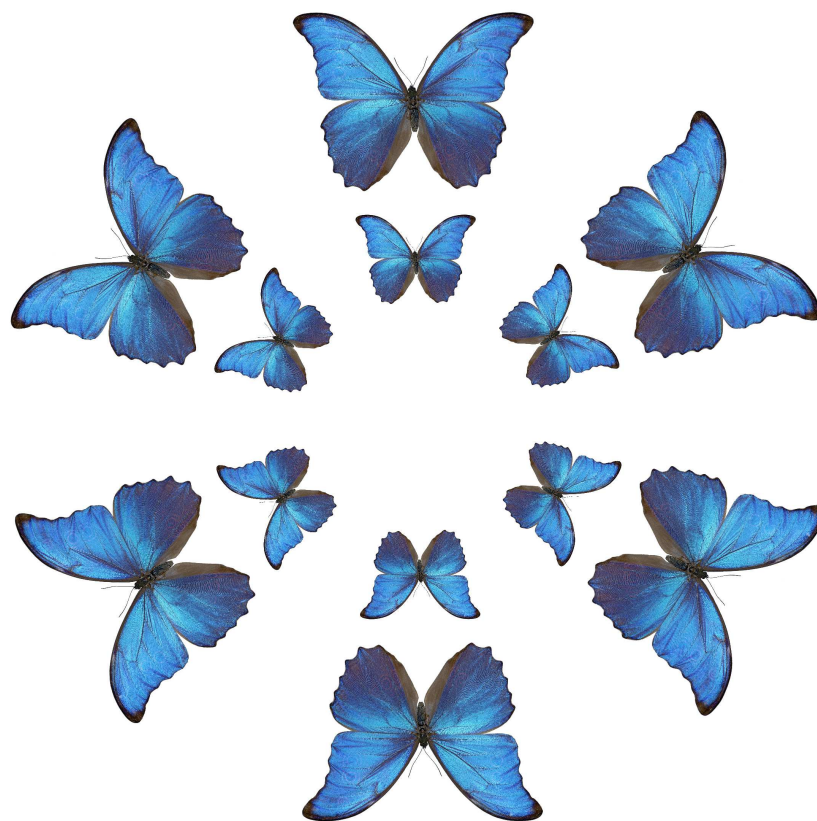
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CONTENTS

1. Introduction

Geometry is part of the everyday lives of children. Suppose one child wants to bounce a ball to a friend. The child has to figure out the *angle* to throw the ball at so that the friend can catch it. The weight of the ball and the force needed to throw it are related to the *surface area* of the ball (if hollow) or its *volume* (if solid). In order to connect geometry to kids' everyday lives, middle school geometry teachers should have a thorough and nuanced content knowledge.

This book is meant to be used for university courses aimed at deepening the content knowledge of (i) teachers currently teaching geometry in middle school; (ii) those teaching lower grades and working towards middle grades math accreditation; and (iii) teachers of special needs children. The text includes workshop problems at the end of each chapter which are meant to be worked on collaboratively. It also contains various problems, some of which are labelled as activities meaning that they are especially amenable to in-class discussion.

Each chapter ends with a section containing problems from various state level assessments. Several of these problems are included as examples of the type of problems that have been used in past assessments on that chapter's topics, and could be used by the teacher as practice or assessment problems for their students. Other problems are included to highlight the fact that sometimes assessments contain problematic (vague) language or misuse terminology. Lastly, some problems are followed by sample student work, which serves to familiarize the teacher-learners with common student reasoning and/or errors; the sample work can also be used in the middle school classroom with the purpose of having the students critique the presented work. The optional topics (refraction, ellipses, etc.) are indicated with asterisks in the section headers. The last chapter contains an instructor's guide to some of the activities, since the motivation for and subtleties in the activities may not be clear at first reading.

The book is designed to be compatible with the Common Core State Standards in Mathematics and related standards. Originally we planned to discuss how each problem related to particular standard in the Common Core; but this turned out to be problematic because of continual changes in the Common Core and the fact that many states have chosen to adopt closely related, but not identical, standards.

2. Distances, lines, and measurement

Imagine that a child rolls a ball to another child who is close by. How can this situation be represented geometrically? The locations of the two children and the ball might be represented as *points*, while the path of the ball would be represented as a *line segment* (if the ball is stopped) or a *ray* (if the second child fails to stop the ball and it keeps going.)

The concepts of points, lines, and so on are idealized versions of what happens in real life, where the positions of the children and ball are much more complicated than just three points, the actual path is curved, and the ball does eventually stop so the path is not a ray.

The objectives of this chapter are: (i) to introduce the concepts of points, lines, rays, line segments and their properties; (ii) to learn what it takes to make a precise definition; (iii) to introduce the concept of congruence at an informal level; (iv) to learn to reason using distances; and (v) to discuss issues involved in measurements, including rules of measurements, choice of units, and working correctly with units.

2.1. **Lines, rays, segments, and distances.**

Children should learn to use clear definitions in discussion with others and in their own reasoning. In geometry it is not possible to be *completely* precise at a level suitable for children, and we will simply try to be as precise as possible. We start by giving descriptions of the most fundamental terms in geometry. They are shown in Figure 1.
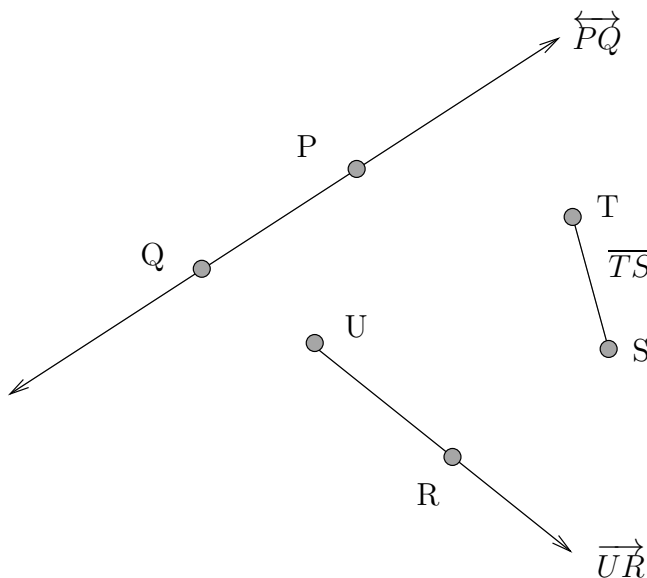

Figure 1. A line, line segment, and ray

A *point* is a location with no width, denoted P, Q, R, etc.

A *line* is a straight path extending infinitely in both directions. The line through points P and Q is denoted $\overleftrightarrow{PQ}$.

A *ray* is a portion of a line on one side of a point on the line. The ray starting at U and passing through R is denoted $\overrightarrow{UR}$.

A *line segment* is a portion of a line between two points on the line. The segment between T and S is denoted $\overline{TS}$.

A *plane* is a flat surface extending infinitely in all directions.

Graphically we represent points with some width, since otherwise the points would be invisible. An arrowhead indicates that a path extends infinitely in that direction as in Figure 1.

Each geometric object above is an *idealization* that represents part of the reality familiar to children. A point could represent the location of the child's desk in a classroom; a line could represent a highway passing through the child's town; a line segment could represent the path of a ball rolled between two children and a ray could represent the path of a kicked ball as it moves away from the kicker.

Any type of geometry requires some set of *axioms*, which are statements that are taken to be true and on which the rest of the theory is built. The descriptions of a line as a straight path, point as a location and so on are not really mathematical definitions but rather informal descriptions that are part of the set of axioms for the kind of geometry considered here. We will not emphasize the axiomatic point of view.

Distances between points

Distances depend on the choice of a *unit segment* whose length we all agree on. For example, in the United States citizens use Imperial units such as *one inch* or *one foot* but also metric units such as *one centimeter* or *one meter*. If a ruler is exactly one foot long, then it represents a *unit segment* of one foot. The *length* of a line segment is the number of unit segments (including parts of unit segments) needed to cover it. For example, the segment below is covered by exactly 8 full units and 2 half units in length, and so has total length 9 units.

FIGURE 2. Dividing a line segment into unit intervals and parts of intervals

The *distance* between two points P, Q is the length of the segment between them, denoted PQ. For example, in Figure 3, the distance between P and Q is approximately 8.9 cm, not 12 cm. This is the distance a person would travel in the *best possible situation* that there is a road, path, etc. from the starting point straight to the ending point.
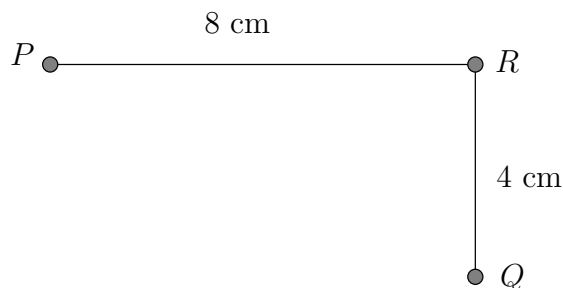
FIGURE 3. The distance between P and Q is the length of a segment that is not shown

Measuring distances

Since points cannot be drawn with zero width, each point is represented as a dot with some small width. The distance between the points is represented by the distance between the *centers* of the dots.

To measure the distance between points in a plane, place a ruler so that the centers of the dots representing the points lie on the edge. Place zero on the ruler at the center of the dot representing the first point, and read off the distance to the center of the dot representing the second point.

Activities involving measuring will often have children using manipulatives. Some of the most accessible manipulatives for children are parts of their own bodies. For example, children might measure the length of one of their fingers to get an idea of length and different units. Children might be asked to guess the length of their finger first, and then measure to assess the accuracy of their guess. Some of the issues involved in such an activity include where the finger starts and how to align the ruler with the object being measured.[1]

FIGURE 4. Measuring the length of a finger

Common classroom objects such as pencils, books, or sides of the desk can also be used for activities involving guessing then measuring. Other useful manipulatives in the geometry class to which we will be referring throughout the book are Magna-Tiles, Zome tools, and Pattern blocks. These are best suited for work with

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[1]Photo credit I. Radu.

two-dimensional and three-dimensional objects, but one could certainly explore the length of the various Zome sticks, or of the edges of a Magna-Tile piece or of a pattern block.

2.2. **Congruence and equality of segments.**

Often in sports games there are two *goals* at the end of a *field*. In order for the game to be fair the goals should be the same *shape* and *size*. Two objects that are the same shape and size are called *congruent*. However, this informal definition can be very misleading since shape and size are somewhat vague terms. For example, do a short and long rectangle have the same shape because they are both rectangles? A better definition that avoids these problems is the following.

Two shapes are congruent if one can be changed into the other by a combination of slides, flips, and turns.

So two line segments are congruent if and only if one segment can be moved without stretching to exactly match the other. Types of motions will be discussed in more detail in Chapter 7.

We write $\overline{PQ} \cong \overline{RS}$ if the segments $\overline{PQ}$, $\overline{RS}$ are congruent. Often congruent segments are indicated in a figure by giving each the same number of hash marks. For example, in Figure 5, segments $\overline{TU}$ and $\overline{TS}$ are congruent, but $\overline{TU}$ and $\overline{US}$ are not.
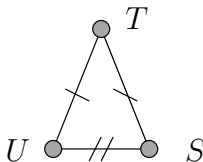

FIGURE 5. Congruent and non-congruent segments

Congruence of segments via lengths

Calling the sentence above the *definition* of congruence means that the sentence above explains the meaning of the word *congruent*. Statements that are true in mathematics based on already known or agreed upon statements are called *theorems*. Here is an example of a theorem about congruence of line segments:

Line segments are congruent if and only if they have the same length.

The following is a viable argument for why the theorem is true. A hands-on version of it is likely to be accessible to middle school students.

We justify the two directions of the theorem separately. First, if two segments are congruent, then one could be moved to match the other, so their corresponding endpoints would also match. Therefore, the two segments have to be of the same length. Conversely, given two line segments with the same length, we can slide the segments so that they have a common endpoint. Then we can rotate one of the segments so the two line segments point in the same direction. Since the two segments have the same length, the second endpoints of the two segments now match. Therefore, the two segments are congruent. See Figure 6 for a pictorial representation of this argument in which the segments are represented by pens.

Figure 6. Demonstration of congruence

The property of being congruent if and only if the lengths are the same does not hold for non-straight paths. By a *path* we mean the set of positions traced out by a moving point. The *length* of a path is the number of units of length (including parts of units) needed to cover it. Two paths may have the same length without being congruent. Figure 7 shows two paths with the same length that are not congruent.
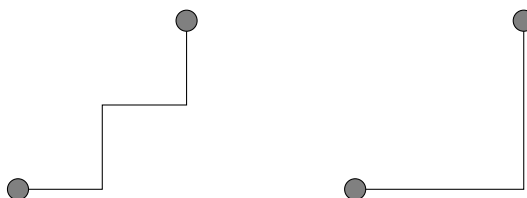

Figure 7. Non-congruent paths with the same length

Congruence versus equality

Two lines are *equal* if they are actually the same line. For example, if P and Q are points, then the line $\overleftrightarrow{PQ}$ is *exactly the same* as (that is, geometrically coincides with) the line $\overleftrightarrow{QP}$, and so we can write $\overleftrightarrow{PQ} = \overleftrightarrow{QP}$.

Similarly, rays and line segments are equal if they geometrically coincide. Imagine that one child rolls a ball to another in approximately a straight line segment, and the other child rolls it back. The path of the ball is approximately a line segment; let's call this segment $\overline{PQ}$ for the first roll and $\overline{QP}$ for the return roll. As line segments, $\overline{PQ}$ and $\overline{QP}$ are *equal*. The rays $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are not equal since they are infinite in different directions. See Figure 8 where $\overrightarrow{PQ} = \overrightarrow{PR}$.

$$\overrightarrow{PQ} = \overrightarrow{PR}$$
$$\overleftrightarrow{PQ} = \overleftrightarrow{PR} = \overleftrightarrow{QR}$$
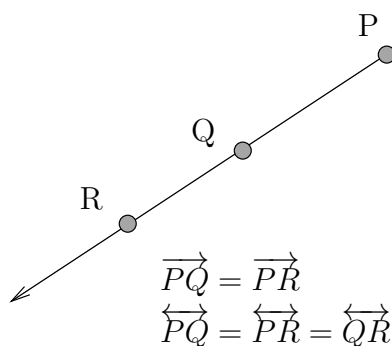
FIGURE 8. Equal rays and lines

Objects can be *congruent* without being equal. Sometimes, when someone talks informally about line segments being "the same", the person may mean that the two segments are congruent. If so, one uses the symbol $\cong$. However, there are some situations where one wants to use strict equality of geometric objects to mean "geometrically coincides with".

Sometimes it is taught that congruence should be used for geometric objects, and equality for their measurements. It might be helpful to tell students that this is usually the case in most, but not all, geometric arguments.

2.3. **Tools for drawing and measuring.**

Children should be familiar with tools in geometry appropriate for their grade or course and to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use and their limitations. They should also be able to use technological tools such as geometry software packages, if age-appropriate, to explore and deepen their understanding of concepts.

Tools for drawing lines, rays and line segments include straightedges and rulers as well as computer packages such as the open-source package GeoGebra. A *straight-edge* is a ruler without distances marked.

If you are viewing this document on a computer, tablet, or phone with internet access, clicking on the following link will open the software package *GeoGebra*.

<p style="text-align:center;color:#8b1a1a">GeoGebra</p>

On the "GeoGebra apps" page, if you click on "Geometry", you will see several tools at the top of the screen.

> The *arrow tool* allows objects to be selected and moved.
>
> The *point tool* allows points to be drawn. GeoGebra automatically labels them A, B, C , and so on.
>
> The *line tool* allows lines to be drawn. Follow the instructions to first choose one point on the line, then another.

Each tool button in the toolbar has a small arrow that allows that tool to be changed to another. For example, clicking on the little arrow in the line tool allows you to change it to a line segment tool, either using two points to specify a line segment or specifying an endpoint and a length.

Other tools allow the creation of polygons, circles, angles, reflections, rotations, translations, and so on. We will discuss these notions in later chapters.

Problem 2.3.1. Compare and contrast the designated shapes as indicated on the following page. For each pair of shapes, find one similarity and one difference; measure the shapes with a ruler or tape measure as necessary. Try to use the vocabulary and notation introduced above, e.g. "in the sixth shape the two line segments $\overline{AB}$ and $\overline{BC}$ have endpoint B in common ..... ".
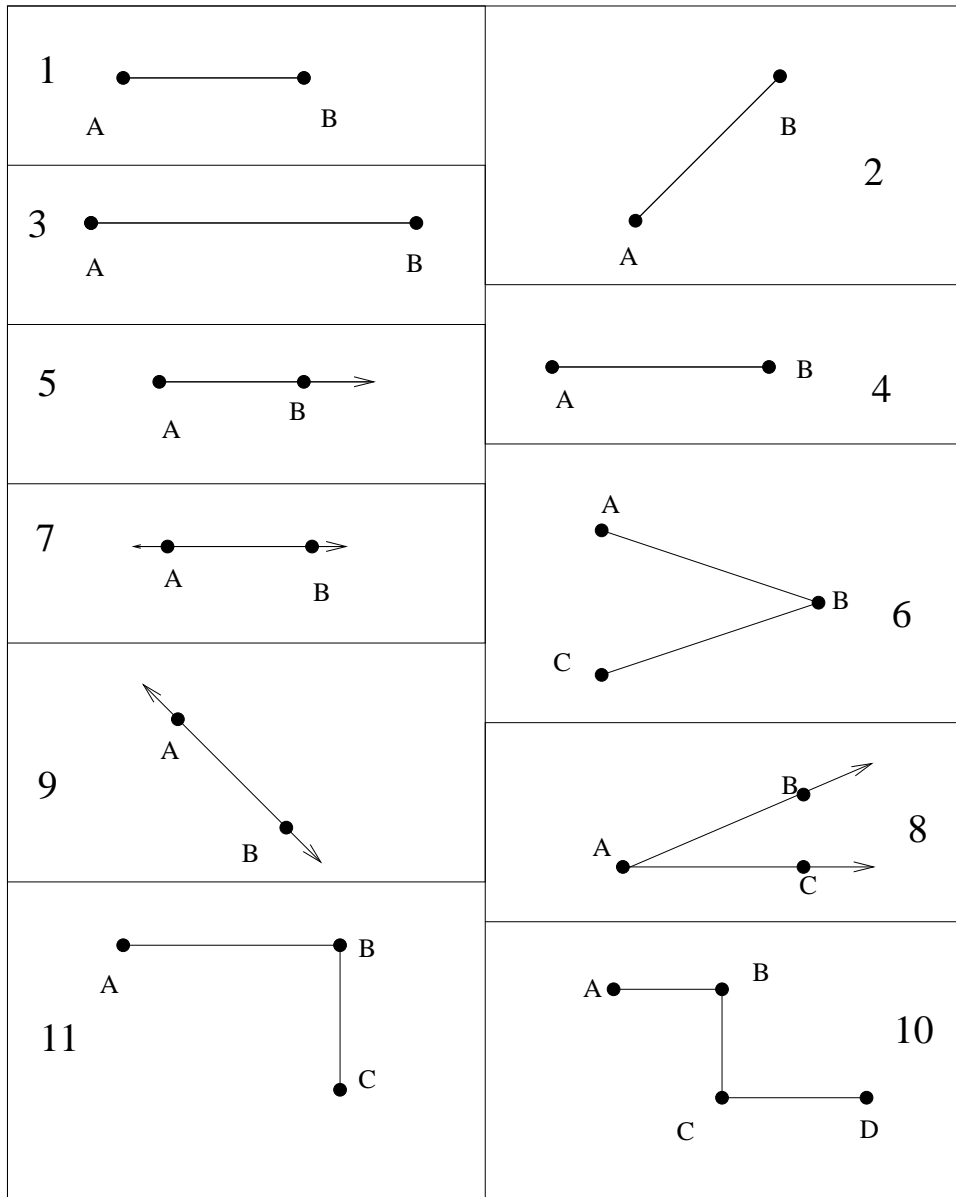
Shape	Shape	Ways shapes are alike	Ways shapes are not alike
Shape 1	Shape 3		
Shape 9	Shape 11		
Shape 2	Shape 4		
Shape 6	Shape 8		
Shape 8	Shape 10		

Working correctly with units

Children should learn to be careful about specifying units of measure in geometry, and to work with units correctly throughout a problem. When adding lengths, the unit should appear on each side of the equals sign, as shown below:

$$2 \text{ in} + 3 \text{ in} + 2 \text{ in} + 3 \text{ in} = (2 + 3 + 2 + 3) \text{ in} = 10 \text{ in}$$

or, in more detail,

$$
\begin{aligned}
2 \text{ in} + 3 \text{ in} + 2 \text{ in} + 3 \text{ in} &= 2 \times 1 \text{ in} + 3 \times 1 \text{ in} + 2 \times 1 \text{ in} + 3 \times 1 \text{ in} \\
&= (2 + 3 + 2 + 3) \times 1 \text{ in} \\
&= 10 \times 1 \text{ in} \\
&= 10 \text{ in.}
\end{aligned}
$$

Many people will omit the units from intermediate steps as a way of saving time; please be aware that this is a "shortcut", that is, not really correct but may be acceptable in some circumstances.

When converting lengths from one unit to the other, units should be used correctly, as in:

$$24 \text{ in} = 24 \text{ in} \frac{1 \text{ft}}{12 \text{ in}} = \frac{24}{12} \text{ft} = 2 \text{ft}$$

or

$$24 \text{ in} = 2(12 \text{ in}) = 2 \text{ft.}$$

This seems to require unnecessary effort now, but it will help later with more complicated volume and area problems, as well as more complicated conversions. Even if a student or teacher chooses to drop units from part of the computation, the person should realize that he or she is taking a shortcut (i.e., writing it wrongly.)

2.4. **Arrangements of lines and points.**

The paths of two children on a playground that meet are said to *intersect*. If the paths intersect, then the children are in danger of colliding. They collide if they meet the *intersection point* at the same time. On the other hand, if they are running side by side then they are running in *parallel* and will not intersect unless they change direction. Children who are lined up in a hallway are *colinear*: they all are in the same line. In what follows we define these terms more precisely and show examples of each in Figure 9.

Intersecting lines   Perpendicular lines   Parallel lines

A bisected line segment   Colinear points   Parallel line segments
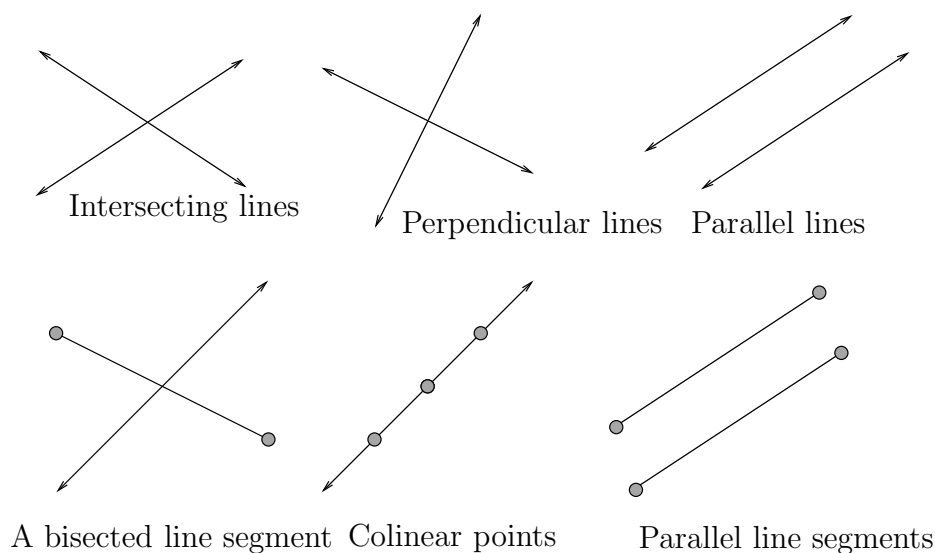
FIGURE 9. Configurations of lines and segments

Two lines *intersect* if they meet, that is, have points in common.

Two lines that do not intersect and lie in the same plane are *parallel*.

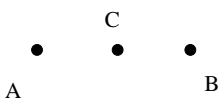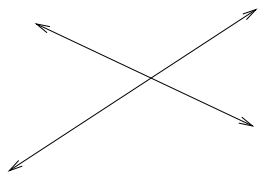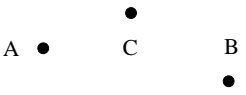Two lines that meet at a right angle are *perpendicular*.
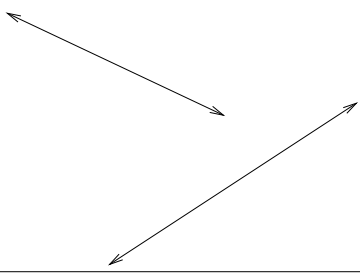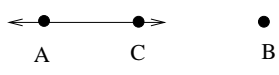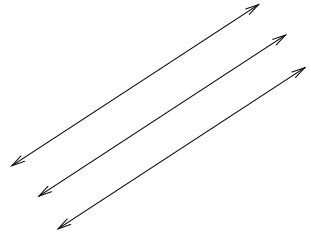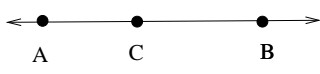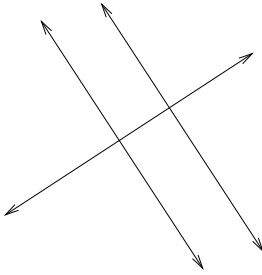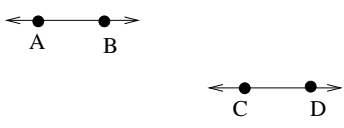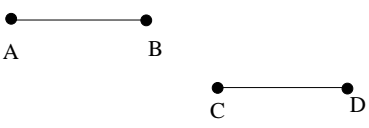
Two line segments are parallel if they are part of parallel lines.

Points are *colinear* if they lie on the same line.

A line segment is *bisected* by an intersecting line if it is cut into two congruent pieces, that is, two pieces of equal length.

For example, two points are always colinear, because there is a line passing through them. Note that the line on which they lie does not need to be drawn for the points to be colinear. Three points may or may not be colinear. If they are not colinear, then they are the vertices of a triangle. (Triangles are presented in a later chapter.)

Problem 2.4.1. Compare and contrast the designated shapes; add points with labels as necessary. For each pair of shapes, find one similarity and one difference.

Shape	Shape	Ways shapes are alike	Ways shapes are not alike
Shape 1	Shape 3		
Shape 5	Shape 7		
Shape 7	Shape 9		
Shape 9	Shape 10		
Shape 2	Shape 4		
Shape 6	Shape 8		

Problem 2.4.2. Consider the drawing with points P, Q, R, S, T, U marked.

T
•

•   •   •
P   Q   R

                •   •
                U   S
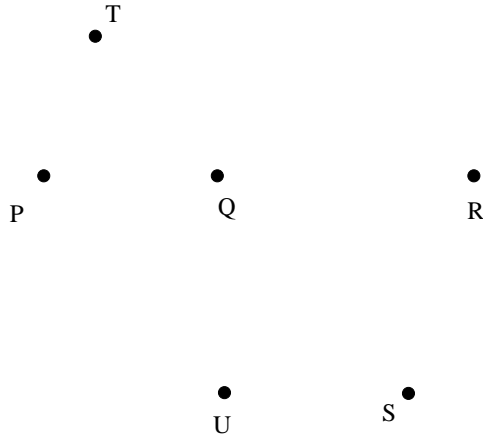
(1) Draw $\overline{PS}$, $\overrightarrow{QS}$, and finally $\overleftrightarrow{RS}$.

(2) Describe (using standard notation) a pair of lines that are perpendicular, so that each line passes through two of the given points. Explain how you know the lines are perpendicular.

(3) Describe (using standard notation) a pair of lines that are parallel, each passing through at least two of the given points. Explain how you know the lines are parallel.

(4) Measure the distances PQ, QR, PR. Add your measurements to find $PQ + QR, QR+PR, PQ+PR$. (Do not adjust so that you get the "right" answer.)

PQ	QR	PR	PQ + QR	QR + PR	PQ + PR

(5) Measure the distances PQ, QS, PS. Add your measurements to find $PQ + QS, QS+PS, PQ+PS$. Explain how these are related to PQ, QS and PS.

PQ	QS	PS	PQ + QS	QS + PS	PQ + PS

2.5. **Reasoning using distances.**

Children should be able to reason abstractly and quantitatively about distances, and relate the language of geometry to situations in their own life. For example, the shortest distance between any two children (in plane geometry) is the length of the line segment between them, thinking of the children as points.

Here is an example of reasoning using distances. Suppose that three children, Pedro, Quinlan, and Rachel, live in the same city or town. If Pedro goes from his house to Rachel's apartment, passing by Quinlan's house, the distance traveled is at least as much as if he went straight to Rachel's in a straight line.
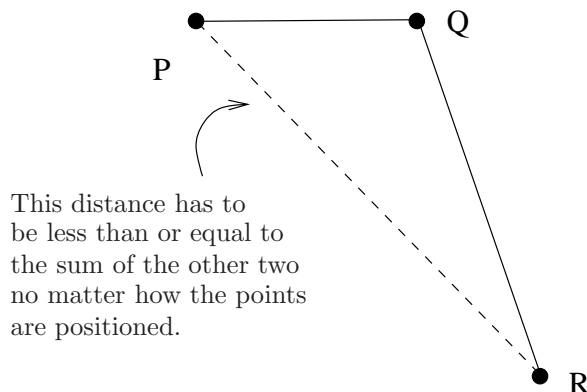

This distance has to be less than or equal to the sum of the other two no matter how the points are positioned.

FIGURE 10. The triangle inequalities

More generally, for any three points P, Q, R, the distance between any two points is less than or equal to the sum of the other two distances. We use the symbol $\leq$ to denote *less than or equal to* or *is at most*, and the symbol $<$ to denote *less than*. The first symbol is somewhat confusingly called an *inequality* and the second a *strict inequality*. So the distances between points P, Q, R satisfy the inequalities

$$PR \leq PQ + QR, \quad PQ \leq PR + RQ, \quad RQ \leq PQ + PR.$$

The points P, Q, R are not colinear if and only if the inequalities above are *strict*, that is,

$$PR < PQ + QR, \quad PQ < PR + RQ, \quad RQ < PQ + PR.$$

These inequalities are known as the *triangle inequalities*.

Because a line segment is always the shortest path between two points, the distances between the points can be used to determine whether points are colinear. For example, if PQ and QR are each equal to 1 centimeter and PR is equal to 2 centimeters, then Q is on the line segment between P and R. The path from P to Q to R has the same length as the straight path from P to R, and so both paths must be the same.

Children should be able to make conjectures in geometry and use logic to explore the truth of their conjectures. Typically conjectures in geometry are made after drawing pictures of various possibilities. Here is an example of a problem for which drawing various possibilities can help with answering the problem.

Problem 2.5.1. Pedro, Quinlan, and Rachel are friends. If the distance from Pedro to Quinlan's house is 200 feet, and the distance from Quinlan's to Rachel's apartment is 300 feet, what are all the possibilities for the distance from Pedro's house to Rachel's apartment?

(1) Draw several possibilities for the locations of Pedro, Quinlan, and Rachel's places.
(2) Use your pictures to make a conjecture for an answer to the question above.
(3) Justify your answer using the triangle inequalities.

Here is a possible answer to part (3). Using the inequality $PR \leq PQ + QR$ gives

$$PR \leq 200 \text{ feet} + 300 \text{ feet} = 500 \text{ feet}.$$

So the distance from Pedro's to Rachel's is at most 500 feet.

The distance from Pedro's to Rachel's is at least 100 feet, since

$$
\begin{aligned}
QR \leq PQ + PR \quad &\text{so} \quad 300 \text{ feet} \leq 200 \text{ feet} + PR \\
&\text{so} \quad 300 \text{ feet} - 200 \text{ feet} \leq PR \\
&\text{so} \quad (300 - 200) \text{ feet} \leq PR \\
&\text{so} \quad 100 \text{ feet} \leq PR.
\end{aligned}
$$

Problem 2.5.2.

(1) Using string and a yardstick measure
  (a) the length of your arms;
  (b) the length of your legs; and
  (c) the distance from your heels to your shoulders.
(2) About what fraction of the length of your leg is the length of your arm?
(3) Now measure the height of the highest place on the wall you can reach. How does it relate to the prior measurements?
(4) Consider the height of the lowest place on the wall you can reach with your hands without bending at your waist and without bending your knees. Without measuring, explain how it relates to the prior measurements?
(5) Now measure the height of the lowest place you can reach without bending at your waist and without bending your knees. Does this confirm your expectation from (4)?

Here are some more problems that encourage making viable arguments.

Problem 2.5.3. True or false? In each case, give a viable argument for your answer.

(1) If two line segments do not intersect, then they are parallel.
(2) If two lines do not intersect, then they are parallel.
(3) If points A, B, C are such that $AB + BC$ is strictly bigger than AC, then A, B, C are not colinear.
(4) If John is 5 feet from Martha and Martha is 6 feet from Sam, then Sam is at least 1 foot away from John.

2.6. **Units and conversion.**

Children might or might not be familiar with distances given in *feet* or *yards*. For example, their heights are usually measured in feet, and some schools ask them to run a *hundred yard dash* or a *hundred meter dash*. They are also familiar with measurements of time in different units, such as *months* or *years*.

Changing from one unit to another is called *conversion*. For example, for distance we have

$$1 \text{ yard} = 3 \text{ feet}$$

while for time we have

$$1 \text{ year} = 12 \text{ months}.$$

Units can be multiplied and divided just like numbers. So for example, dividing 1 yard $=$ 3 feet by 1 yard gives

$$\frac{1 \text{ yard}}{1 \text{ yard}} = \frac{3 \text{ feet}}{1 \text{ yard}}.$$

Simplifying we get

$$1 = 3 \text{ ft}/ \text{ yard}.$$

The quantity 3 ft/ yard is called a *conversion factor*. Because the conversion factor is equal to one and multiplying by one does not change a quantity, one can multiply by the conversion factor to convert from yards to feet. For example, if a child runs a one-hundred yard dash, then in feet the child has run

$$
\begin{aligned}
100 \text{ yards} \ &= \ (100 \ \times \ 1 \text{ yard}) \ \times \ \frac{3 \text{ ft}}{\text{yard}} \\
&= \ \frac{100 \ \times \ 3 \ \text{ft} \times \ 1 \text{ yard}}{1 \text{ yard}} \\
&= \ (100 \ \times \ 3) \text{ ft} \times \frac{1 \text{ yard}}{1 \text{ yard}} \\
&= \ 300 \ \text{ft}.
\end{aligned}
$$

The multiplication by the conversion factor is justified because the conversion factor is equal to one. The conversion factor 3 ft/yard is an *exact* conversion factor. Other factors are *approximate*, for example, 3.3ft/meter is an approximate conversion factor which means that

$$1 \approx 3.3\text{ft}/\text{m}.$$

So

$$
\begin{aligned}
100 \text{ m} \ &\approx \ 100 \text{ m} \ \times \ 3.3 \ \text{ft}/ \text{ m} \\
&\approx \ (100 \ \times \ 3.3) \text{ ft} \times \ \text{m}/ \text{ m} \\
&\approx \ 330 \ \text{ft}.
\end{aligned}
$$

Note that we used the squiggly $\approx$ instead of the straight $=$ throughout the computation, because the final answer 100 m $\approx$ 330 ft is only an approximate equality. However, it would also be correct to write an equals sign on the second and third lines above, as long as one remembers that the result of the computation is still an approximation of the original quantity.

Here is a sample conversion of a time measurement: A child who is 8 years old has age given in months by

$$8 \text{ years} = 8 \text{ years} \times \frac{12 \text{ months}}{1 \text{ year}} = 96 \text{ months}.$$

Here are some commonly used conversion factors. We use the *approximately equal symbol* $\approx$ if two quantities are only approximately equal, for example

$$1 \text{ mile} \approx 1.6 \text{ kilometers}.$$

We use the equals sign $=$ if two quantities are *exactly equal*, for example

$$1 \text{ yard} = 3 \text{ feet}.$$

It is also correct to write

$$1 \text{ mile} = 1.6 \ldots \text{ kilometers} .$$

Here are some common conversion factors and symbols for units:

1 inch = 1"	=	2.54 centimeters
1 foot = 1'	$\approx$	0.305 meters
1 yard	$\approx$	0.914 meters
1 mile	$\approx$	1.609 kilometers
1 nautical mile	$\approx$	1.852 kilometers
1 centimeter	$\approx$	0.39 inches
1 meter	$\approx$	3.28 feet
1 kilometer	$\approx$	0.62 miles
1 kilogram	$\approx$	2.2 pounds
1 pound	=	16 ounces

Two methods for unit conversion are *substitution* and *ratios*. In substitution, one unit is substituted for the corresponding amount of another unit. In the ratio method, the ratio of units equal to one is multiplied and then the units in numerator and denominator are cancelled. Both methods are explained in more detail in the following example.

Here is a sample problem: *Convert 10 miles to kilometers.* A *teacher's solution* using ratios is given below:

$$\begin{aligned} 10\text{mi} \quad &= \quad 10 \times 1\text{mi} \\ &\approx \quad 10 \times 1\text{mi}\frac{1.6\text{km}}{1\text{mi}} \\ &\approx \quad (10 \times 1.6)\text{km} \\ &\approx \quad 16\text{km}. \end{aligned}$$

To go from the first line to the second line, we used that the ratio $\frac{1.6\text{km}}{1\text{mi}}$ is approximately one, and multiplying by 1 does not change a quantity. The quantities 1mi from the numerator and denominator were then cancelled, and 10 was multiplied by 1.6 to obtain a final answer.

By a *teacher's solution* we mean a solution that is presented in a way so that all of the audience can understand, using correct notation.

Using substitution, an answer is:

$$10\text{mi} = 10(1\text{mi}) \approx 10(1.6 \text{ km}) \approx 16 \text{ km}.$$

A commonly-used variation on the ratio method is the method of *proportions* in which one solves for the unknown quantity:

$$\frac{1 \text{ mi}}{1.6 \text{ km}} \approx \frac{10 \text{ mi}}{X \text{ km}}$$

so

$$X\text{km} \approx (10 \text{ mi})\frac{1.6 \text{ km}}{1 \text{ mi}} \approx 16\text{km}.$$

There are many variations on this method. The proportion should be formulated so that on each side of the equation, the numerator and denominator of the fraction are equivalent expressions for the same quantity. Additionally, the units in the proportion should be listed in such a way that when solving, the "old" units are canceled, as shown in the example above. Given these subtleties involved in formulating the correct proportion for the desired conversion, students often make mistakes using this method, and for this reason we do not recommend it.

Here is a sample problem on conversion using ratios: Convert 100kmh to mph. An answer using ratios is given below:

$$100\text{kmh} = \frac{100\text{km}}{\text{hr}} \approx \frac{100\text{km}}{\text{hr}} \left(\frac{.62\text{mi}}{\text{km}}\right) \approx \frac{62\text{mi}}{\text{hr}} \approx 62\text{mph}.$$

Converting temperatures

Most children get their temperature taken when they get sick. In the United States the temperature is usually taken in *degrees Fahrenheit*, but international thermometers usually use *degrees Celsius*. In fact, there are three widespread systems for measuring temperatures: Fahrenheit, Celsius, and Kelvin. We discuss only converting between the first two since Kelvin is used mostly for scientific purposes.

Celsius is designed so that $0°C$ is the freezing point of water and $100°C$ is the boiling point.

Fahrenheit is designed so that $32°F$ is the freezing point of water and $212°F$ is the boiling point.

Originally, Fahrenheit chose $96°F$ to be the human temperature. He measured under the arm, so there was a slight difference between that and today's measurement of $98.6°F$. He chose the numbers so that $64°F$ degrees separate human temperature (approximately) and the freezing point of water.

The conversion from Celsius to Fahrenheit is a *shifted conversion*: temperature in degrees F is
$$T_F = 32°F + (9°F/5°C)T_C.$$

Problem 2.6.1.   (1) Convert $5°C$ to Fahrenheit.
  (2) Convert $33°F$ to Celsius.
  (3) Suppose the temperature goes up by one degree Celsius. How much is the corresponding rise in Fahrenheit?

2.7. **Workshops.**

Workshop 2.7.1.

(1) Using either ruler/GeoGebra/Magna-Tiles/Zome tools draw or make a model or floor plan of your house or apartment or classroom. (The plan should not be too complicated, that is, it should not take long to construct or draw. On the other hand, it should not just be a rectangle. Try to include, for example, the location of a window, door, or desk. If GeoGebra is used, please include a print-out of the GeoGebra diagram.)

(2) Label the corners and line segments of your model or drawing using standard notation as above.

(3) Measure the length of two line segments in your plan, in (i) inches and (ii) centimeters.

Segment	Length in Inches	Length in Centimeters

(4) Convert the measurements in inches above to centimeters, using the rule 1 in $\approx$ 2.54 cm. Example: 5in $\approx$ 5in(2.54cm/in) $\approx$ 12.7cm. Make sure to give a *teacher's solution*, that is, a solution that communicates your work clearly to all members of the audience.

(5) Does your conversion match your actual measurement? Why or why not?

(6) What possible errors might beginning learners make using a ruler?

Workshop 2.7.2.

(1) Can you find points A, B, C so that the distances between them are
  (a) $AB = 2$ cm, $BC = 3$ cm, $AC = 4$ cm?
  (b) $AB = 1$ cm, $BC = 4$ cm, $AC = 7$ cm?
  (c) $AB = 4$ cm, $BC = 1$ cm, $AC = 9$ cm?
  (d) $AB = 8$ cm, $BC = 1$ cm, $AC = 9$ cm?
  If possible, draw the triples of points so that the distances are those given above. If not possible, explain why not.
(2) Suppose that four points have the property that three pairs have distance 1 in, two pairs have distance 2 in, and one pair has distance 3 in. (That is, of the distances AB, AC, AD, BC, BD, CD, three are 1 in, two are 2 in, and one is 3 in.) Draw a possible picture of the four points.
(3) Is there more than one possible configuration of 4 points meeting the conditions in (2)? Explain your answer.

Workshop 2.7.3.

Do the following conversions using at least two different methods (e.g. ratios, substitutions). In each case, explain your method in one or two sentences. Make sure to give a *teacher's solution*, that is, a solution which communicates your work clearly to all members of the audience.

(1) 10ft to inches.
(2) 10.2ft to inches.
(3) 40 inches to feet.
(4) 1000ft to meters.
(5) 65mph to km/hr. Note that 1 mph = 1 mi/hr.
(6) 10lbs to kg.
(7) challenge: $11.999\ldots$ inches to feet.

Workshop 2.7.4.

(1) The average body temperature is 98.6 degrees Fahrenheit. Convert this to degrees Celsius.
(2) During the last minute, the temperature change has been approximately $0°C$. Find the corresponding temperature change in degrees Fahrenheit. (Hint: a change of $0°C$ means there has been *no change.*)
(3) The global average temperature increase in the period 1900-2000 was approximately $0.7°C$ Find the temperature rise for 1900-2000 in degrees Fahrenheit.
(4) Can you give a formula that allows you to calculate a change in temperature in Fahrenheit, based on a change in temperature in Celsius?
(5) Is there a temperature for which the number of degrees F equals the number of degrees C?

2.8. Assessments and sample student work.

Here is a typical assessment problem on lengths and converting units of length.

Problem 2.8.1. (Adapted from [NJ]) Kathryn is using a piece of cloth that is 23 yards in length for two projects. She cuts off a piece from the cloth that is 2 yards in length to keep for her second project.

(1) What is the length, in feet, of the piece of cloth Kathryn is using for her first project? Show your work or explain your answer.
(2) Kathryn's second project uses at least 1/2 and no more than 3/4 of the piece of cloth that she cut from the original piece. What is a possible length, in inches, for the cloth used in the second project? Show your work or explain your answer.

Many student mistakes on the problem above are caused by misreading. Others are caused by conversion errors, for example, 21 yards divided by 3 gives 7 feet. The conversion errors can be avoided by working correctly with units, and having the students cancel units in the numerator and denominator. Two sample responses are shown below. Note that even responses marked with full 3/3 scores had mistakes, e.g. are not using units correctly.

3. Angles

A child who bounces a ball to a friend knows that the direction of the ball will change when it bounces. In order to send the ball to the friend the child has to figure out what *angle* to throw it at. In everyday language an *angle* can mean an amount of turning, the difference between two directions, or a shape formed where lines, line segments, or rays meet. The Common Core State Standards define an angle as a *shape* formed by rays with a common endpoint. Angles are usually measured in degrees (or in later mathematics, radians), using a *protractor*. Measuring angles and drawing angles of a given measure can be used to predict the paths of reflecting objects such as light rays bouncing off mirrors or objects bouncing off walls.

The objectives of this chapter are: (i) to appreciate different aspects of the concept of an angle; (ii) to discuss why rays instead of line segments are used in the formal definition of angles; (iii) to discuss how the sizes of angles are compared; (iv) to review angle measurement in degrees, discuss correct use of protractors, and briefly mention angle measurement in radians; and (v) to discuss a few basic facts involving angles and learn to reason with them.

3.1. **Angles, angle measures, and protractors.**

A child on a skateboard who does a full turn is said to do a 360 *degree rotation*. So for most children, an angle will most naturally represent an amount of turning. For example, telling a child to turn to face the other way is to tell them to do a 180 *degree rotation*.

The Common Core State Standards in geometry *do not* define an angle as an amount of turning; instead, an angle is defined to be a geometric shape:

An *angle* is a geometric shape formed by a pair of rays starting at the same point, called the *vertex* of the angle.

The common core is a bit vague about which shape is meant. We interpret the angle as the wedge shape that is "between" the two rays with a common starting point as in Figure 11. Usually it is clear what is meant by "between".
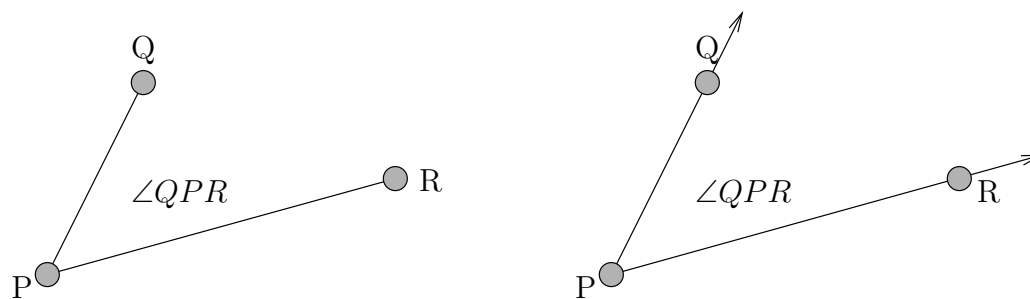
FIGURE 11. An angle

The common starting point of the two rays is called the *vertex* of the angle. The two rays starting at the vertex are called the *sides* of the angle.

Each pair of rays with the same starting point defines *two* angles, depending on whether the angle is formed by going from one ray to the other ray clockwise or counterclockwise. If $\overrightarrow{QP}, \overrightarrow{QR}$ are the rays as in Figure 12 below (that is, not going in opposite directions) then the smaller angle is denoted $\angle PQR$.
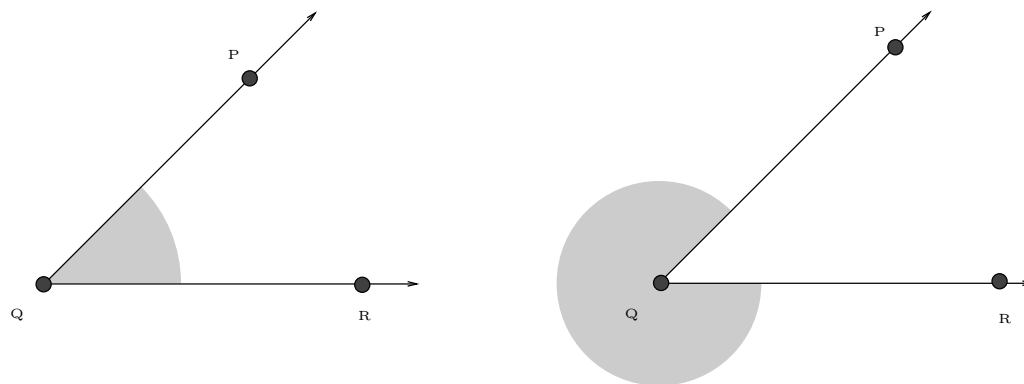

FIGURE 12. Two different angles formed by the same (up to a slide) rays

The bigger angle is the *reflex angle* defined by the rays. There is no standard notation for the reflex angle defined by two rays.

If the rays go in opposite directions, so that the two rays form a straight line, then we say that the angle is *linear*. In this case, there are still two angles formed by the rays, but neither is reflex, as shown in Figure 13.

The *measure* of an angle is the amount of turning it represents, often measured in *degrees* where 360 degrees represents a full turn. In other words, 1 degree represents 1/360 of a full turn, 90 degrees represent one-quarter of a full turn, and so on. Note that a 540 degree turn leaves a person facing the same direction as a 180 degree turn, but a 540 degree turn is not the same as a 180 degree turn for a skateboarder.

Apparently, the number 360 was chosen in Babylonian mathematics because it was believed to be approximately the number of days in one year, and so it took about 360 days for the Sun to go around the Earth in a circle.
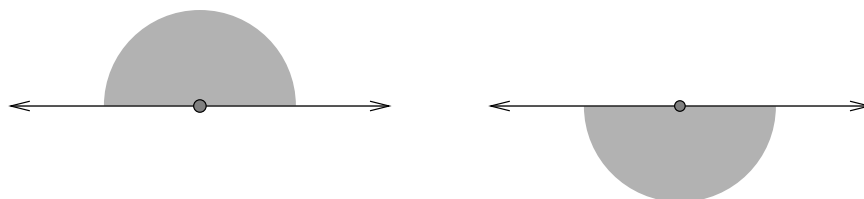

FIGURE 13. Two different linear angles

The symbol ° is used to mean degrees. The measure of an angle is denoted with a small m, that is, $m\angle PQR$ is the measure of the angle $\angle PQR$.

In more advanced courses, angles are measured in *radians*. The *radian measure* of an angle involves *arc length*, which is discussed in Chapter 6.

Two angles are *congruent* if they have equal measures, or equivalently, if one can be made to coincide with the other by sliding and turning (see Chapter 7 below for more details on motions). We write $\angle PQR \cong \angle STU$ if the angles $\angle PQR, \angle STU$ are congruent.

We can also talk about angles coinciding, which in the strict sense means that the angles are overlayed. For example, in Figure 14, the angles $\angle PQR, \angle STU$ are *congruent*, but the angles $\angle PQR, \angle VQW$ are *coincident*, that is, equal.
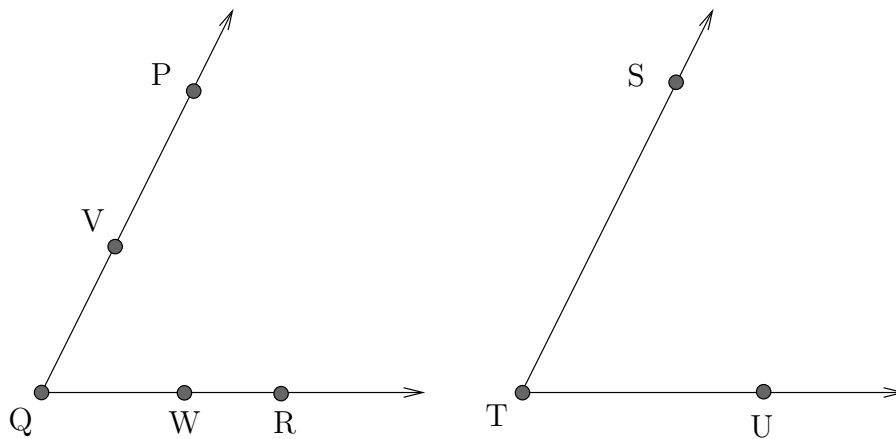

FIGURE 14. Coincident versus congruent angles

We write $\angle PQR = \angle VQW$ to mean that the angles $\angle PQR$ and $\angle VQW$ coincide.

Protractors

Angles are measured or drawn using a *protractor*, right angle, or computer packages such as GeoGebra. As shown in Figure 15, part of the protractor is usually a semicircle, with opposite points marked 0 and 180 degrees. The center of the protractor is the midpoint between the opposite points. Note that it is often *not* on the edge of the protractor. Each degree is represented by 1/360 of a circle drawn on the protractor. This means that if one protractor is larger than another, then the degree markings on the larger protractor are further apart than the degree markings on the smaller protractor.[2] Some online assessments have on-line protractor tools which are meant to simulate the use of an actual protractor.
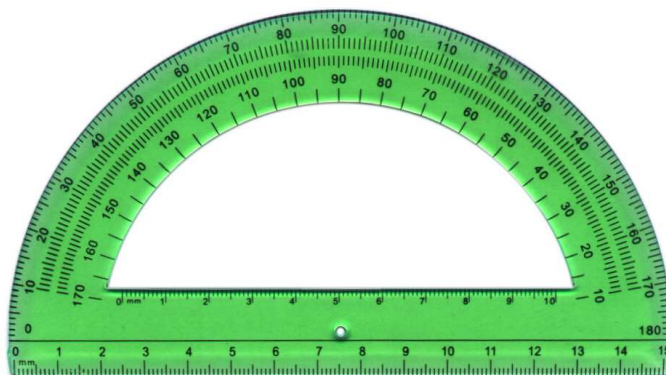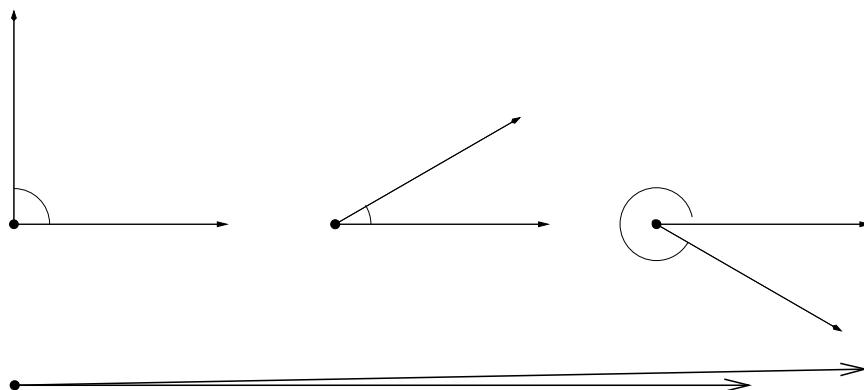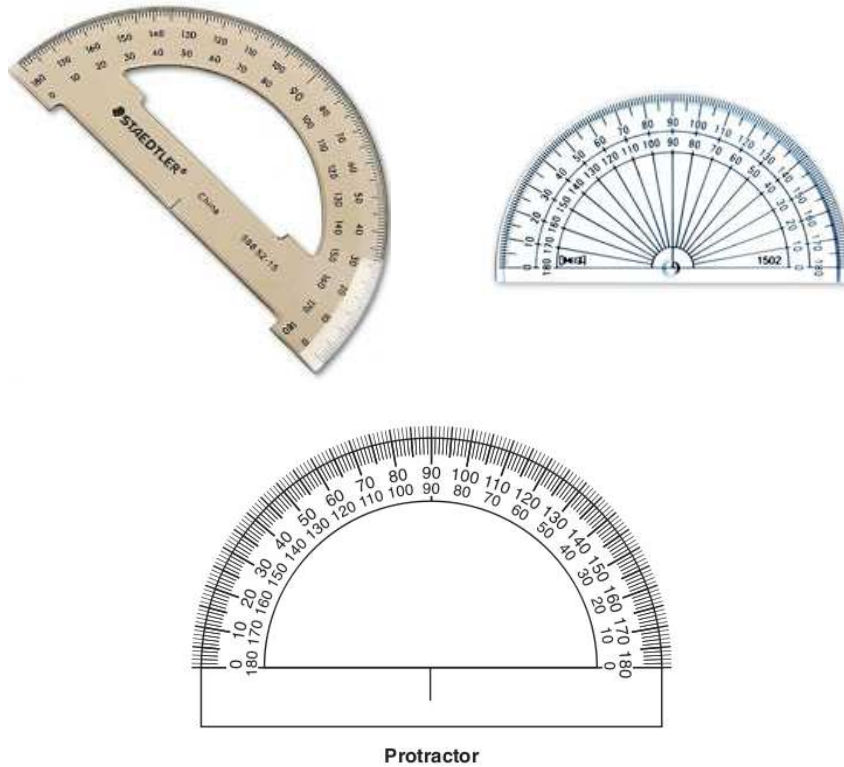

FIGURE 15. A common protractor

Problem 3.1.1. How many degrees do you think the following angles have? Make a conjecture without using a protractor.

---

[2]Photo credit Luigi Chiesa. Used under GNU Free Documentation License.

Problem 3.1.2. Mark each of the centers of the protractors shown with a star. [3]

Protractor

Problem 3.1.3. Using your prior knowledge about protractors, measure the angles in Problem 3.1.1 using a protractor.

Measuring angles with a protractor

To measure an angle that is not reflex, place the center of the protractor over the vertex of the angle, so that 0 degrees marking on the protractor matches up with the first ray. The number on the protractor that is closest to the second ray is the measure of the angle. Be careful to place the center of the protractor over the vertex of the angle. If the angle is reflex, then the measure is 360 degrees minus the measure of the non-reflex angle with the same rays.

On state assessments students have to use a protractor cut out from a piece of paper. For example, here is a New Jersey assessment protractor.[4]
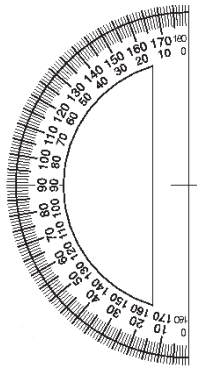

FIGURE 16. Another assessment protractor

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[4]Image from [NJ] used under Fair Use guidelines.

More vocabulary about angles

Sometimes we want to think of an angle as having a direction, that is, clockwise or counterclockwise. In this case, there is a slightly different definition of angle: A *directed angle* is an angle with a specified direction (that is, clockwise or counterclockwise). It can be specified by choosing one of the rays as the *starting ray* and the other ray as the *ending ray*. See Figure 17.
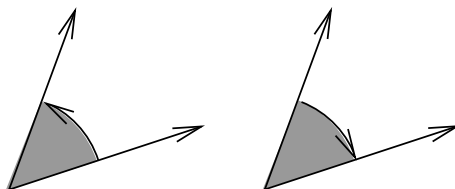
FIGURE 17. Directed angles

We also want to talk about angles formed by line segments with a common endpoint. If two line segments $\overline{PQ}, \overline{PR}$ have a common endpoint P, then the *angle formed by the segments* $\overline{PQ}, \overline{PR}$ will mean the angle $\angle QPR$, that is, formed by the rays $\overrightarrow{PQ}, \overrightarrow{PR}$. However, the definition of angles uses rays instead of segments since the definition of congruence of angles given above only works with rays.
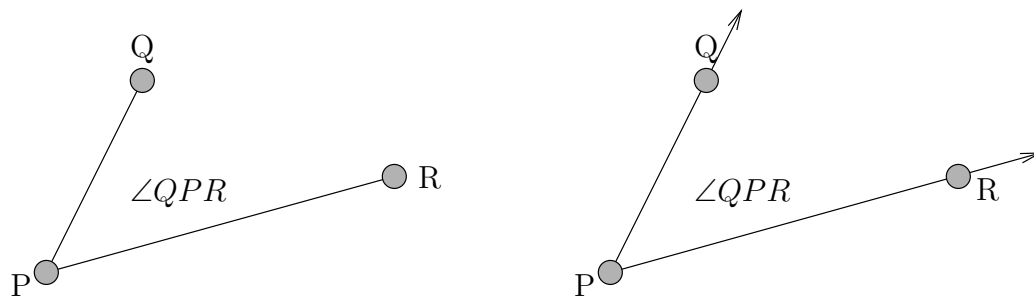
FIGURE 18. Notation for angles

Note that size of the angle is not determined by - or even related to - the length of the segments representing the angle. In Figure 19 we give an example of line segments of different sizes which form the same angle. Here S is a point on $\overrightarrow{PQ}$, not equal to P, and T is on a $\overrightarrow{PR}$, not equal to P. The angles are coincident, that is, $\angle QPR = \angle SPT$, and so have the same measure even though the line segments are different sizes.

Sometimes angles are denoted by numbers or letters as in Figure 20. Using our notation for coincident angles, we have on the left

$$\angle a = \angle PQR.$$

On the right $\angle 1$ is congruent to $\angle PQR$:

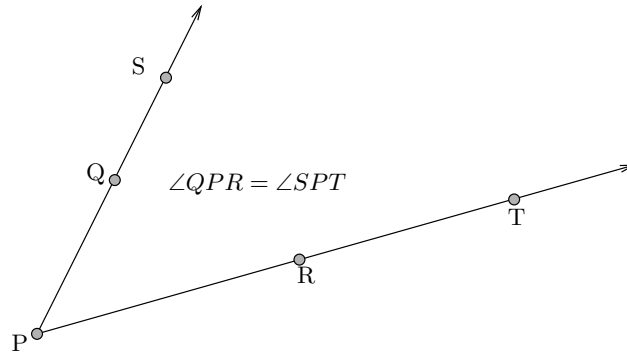$$\angle 1 \cong \angle PQR$$

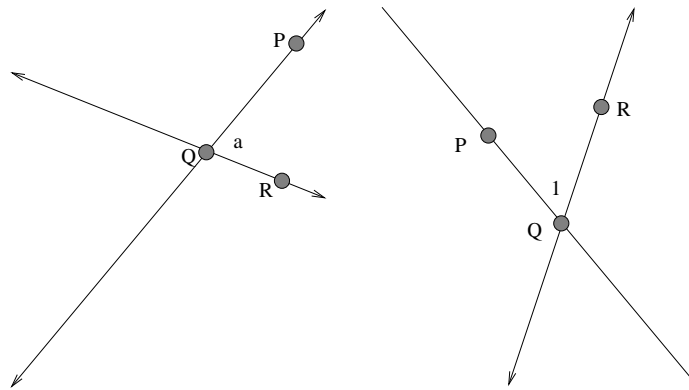FIGURE 19. More notation for angles

FIGURE 20. Different notation for angles

and so they have the same measure:

$$m\angle 1 = m\angle PQR.$$

In different situations, a lower case letter or number may be used to indicate the *measure* of the angle as in Figure 28. The meaning of notation should hopefully be clear from the context of the particular problem.

Interior angles of a triangle

Any triangle has three *interior* angles, that is, angles on the inside of the triangle. If the vertices of the triangle are labeled P, Q, R, then these angles are denoted $\angle PQR, \angle QPR, \angle QRP$ in Figure 21.

P
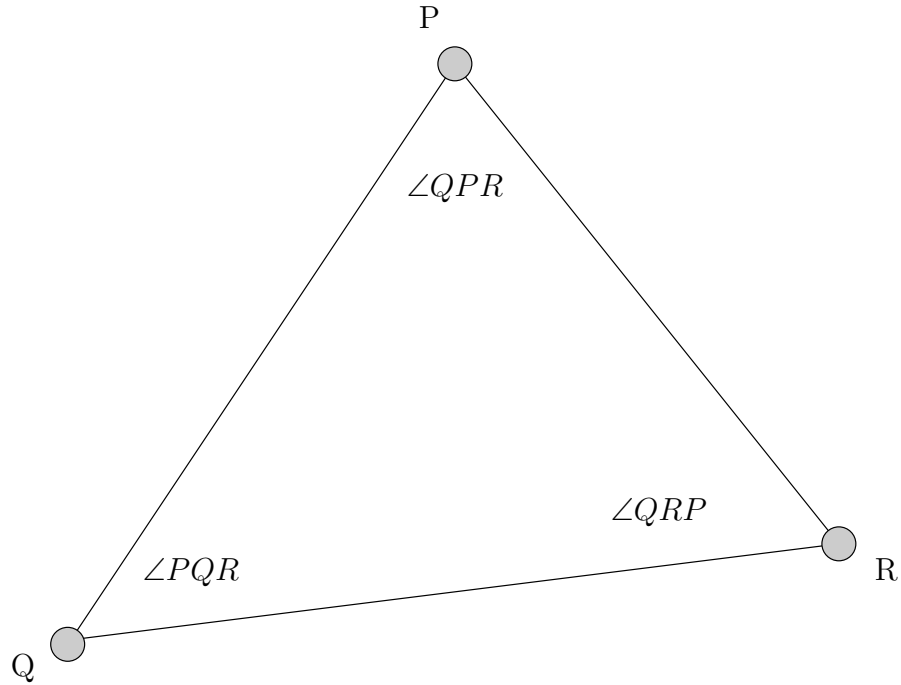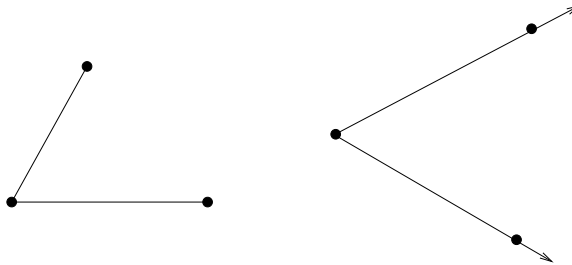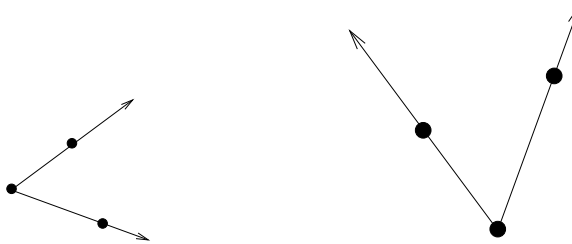
$\angle QPR$

$\angle QRP$

R

$\angle PQR$

Q

FIGURE 21. Interior angles of a triangle

The following problems provide practice with using terminology and notation about angles precisely.

Problem 3.1.4.   (1) Label points and identify any angles in the figure below using standard notation. For each angle, identify the pair of rays that form the angle.
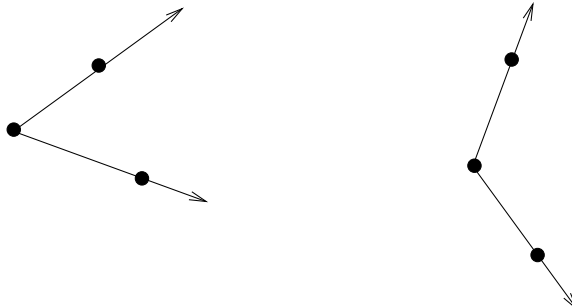
(2) Compare and contrast the angles below.

| Ways shapes are alike	Ways shapes are not alike

(3) Consider the angles shown below. Draw an angle whose measure is the sum of the measures of the shown angles and an angle whose measure is the difference of the measures of the shown angles.

Problem 3.1.5. Consider a rectangle with vertices A, B, C, D. Which of the following rays form an angle? (i) $\overrightarrow{AB}$ and $\overrightarrow{AC}$. (ii) $\overrightarrow{AB}$ and $\overrightarrow{DA}$. (iii) $\overrightarrow{EA}$ and $\overrightarrow{EC}$, where E is the point shown.

A ●――――――――――――――● B

● E

D ●――――――――――――――● C
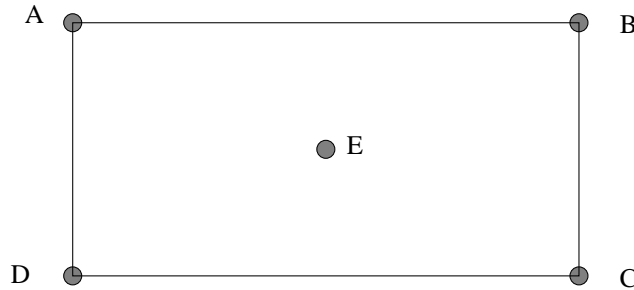
Problem 3.1.6. (a) Measure an angle formed by you and two of your classmates seated around the table, using available tools. Use suitable notation in a diagram for the angle below. (String may be used if available.)

For parts (b) and (c) use the following diagram:

C ●
  B
  ●
O ● ● A
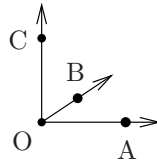
(b) Find the measure of $\angle COA$. Is it possible to find the sum of the measures of $\angle AOB, \angle BOC, \angle COA$ by only knowing the measure of $\angle COA$? If so, find it. If no, explain why.

(c) Measure the angles $\angle AOB, \angle BOC$.

(d) Find the sum of the measures of $\angle AOB, \angle BOC, \angle COA$.

(e) Does this match the sum in your answer from (b)? Why or why not?

3.2. **Acute, right, obtuse, adjacent, opposite.**

One way to think of angle is as a measure of *sharpness*. For example, children playing with blocks that have small angles are more likely to get hurt than those playing with blocks with large angles. Many blocks have only *right angles* which are those formed by perpendicular lines.

> An angle is *acute* if the angle has measure less than 90 degrees, or, informally, if the angle is less than 90 degrees.
>
> An angle is *right* if the angle measure is 90 degrees. Conventionally, a right angle is indicated with a small pair of perpendicular lines at the angle, as shown in Figure 22.
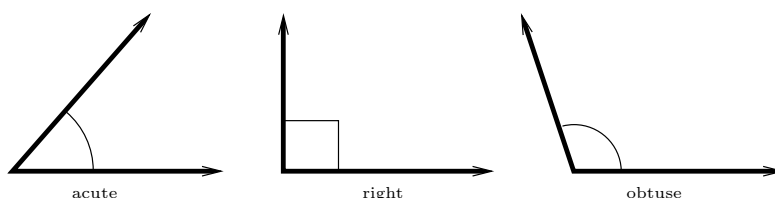

acute   right   obtuse

FIGURE 22. Types of angles

> An angle is *obtuse* if the angle measure is more than 90 degrees and less than 180 degrees.
>
> An angle is *reflex* if the angle measure is more than 180 degrees and less than 360 degrees.
>
> Of the two angles determined by a pair of rays with the same endpoint, exactly one is a reflex angle unless both are 180 degrees. Earlier we defined a reflex angle as the larger of the two shapes formed by two rays meeting at a vertex; the description above is an equivalent way of expressing the same idea.
>
> A *straight* or *linear angle* is one that measures 180 degrees, that is, one that is formed by rays pointing in opposite directions as explained above Figure 13. [5]
>
> Two angles are *adjacent* if they share a ray and do not (otherwise) overlap. See Figure 23 for two examples of adjacent angles.
>
> When two lines intersect, four angles are formed. The angles directly opposite to each other are *opposite angles*. Angles 1 and 3 and also angles 2 and 4 in Figure 24 below are opposite.
>
> Two angles are *complementary* if their measures add up to 90 degrees, and *supplementary* if their measures add up to 180 degrees. For example, angles 1 and 2 in Figure 24 are supplementary.

---

[5]The notation $\angle PQR$ is unclear for linear angles, since it isn't clear which angle (that is, which side) is referred to. One could resolve this confusion, for example, by going counterclockwise from $\overrightarrow{QP}$ to $\overrightarrow{QR}$.
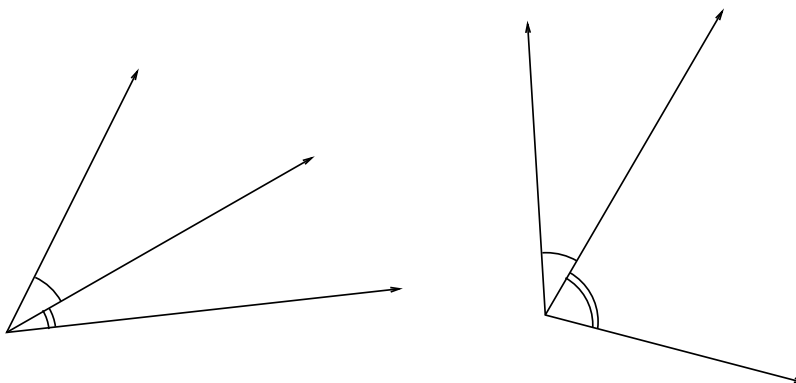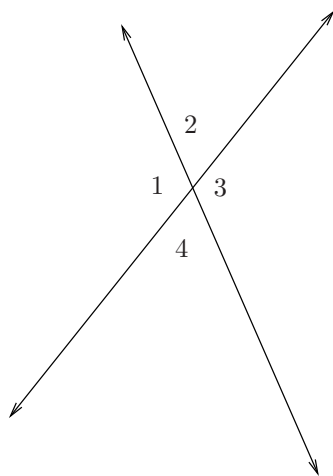
FIGURE 23. Adjacent angles

FIGURE 24. Opposite angles

Both children and professional mathematicians can have trouble remembering this vocabulary. Unfortunately, during assessments children don't have the chance to ask about vocabulary. A common trick for remembering the difference between complementary and supplementary is:

> *complementary* comes first in the dictionary and complementary angles have measures that sum to 90 degrees;[6]

> *supplementary* comes later in the dictionary and supplementary angles have measures that sum to 180 degrees.

[6]Here is another way of remembering: Q: What did one complementary angle say to the other? A: You're (a)cute!

So the order of the terms in the alphabet is the same as the numerical order of the sum of angle measures.

Note the difference between *adjacent* and *supplementary* angles: Any two adjacent angles at the intersection of two lines are supplementary, but supplementary angles need not be adjacent since supplementary angles do not necessarily share a side.

The following problem requires a bit of familiarity with the game of baseball. The problem is meant to introduce the language of "angle formed by two rays" and similar terminology that has sometimes appeared on state assessments.

Problem 3.2.1. A picture of a baseball diamond is shown below. We think of the bases and posts as points.
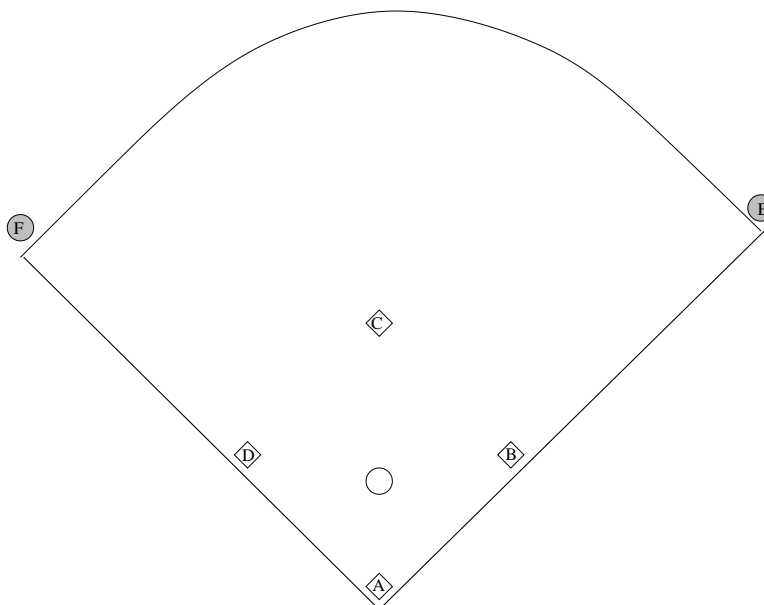
(1) Suppose a ball is hit from home plate towards second base. Is its path best represented by (a) the line $\overleftrightarrow{AC}$, (b) the ray $\overrightarrow{AC}$ (c), the ray $\overrightarrow{CA}$, or (d) the segment $\overline{AC}$? Draw your answer on the picture.

(2) The runner runs from home plate to first base. Is the runner's path best represented by (a) the line $\overleftrightarrow{AB}$ (b) the ray $\overrightarrow{AB}$ (c) the ray $\overrightarrow{BA}$ or (d) the segment $\overline{AB}$? Draw your answer on the picture.

(3) Find and name two rays in the picture that form a perpendicular angle.

(4) Find and name two rays in the picture that form a linear angle.

(5) Find and name two segments that do not intersect.

Distances between shapes

The distance between two shape is the *minimum* distance between a point in the first shape and a point in the second shape. See Figure 25.
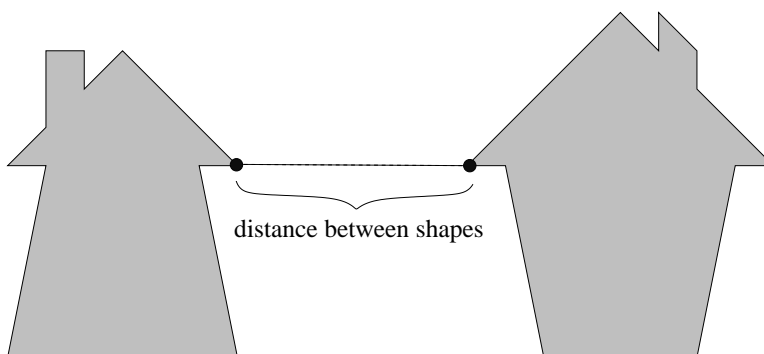

FIGURE 25. Distance between shapes

In particular the *distance from a point to a line* is the length of the line segment perpendicular to the line, that begins at the point. In order to find the distance of a point to a line, find the line perpendicular to the given line and passing through the point, as in Figure 26.
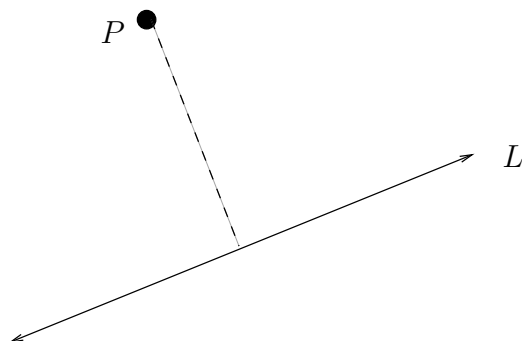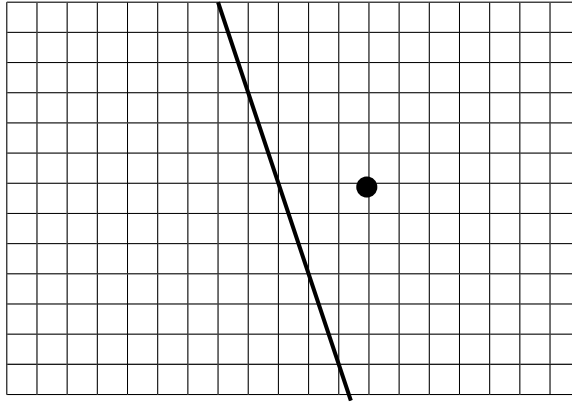

FIGURE 26. The distance of a line to a point

Problem 3.2.2. (1) Draw a line segment perpendicular to the given segment through the given point. Explain your construction. (Why is it perpendicular?) You may use a right angle and ruler, as necessary.

(2) Find the distance of the point to the line, by measuring.

Children should consider the available tools when solving a mathematical problem and attend to precision when measuring. The following problem does not specify what tools to use, which allows the solver to make that decision as part of the problem solving process.

**Problem 3.2.3.** Find the distance of the given point to the given line.
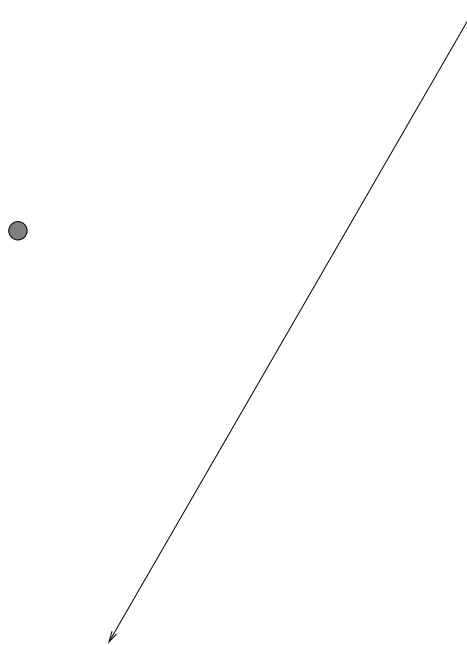
3.3. **Reasoning with angles.**

Children should be able to *construct viable arguments* that explain why basic facts in geometry are true. The following problems are meant for discussion and may use facts that will be discussed shortly afterwards:

Problem 3.3.1. For each of the following statements, decide whether it is true and explain your answer:

(i) If two lines intersect at a point, then any two adjacent angles sum to 180 degrees (or more precisely, have measures that sum to 180 degrees.)

(ii) If two lines intersect at a point, then any two opposite angles are congruent.

Problem 3.3.2. (Adapted from [SM]) In the figure, $XW = XY, m\angle WXY = 38°$ and X, Y, Z are colinear. Find the measures of $\angle XWY$ and $\angle ZYW$.
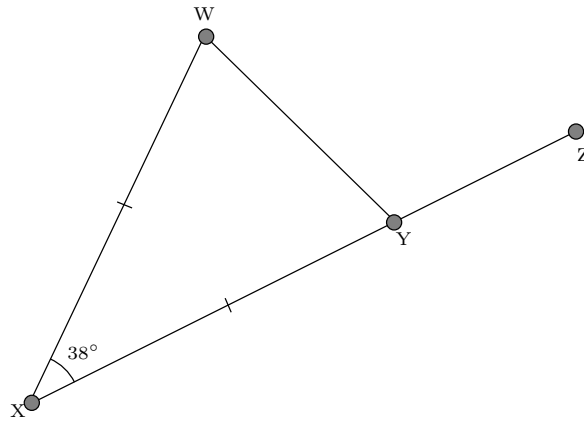
Moving angles

Two angles are congruent if one can be obtained from the other by translation (sliding) and/or rotation (turns), as we already mentioned on page 31. In this case the measures of the two angles are equal. The fact that the measure of an angle does not change when translated or rotated is called the *moving principle* for angles. Translations and rotations are discussed more in Chapter 7.
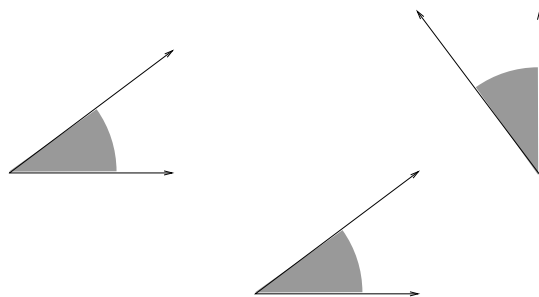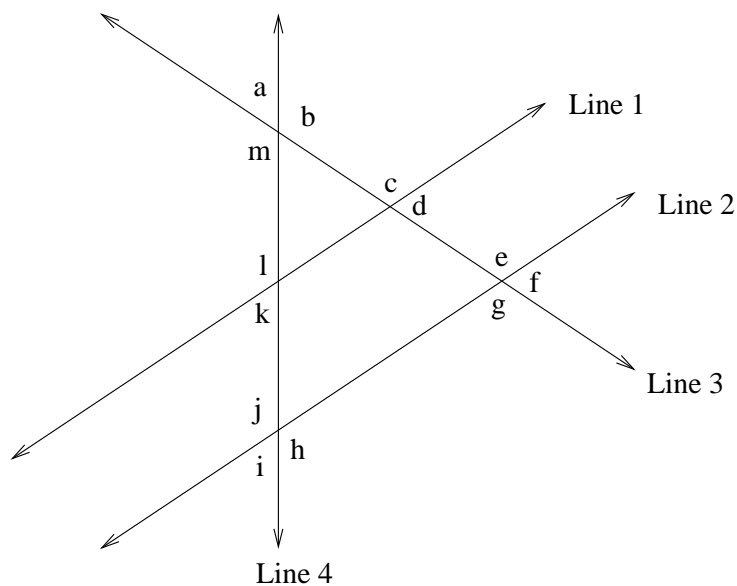

FIGURE 27. Congruent angles

Problem 3.3.3. In the picture below, which of the following angles are related to the angle marked c by slides and turns? Explain your answer (the direction of slide, amount of turn, and so on.)

We discuss two consequences of the moving principle for angles. First, suppose that two lines meet at a point as in Figure 28. Turning around that point by 180 degrees changes each angle to its opposite angle. It follows that *opposite angles have equal measure.*
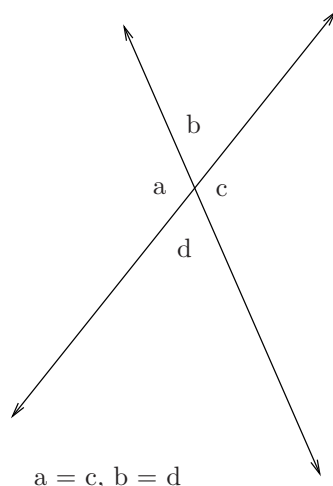
b

a   c

d

a = c, b = d

FIGURE 28. Opposite angles have equal measures denoted a, b, c, d

If you have internet access, a GeoGebra demonstration can be found below:

Opposite angles are congruent

Students can explore the relationship between opposite angles in a variety of hands-on ways. One approach involves linked ice pop sticks (or pencils) as in Figure 29. By keeping one stick fixed and rotating the other one around the "intersection point" (where the sticks are linked), students can observe how opposite angles change in the same way, at the same pace (that is, either both increase or both decrease in measure as the rotating stick keeps moving). This exploration helps students build a "common sense" understanding of opposite angles and experience the dynamic side of geometry, which allows them to "view" a large number of examples in a short amount of time. Another hands-on exploration of opposite angles can be designed using a sheet of paper with two intersecting lines drawn on it. Students can be asked to formulate a conjecture regarding the relationship between two opposite angles indicated on the paper, and then to find a way to convince others that their conjecture is correct. Some may fold the paper to make one angle overlap the other one; others may cut out the angles and use the rotation argument discussed in the paragraph above. Both ways are valid and provide an opportunity to discuss how the moving principles for angles was applied in each case. Note that the folding argument involves a reflection, so it is not a direct application of the moving principle as stated on page 46.[7]
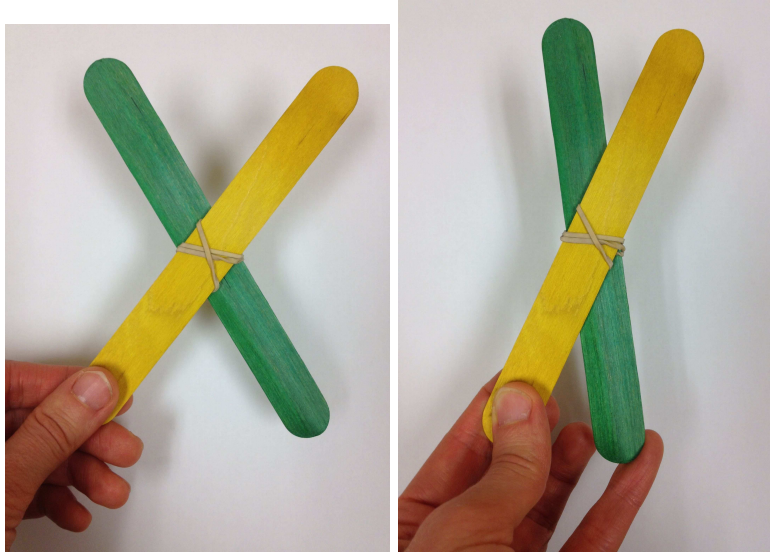
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[7]Photo credit I. Radu

FIGURE 29. Popsicle stick exploration of opposite angles

We now discuss the second application of the moving principle for angles. Suppose two parallel lines are cut by a third line as in Figure 30. The angles at the first intersection point can be slid along the third line to match the corresponding angles at the second intersection point.
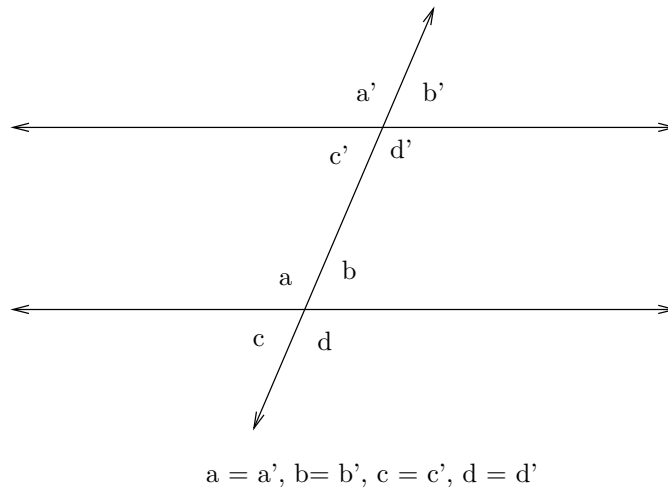

a = a', b= b', c = c', d = d'

FIGURE 30. Angles formed by parallel lines

It follows that the angles at the first intersection point have the same measure as the corresponding angles at the second intersection point. This property is logically equivalent to Euclid's *parallel postulate*.

If you have internet access, a GeoGebra demonstration can be found below:

<p style="color:red; text-align:center">Corresponding angles are congruent</p>

The angle sliding argument discussed above (and demonstrated by the GeoGebra applet) can also be explored in a low-technology manner, using paper materials. As shown in Figure 31, all one needs is a drawing of parallel lines (Lines 1 and 2) intersected by a third line (Line 3), and a cutout angle of equal measure to one of the angles from the drawing. The cutout angle can be slid along Line 3 until it overlaps with an angle corresponding to the original one; the cutout could also be rotated around one of the intersecting points to show (or reinforce) the congruence of opposite angles. This activity offers an opportunity for students to practice terminology by explaining what they are doing at each step, and to relate the exploration to the moving principle for angles, thus linking the informal approach to its theoretical background.[8]

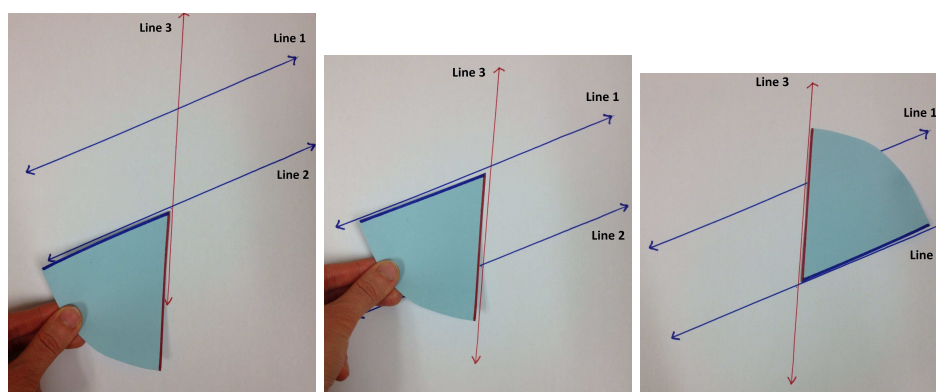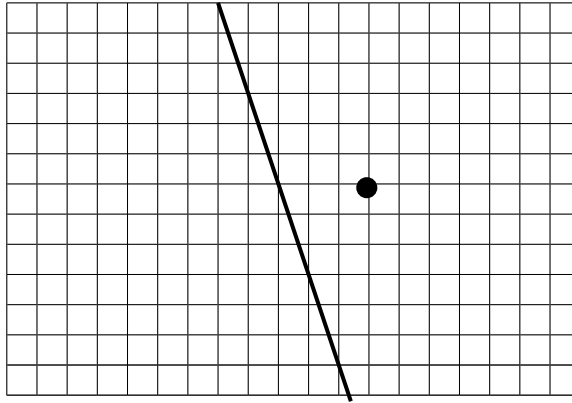FIGURE 31. Exploring corresponding angles with paper cutouts

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[8]Photo credit I. Radu

Problem 3.3.4. (1) Draw a line segment parallel to the given segment through the given point. Explain your construction. (Why is it parallel?) You may use a right angle and ruler, as necessary.

(2) Draw a parallel line one inch away from the given one. How do you know that your drawn line is parallel?
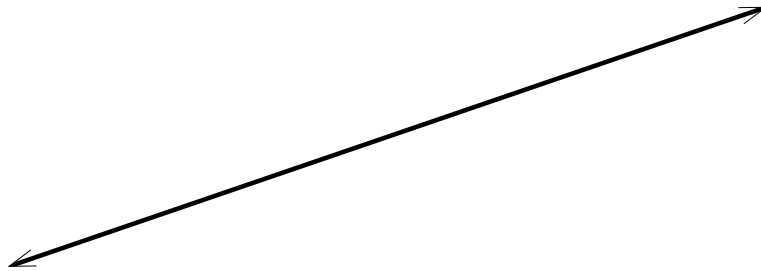
Dividing angles

Suppose we divide up an angle into smaller angles. In this situation, the measure of the larger angle is the sum of the measures of the smaller angles. For example, we can divide a linear angle (measure of 180 degrees) into two right angles (measures of 90 degrees).

This principle is sometimes called the *additivity principle* for angles: If two angles $\angle BAC$ and $\angle CAD$ share a vertex and ray and lie in the same plane so that they make up a larger angle $\angle BAD$, then $m\angle BAC + m\angle CAD = m\angle BAD$ as in Figure 32.
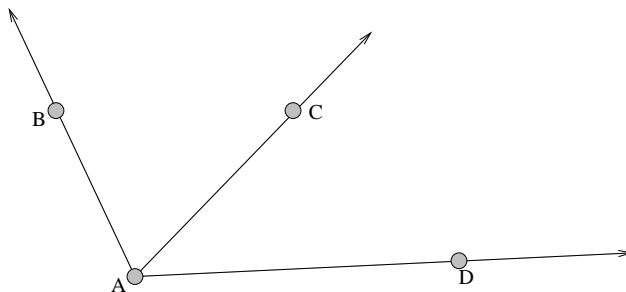
FIGURE 32. Additivity principle for angles

Problem 3.3.5. Suppose that three lines in the plane are such that no two lines are parallel as in Figure 33. Explain how to slide and turn the interior angles of the triangle below so that they make up a linear angle. Explain how you know that the constructed angle is linear.
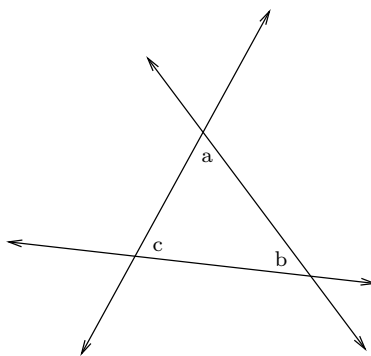
FIGURE 33. Sum of the measures of the interior angles of a triangle

The problem above shows that the interior angles of a triangle make up a linear angle. By the additivity principle for angles, the sum of the measures of the interior angles is 180 degrees.

If you have internet access, you can click on the following link to see a GeoGebra illustration of the fact that the interior angles of a triangle sum to 180 degrees: (Drag the black points to animate the figure.)

Interior angles of a triangle

This property of the interior angles of a triangle is discussed more below in Workshop 3.7.1. A hands-on way of exploring the sum of the measures of the interior angles in a triangle involves paper cutouts and translations (sliding). In Figure 34, angles 1 and 1' are congruent, as are angles 2 and 2', respectively (because we already know that opposite angles are congruent). Sliding cutout angles 1' and 2' along sides of the triangle, we can create what appears to be a linear angle "at the top" of the triangle. At the middle school level, classroom discussion should relate this activity to the moving principle for angles, as well as engage students in finding reasons to believe that the created angle is a linear angle in general (meaning, for any triangle). [9]
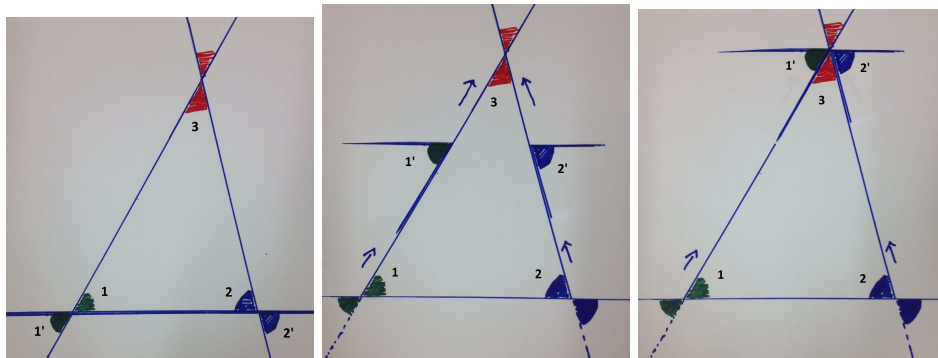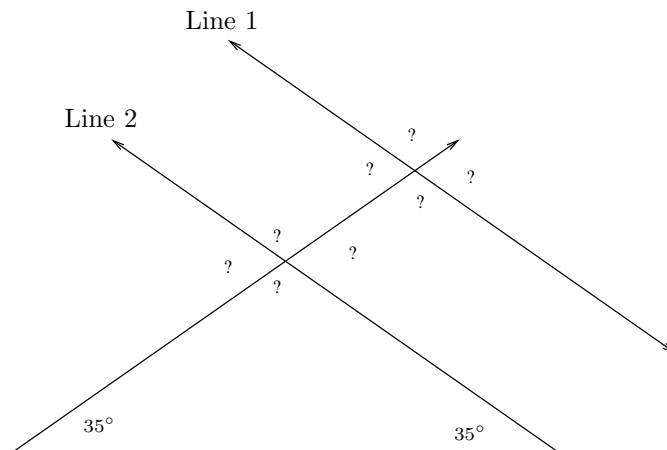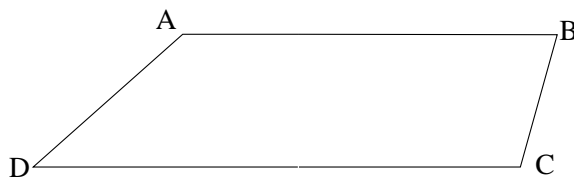

FIGURE 34. Exploring the sum of angle measures in a triangle.

Problem 3.3.6. Fill in the missing angles, without measuring. Line 1 and Line 2 are parallel.

Problem 3.3.7. (Adapted from [SM]) In the figure below, $\overline{AB}$ is parallel to $\overline{CD}$, $m\angle ABC = 82°$, and $m\angle ADC = 48°$. Find $m\angle BAD$ and $m\angle BCD$.

---

[9]Photo credit I. Radu

Summary

In the discussion above we covered the following four facts about angles that are often used in reasoning about geometry in a plane.

(1) (Opposite angles) Opposite angles have equal measures.
(2) (Version of the Parallel Postulate) If two parallel lines are cut by a third line, then the angles at the first intersection point have the same measure as the corresponding angles at the second intersection point.
(3) (Interior angles of a triangle) The measures of the interior angles of a triangle sum to 180 degrees.
(4) (Additivity Principle) If two angles $\angle BAC$ and $\angle CAD$ share a vertex and ray and lie in the same plane so that they make up a larger angle $\angle BAD$, then $m\angle BAC + m\angle CAD = m\angle BAD$.

For the following problem one needs to know the definition of *angle bisector:* It is a ray with the same vertex as the angle that divides the angle into two angles of equal measure, as in Figure 35.
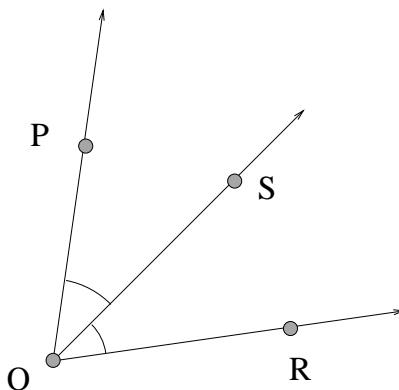

FIGURE 35. An angle bisector

Regarding the relevant standards, bisectors are only covered in the High School Common Core State Standards, although presumably students should be familiar with the idea of bisector before then.

3.4. **Reflections.**

When light hits a smooth mirror or reflects off water, it reflects with *angle of reflection* equal to the *angle of incidence*, that is, the angle that the light beam hits the surface with. Both angles are measured from the *normal line*, that is, the line perpendicular to the surface. The angle of incidence is often denoted i and the angle of reflection r as in Figure 36.

normal line

path of light

i   r

reflecting surface
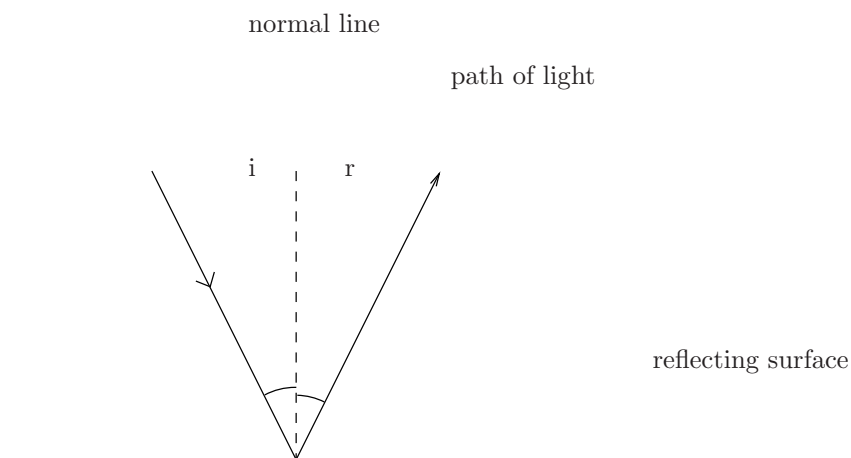
FIGURE 36. Angles of incidence and reflection

To find the path of a light beam reflecting off a mirror, seen from the side, perform the following steps: (a) draw the normal to the line representing the mirror using a protractor; (b) measure the angle of incidence; and (c) use a protractor again to draw an angle of the same measure on the other side of the normal as in Figure 36.

Problem 3.4.1. Draw the paths of the light rays shown. In each case, indicate the angle of incidence.

first ray   second ray   third ray

mirror
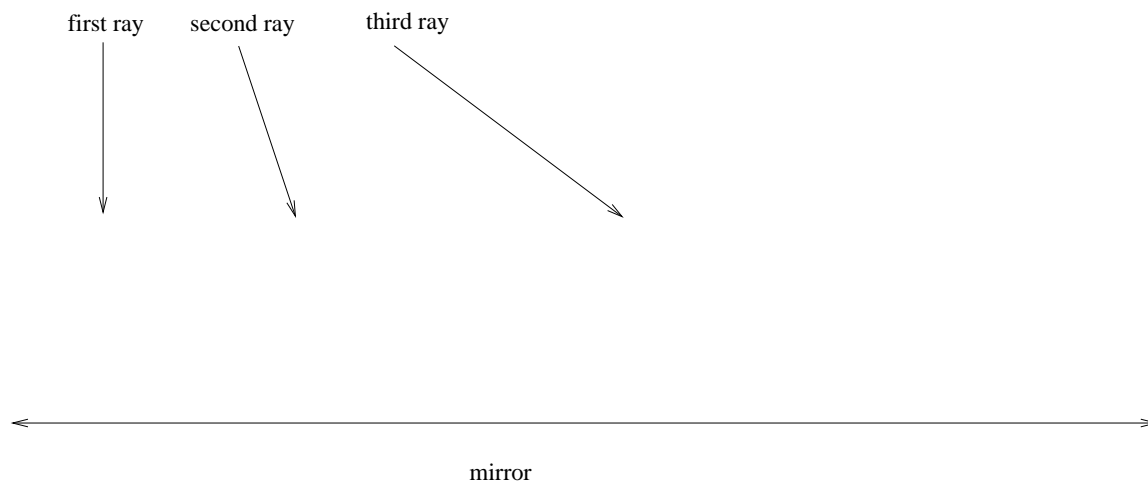
3.5. **Refraction***.

When light hits another transparent object, such as water or glass, the part that passes through does so with *angle of refraction* approximately equal to the *angle of incidence* divided by the *index of refraction*.[10] The index of refraction is about 1.5 for glass.

normal line

path of
light ray   i

surface of water or glass

r

angle of refraction r $\approx$ angle of incidence i / index of refraction p
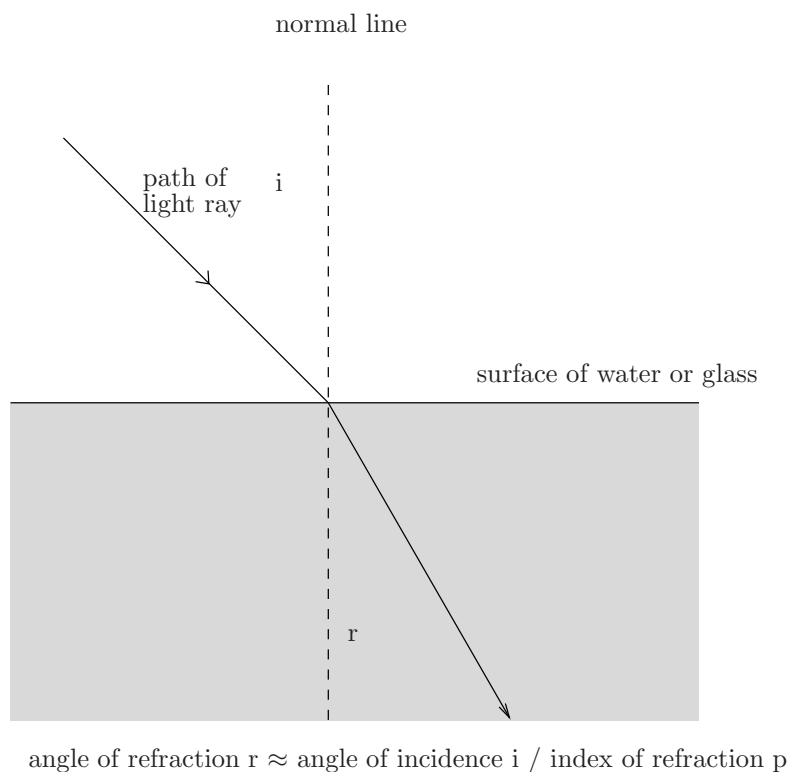
FIGURE 37. Refraction

For example, if the angle of incidence is $45°$, as shown in the figure above, then the angle of refraction is about $45°/1.5 = 30°$.[11]

When the light leaves the glass, it changes angle again. The angle of the outgoing ray r has measure approximately equal to the measure of the incoming ray i *times* 1.5, so that the angle of the ray is that before it entered the glass. See Figure 38.

---

[10]The precise relationship is called *Snell's law* and involves the sinusoidal function, which we are approximating by the identity to keep things at the middle school level.

[11]Since we have approximated the sinusoidal function by the identity, this is off by about 2 degrees from the actual answer.
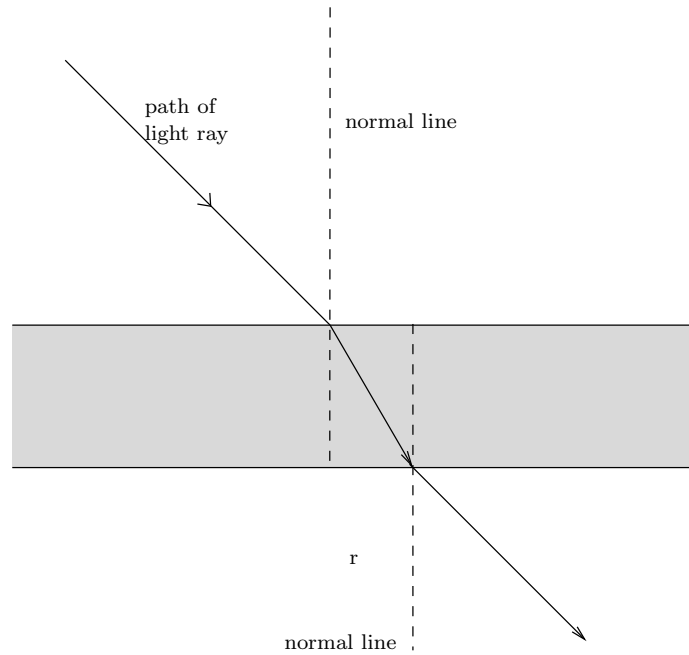
FIGURE 38. A light ray entering and then leaving a slab of glass

Problem 3.5.1. Draw the paths of the light rays shown until the ray meets the other side of the glass. (It changes direction again as it leaves the glass, but you do not need to show the path after it leaves.) In each case, indicate the angle of incidence.

first ray

second ray

third ray

glass
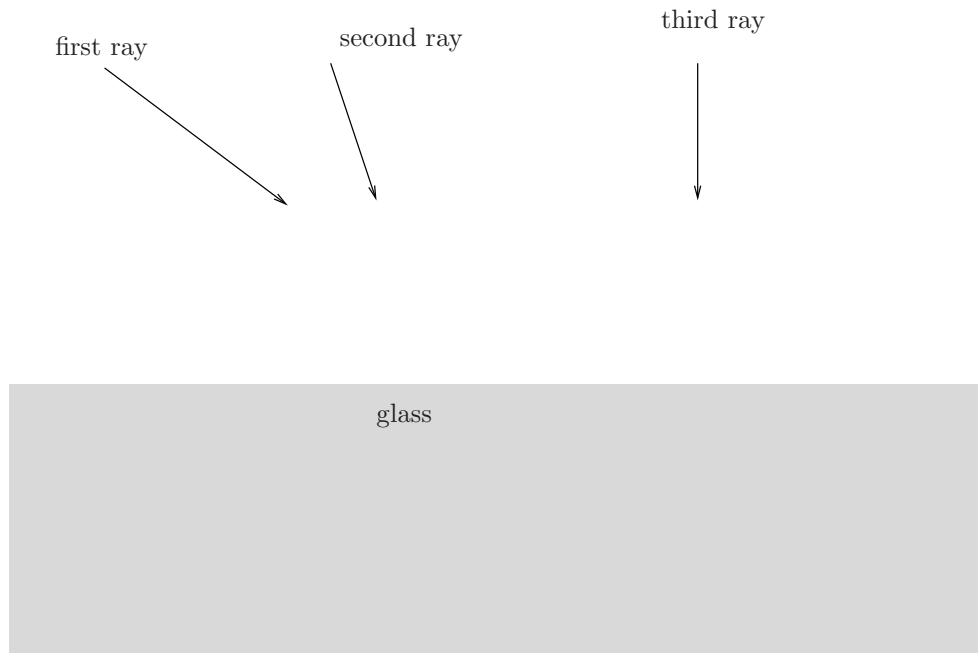
Problem 3.5.2. (Challenge) Your friend throws you in a pool. From the bottom of the pool does your friend look taller or shorter than usual? Explain your answer using a picture.

3.6. **Optics of curved objects**[*].

For a curved mirror or curved lens, the reflection/refraction angle is measured from the *normal (or perpendicular) line* to the surface. This is the line perpendicular to the *tangent line* to the mirror, that is, the line that has the same *direction* as the mirror at the reflection/refraction point. The angle of incidence is the angle between the incoming ray and the normal line. Figure 39 shows a picture of reflection off a curved mirror, while Figure 40 shows a picture of a ray of light as it refracts through a lens.

path of light

i

normal line

r

tangent line
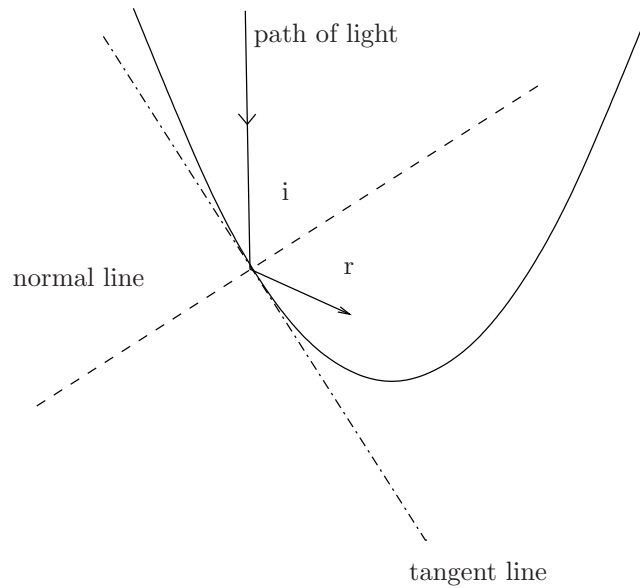
FIGURE 39. Reflection off a curved object

Problem 3.6.1.

(a) Draw the path of a light ray reflecting off the following mirror. (Actually, it reflects a second time but you need not draw the second reflection.)
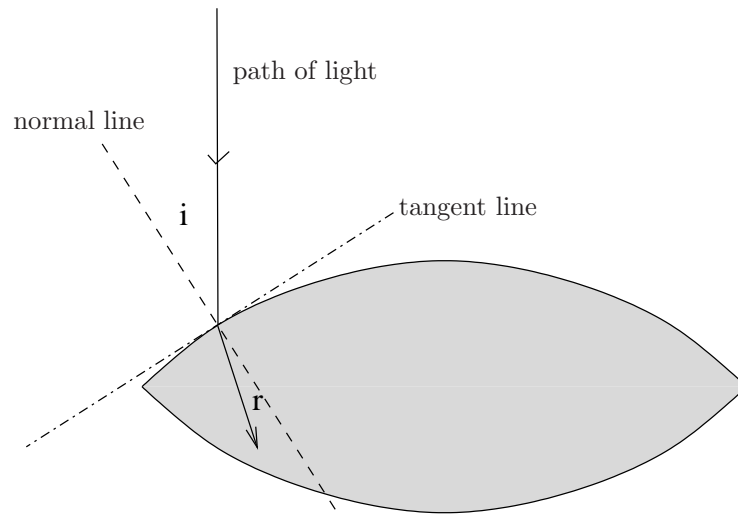
FIGURE 40. Refraction through a curved object
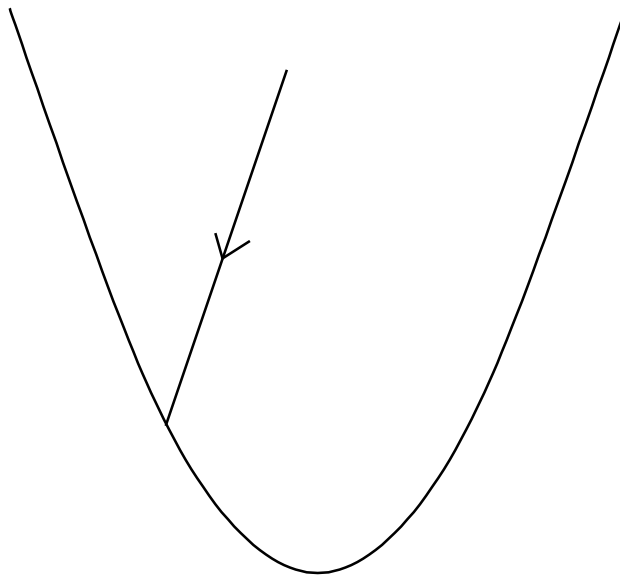
(b) Suppose the shape above was not a mirror, but represents the boundary between air and a piece of glass. Indicate the path of the light ray as it enters the glass. (Write down the index of refraction you are using.) Label your answer clearly to distinguish it from your answer to part (a).

Problem 3.6.2. Draw the paths of the light rays shown, as they pass through the lens. Each time the ray passes through or leaves the lens, indicate the angle of incidence.
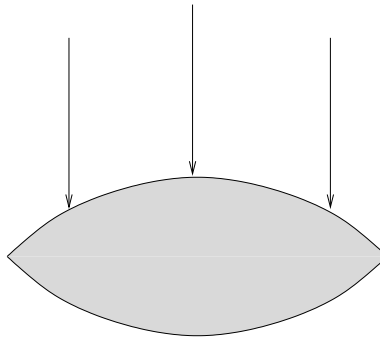
There are situations where light does not travel in a straight line. For example, objects like black holes bend light, so that two images of an object can appear on the other side of a black hole. Also, light travels along fiber-optic cable, which clearly bends; this is how many telephone and cable services now work.

Problem 3.6.3. (Challenge) Consider a person viewing an object through a lens, as shown. Will the object look larger or smaller to the viewer, than if the lens were not there? Explain your answer with a drawing showing how light travels through the lens.
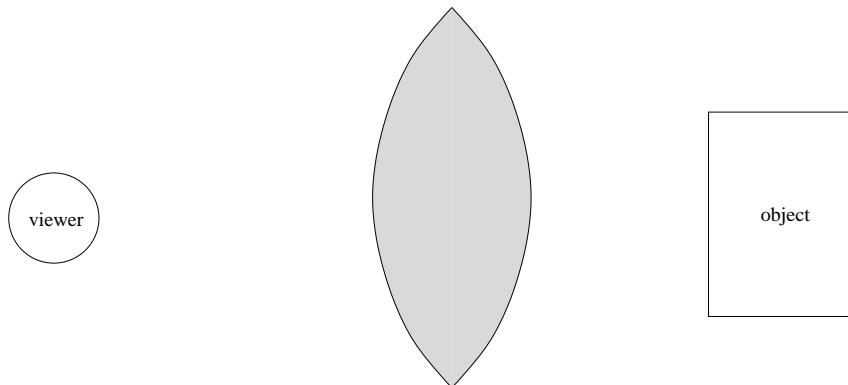
3.7. **Workshops.**

Workshop 3.7.1.

(1) Suppose Pat is standing at point A and is facing point B. He then walks
to point B and turns counterclockwise to face C. He walks to point C and
turns counterclockwise to face point A. Finally, he walks to point A and
turns counterclockwise to face point B again. How much has Pat turned
altogether? How can this be used to show that the measures of the interior
angles of a triangle sum to 180 degrees?

C

A   B
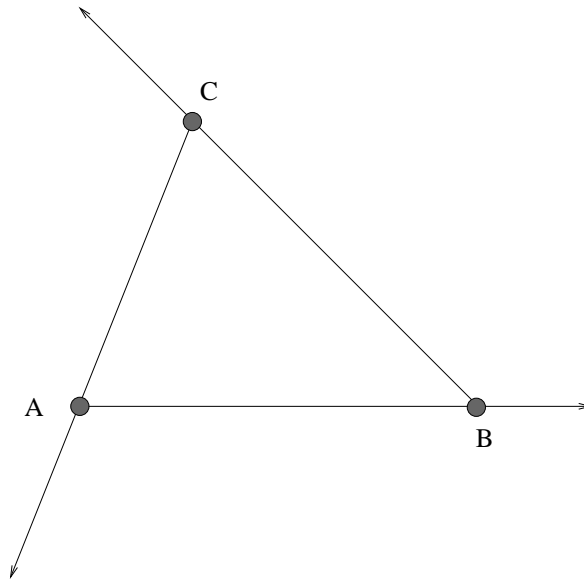
(2) In the figure below, Lines 1 and 2 are parallel, A is a point on Line 1 and
B and C are points on Line 2. Use the material discussed in this section
to explain why the sum of the measures of the interior angles of a triangle
is 180 degrees.
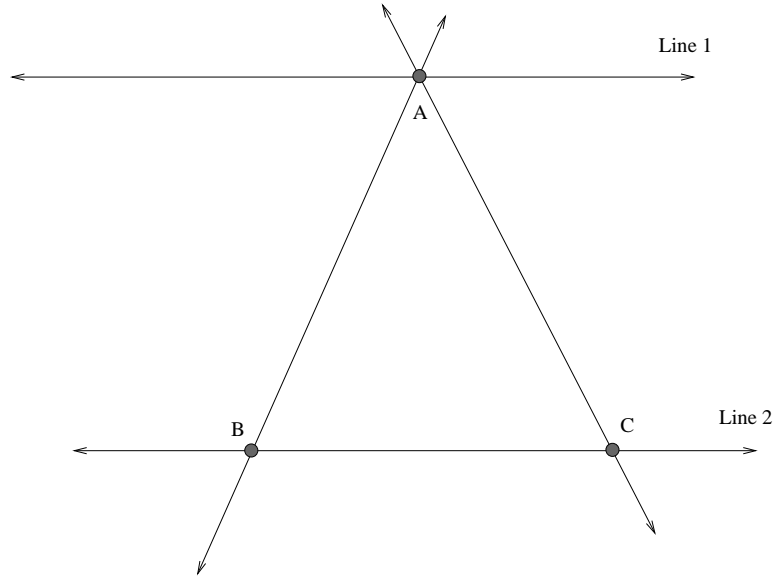
(3) Take a triangle and fold over the corners over the edges of a rectangle as shown in the figure below. (Your triangle should probably be bigger than the triangle in the figure. You may want to use colored paper or similar.) Explain *carefully* how to fold so that the three corners meet at a point. (How did you know where to fold each time?) Explain how this is relevant to the fact that the sum of the measures of the angles of the original triangle must be 180 degrees. (If this problem is being submitted, please submit your folded triangle.)
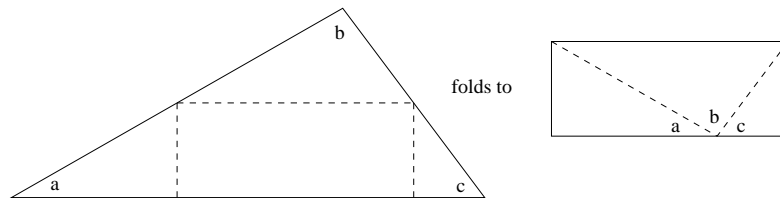
(4) Take a paper triangle and tear off a piece around each vertex. Now arrange the corners to make a linear angle. (A large triangle, possibly made of colored paper, works best, possibly with masking tape so that the corners can be re-attached.) Discuss how this method compares to the others in this problem. What are some advantages or disadvantages of using each method in a middle school classroom? (If this work is being submitted, please submit your triangle with corners torn off.)

Workshop 3.7.2.

(a) Johnny walks to school by walking 100 feet north, turning left 90 degrees, walking 200 feet, turning right 30 degrees, walking 100 feet, turning 222 degrees right, walking 50 feet. Draw his path using the scale that 100 real feet equals 1 map inch.

N

(b) Pirate Jack has hidden his treasure according to the map shown. Your friend is on Jack's island (equipped with a protractor) and you have to communicate to him, over cell phone, how to get to the treasure. What instructions would you give (based on one step per centimeter on the map)?
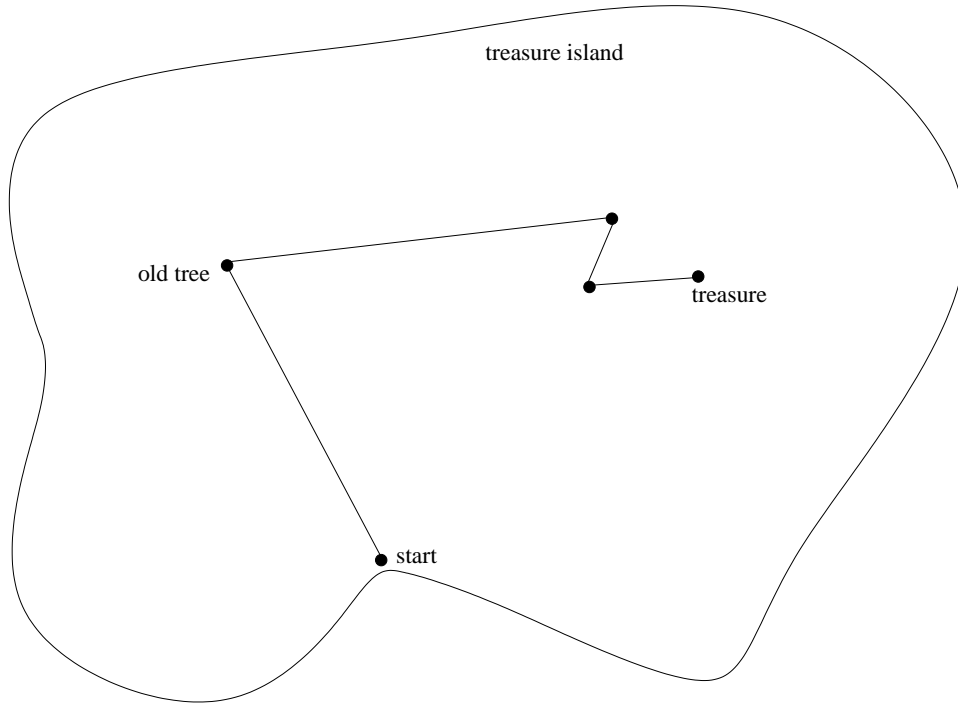
Workshop 3.7.3.

(a) In the games of pool or billiards, a ball bounces off the edge of the playing table with only a small loss of speed. Draw the path of the ball shown, with initial direction shown by the line segment in the figure, until it bounces twice. Explain each step of your construction.
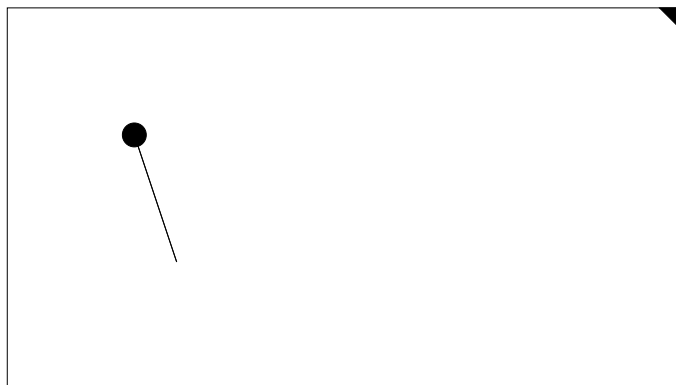
(b) Can you draw a path of the ball so that it bounces off a wall once and goes into the corner pocket? (This is a challenging problem if one is to use only the material used so far; we will revisit this problem later.) Explain the logic of each step in your construction.
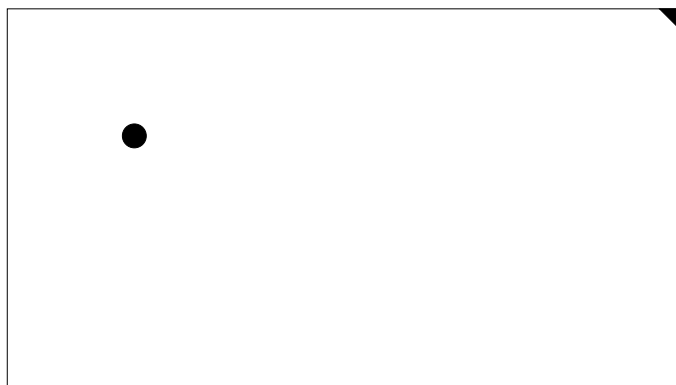
(c) Indicate where to place mirrors in the periscope, so that anyone looking in it can see out. (The mirrors do not have to be parallel to the walls.) Draw the path of a possible light ray passing through the periscope. Explain each step in your construction.

viewer
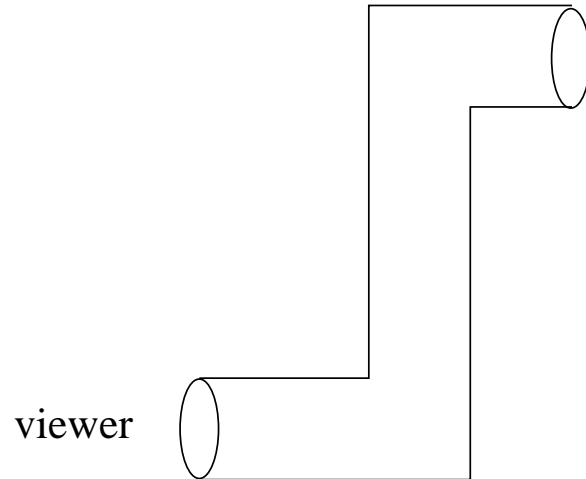
(d) Indicate where to place mirrors in the following more complicated periscope so that anyone looking in it can see out. (The mirrors do not have to be parallel to the walls.) Draw the path of a possible light ray passing through the periscope. Explain each step in your construction.

viewer
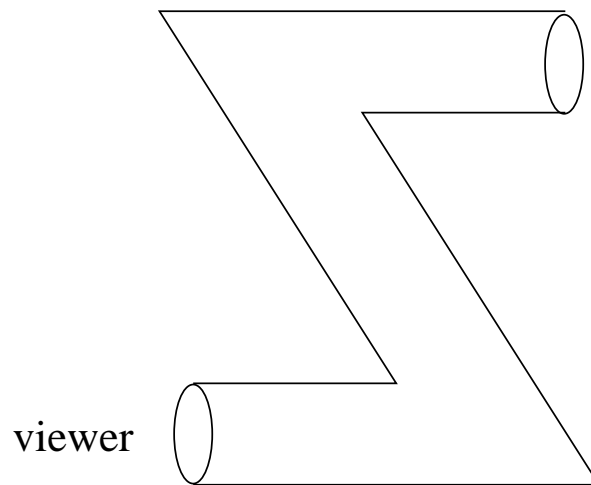
(e) (This part requires two small mirrors and a laser pointer.) Check your design by using the mirrors and laser pointer to "build" a periscope laid out horizontally.

3.8. Assessments and sample student work.

Many assessment problems on angles test the vocabulary on right angles, perpendicular lines, and parallel lines. However, children are not always familiar with the assessment vocabulary. Here is an example from [NJ].

Problem 3.8.1. In the figure below: (1) Name two rays that form a right angle. (2) Name two lines that intersect each other but are not perpendicular. (3) Name two lines that appear to be parallel.
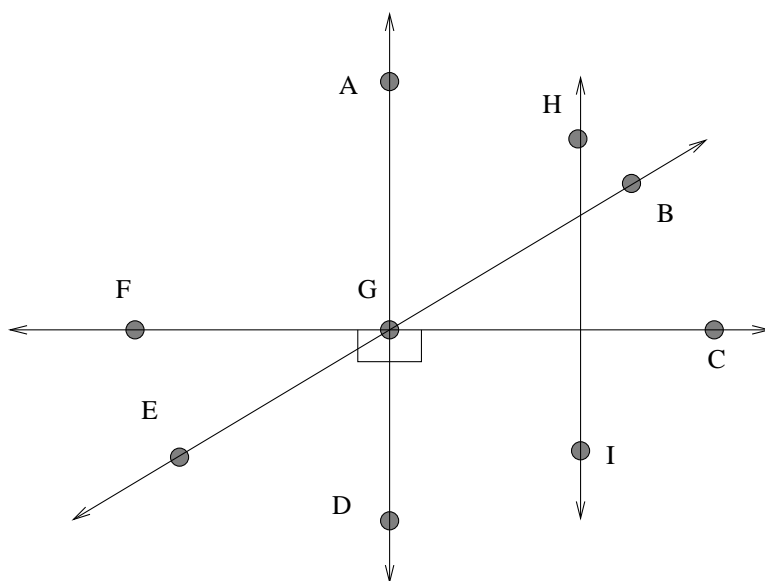
Sample Answer from [NJ] with Score 3/3: *Possible rays are GA and GF, GA and GC, GC and GD, GD and GF or other combinations. Possible intersecting but non-perpendicular lines are AD and BE, CF and BE, DA and EB, FC and EB, HI and EB or other combinations. Possible parallel lines are AD and HI or DA and IH.*

Note that the test writers do use but do not insist on the use of notation $\overleftrightarrow{GA}$, $\overrightarrow{GA}$ and so on. They write [NJ] "It was not necessary for students to use the ↔ or → to denote rays and lines. However, the correct 2 letters were necessary. There were a number of students who used commas in between letters, or also used 3 letters, which is incorrect notation."

Sample Answer from [NJ] with Score 2/3: *(1) F,G and E,G. (2) EB and HI. (3) AD and HI.*

The response shows nearly complete understanding of the problem's mathematical concepts. The student incorrectly answers (1), and answers correctly parts (2), (3).
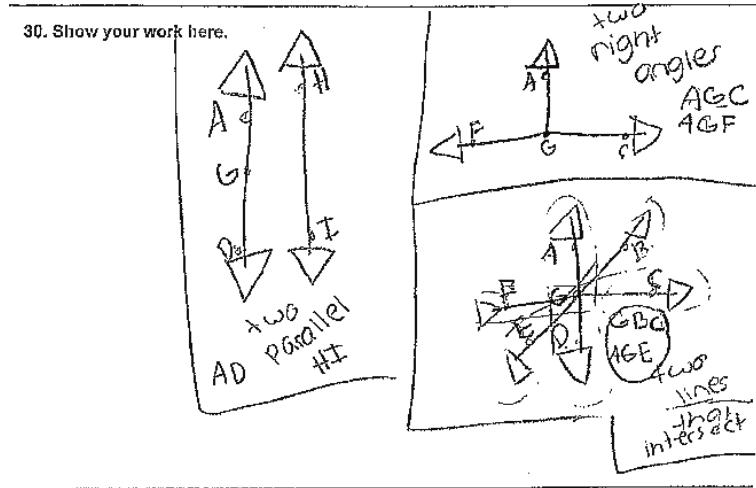
Score: 1/3.

Sample Answer from [NJ] with Score 0/3:   *Two rays that form a right angle are A, C. Two lines that intersect are F, C and A, D. Two lines that are parallel are A, D and H, I.*

Sample Answer from [NJ] with Score 0/3:
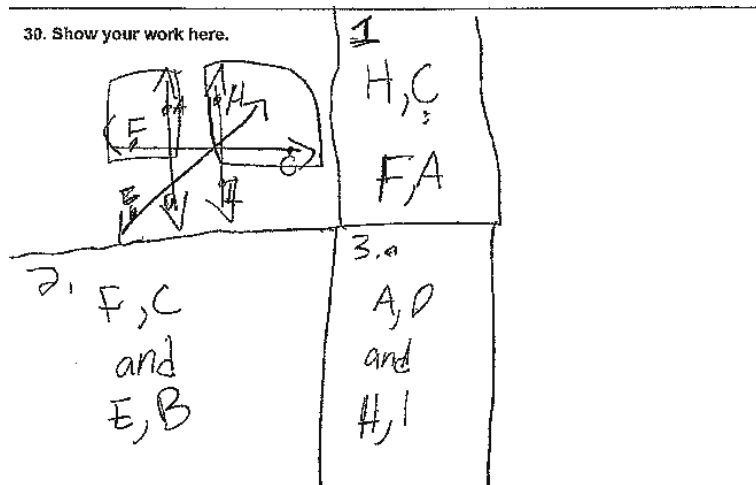
Note: The sample responses show that few of the students are really familiar with the concept that an angle is made of two rays with a common vertex.

4. Polygons

Paths taken by children while walking often consist of a sequence of straight line segments. Such a path is called *polygonal*. Other polygonal paths include boundaries of regions such as many playgrounds, gardens, and yards.

Elementary geometry often does not distinguish between the region inside a shape and the shape itself, since the difference is usually clear from the context. For example, we might say that a park has a square shape when in fact we mean that the boundary of the park is a square region; similarly, saying that a bike path is a square means that the bike path consists of four segments of equal lengths at right angles. To distinguish the two possibilities graphically, regions should be *shaded*.

The objectives of this chapter are: (i) to work towards a definition of polygons; (ii) to review basic terminology and properties of triangles and quadrilaterals; and (iii) to develop a better understanding of the sum of the measures of the interior and exterior angles of polygons.

4.1. **Polygons, vertices, edges.**

Children who walk or drive to school often take a path which is a sequence of straight line segments and changes of direction. Such a path is called a *polygonal path*: a shape formed by a sequence of line segments, such that any two consecutive line segments intersect in an endpoint. The endpoints of the line segments in a polygonal path are called *vertices* and the line segments in a polygonal path are called *edges.*

Figure 41 shows polygonal paths in a plane with five edges.
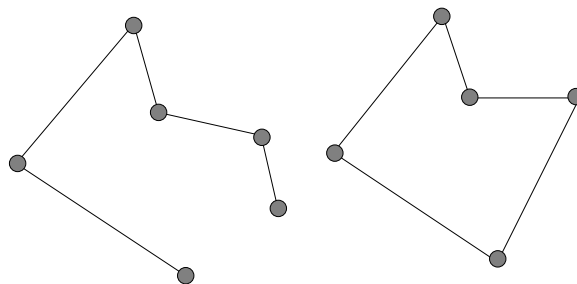
Figure 41. Polygonal paths

A polygonal path is *closed* if the last point in the last segment is the first point in the first segment. For example, in Figure 41, the path on the right is closed but the one on the left is not.

A polygonal path is *non-self-crossing* if it does not cross itself.

A *polygon* is a polygonal path that satisfies all of the following conditions:

Self–crossing   Self–crossing   Non–self–crossing
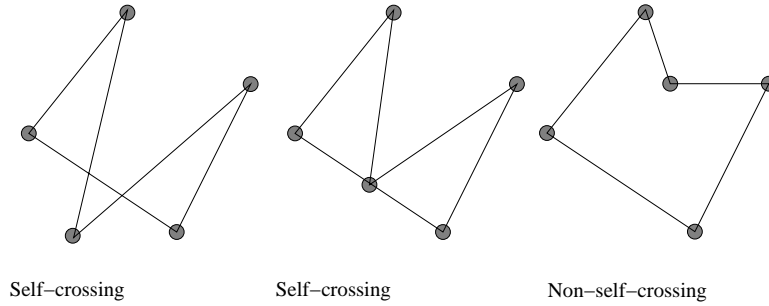
FIGURE 42. Self-crossing (left and middle) and non-self-crossing (right) paths

(1) closed;
(2) non-self-crossing; and
(3) no two edges that meet in a vertex form a linear angle.

In other words, at every vertex there is some change in direction.
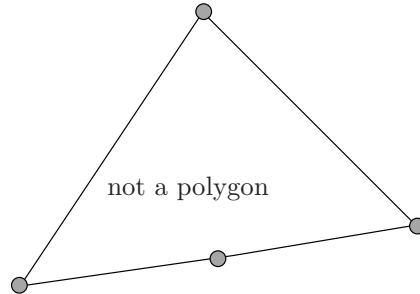
not a polygon

FIGURE 43. A polygonal path with no change in direction at a vertex

a polygon   not a polygon (curved sides)   not a polygon (not closed)
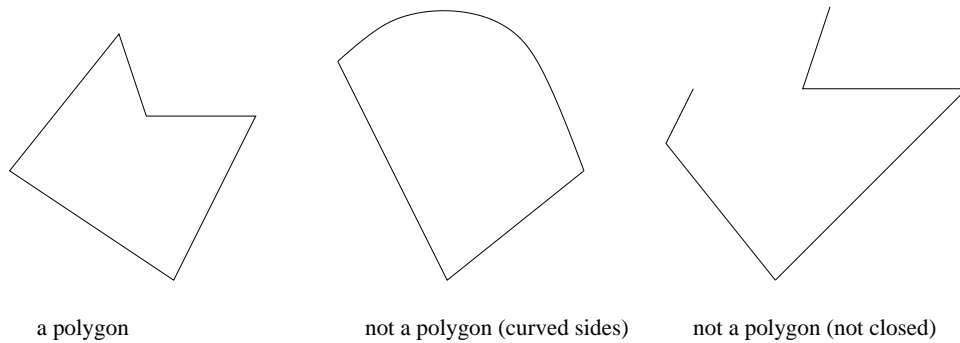
FIGURE 44. Figures that are polygons and not polygons

Other authors have different definitions of polygon; unfortunately, there is no "standard definition". For the rest of this book, we will use polygon in the sense defined above.

An *n-gon* is a polygon with *n* edges, where *n* is some positive integer.

The number of edges of a polygon is always equal to the number of vertices: if we choose a direction to go around the polygon, each edge is followed by a vertex, so there is a one-to-one correspondence between vertices and edges.

A polygon is *rectilinear* if all of the angles are right angles.
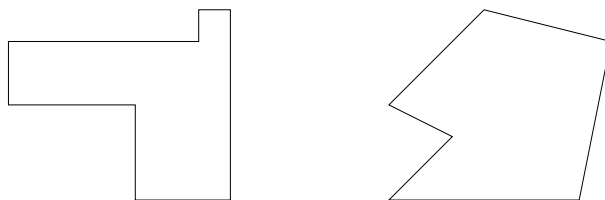
FIGURE 45. Rectilinear (left) and non-rectilinear (right) polygons

4.2. **Regions.**

A *region* in a plane is a set of points that is *two dimensional* in the sense that starting at any point in the region, one can move a small distance in any direction contained in the plane and stay inside the region. Examples of regions include geographic regions such as states, counties as well as yards, playgrounds, and parks. The inside of a polygon is a *polygonal region* while the polygon itself is not a region.

The *boundary* of a region in a plane is its edge, or more technically, the set of points such that *any* circle drawn around the point contains in its interior points both points inside and outside the region. For example, the boundary of a country is its *border*. The boundary of an ocean is its *shore*. The boundary of a river is its *bank*.
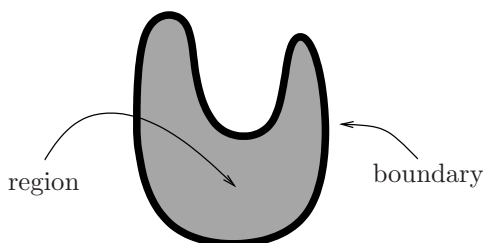
region   boundary

FIGURE 46. A region and its boundary

Polygonal regions can be built easily with commonly available manipulatives such as pattern blocks, Magna-tiles, or paper cutouts. Many engaging activities for middle school students can be done with these manipulatives, as will be discussed in more detail on page 77.

We draw regions by *shading* them. For example, a shaded square indicates the set of points *inside* the square.
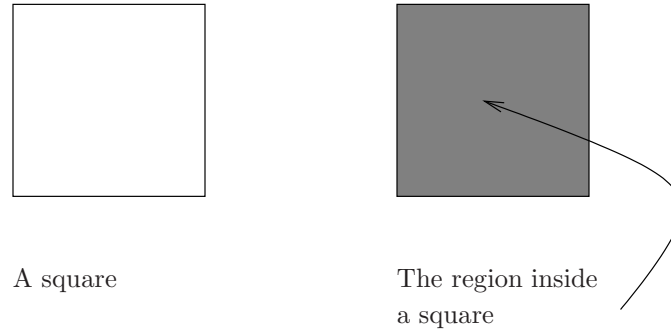
FIGURE 47. A square and a square region

Any polygon divides the plane into two *regions*, the *region inside the polygon* and the *region outside the polygon.*

A region is *convex* if the line segment between *any* two points inside the region stays inside the region. For example, the state of Colorado is convex, but the state of Florida is not convex, because a line segment from Panama City to Miami goes over the water. [12]

FIGURE 48. Florida is a non-convex region

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[12]Retrieved from http://edr.state.fl.us/content/area-profiles/county/index.cfm. Used under fair use guidelines.

Interior and exterior angles

The *interior angle* of a polygon at a vertex is the angle pointing to the inside. Note that an interior angle can be more than 180 degrees, that is, it can be a reflex angle as in Figure 49.

a reflex interior angle
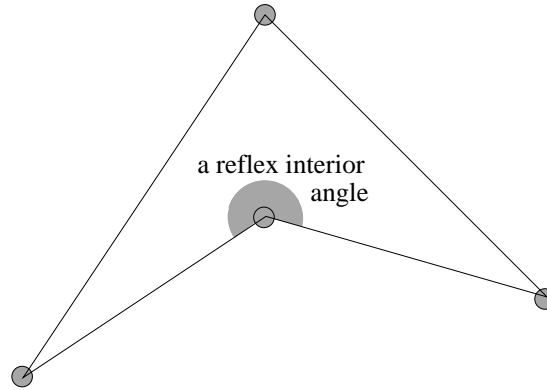
FIGURE 49. A polygon with a reflex interior angle

An *exterior angle* of a polygon is an angle formed by an edge and a line extended from an adjacent edge as in Figure 50.
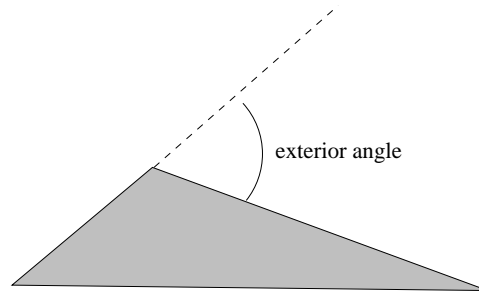
exterior angle

FIGURE 50. An exterior angle

Note that if the polygon is not convex then the internal and external angles may overlap as in Figure 51.
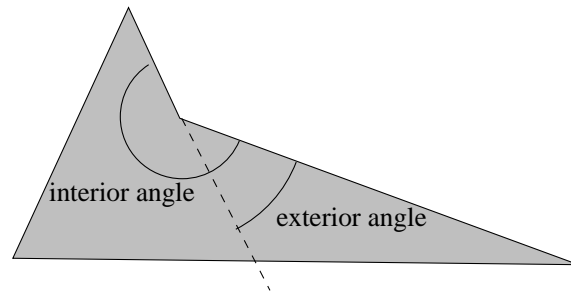
FIGURE 51. An exterior angle of a non-convex polygon

If the polygon is convex, then the sum of the measures of the interior and exterior angles at each vertex is 180°. If the polygon is not convex then this is still true if one takes the angles to be *directed* as on page 35, so that some angles may have negative measure. For example, in Figure 51, if the indicated interior angle has measure 220°, the exterior angle has measure −40° as a directed angle, then the sum of measures of the interior and exterior angles is 180°.

Problem 4.2.1. Compare and contrast the designated shapes. For each pair of shapes, find one similarity and one difference. (Recall that if the shape is shaded then it represents a *region*.)
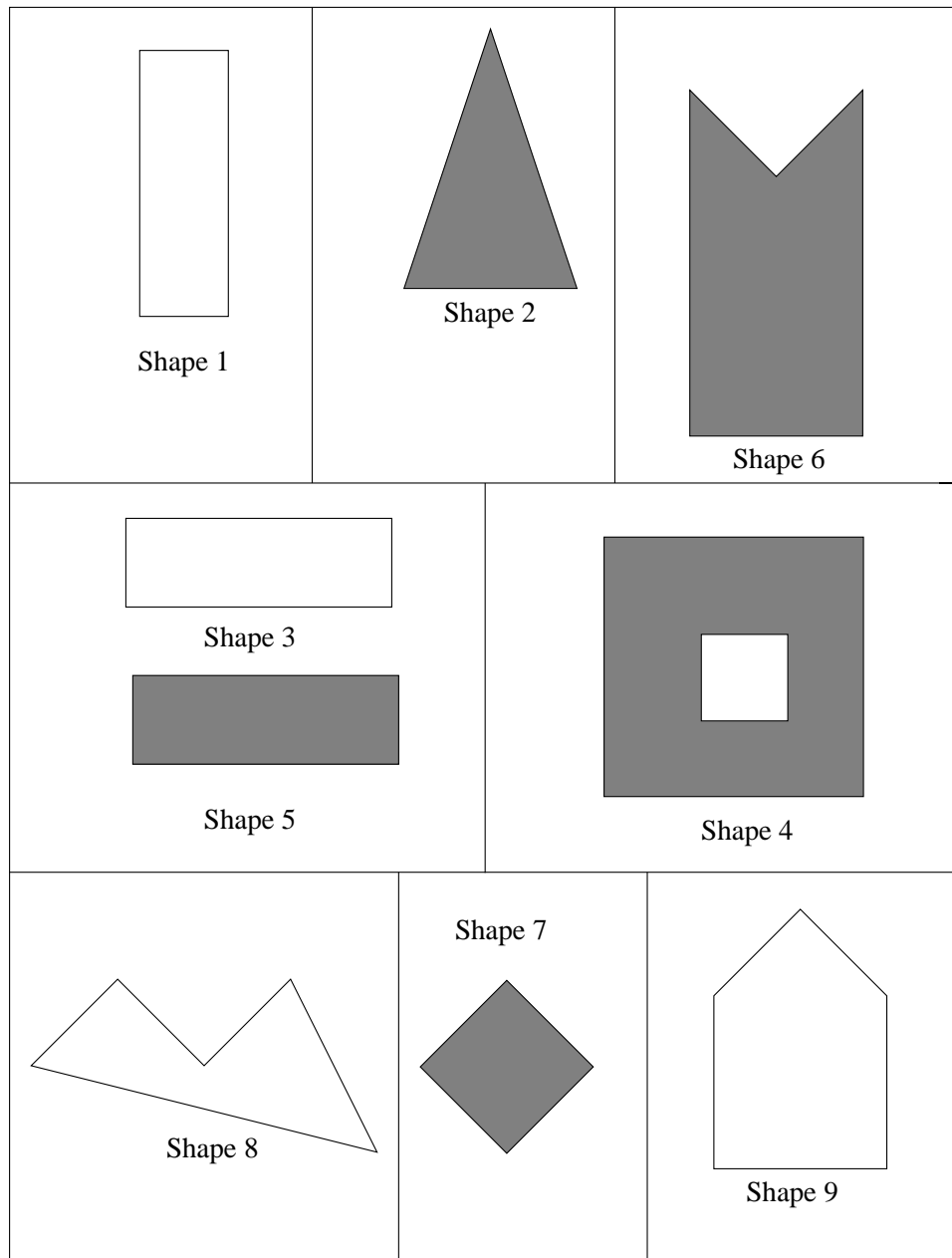
Shape 1

Shape 2

Shape 6

Shape 3

Shape 5

Shape 4

Shape 8

Shape 7

Shape 9

Shape	Shape	Ways shapes are alike	Ways shapes are not alike
Shape 1	Shape 3		
Shape 5	Shape 7		
Shape 7	Shape 9		
Shape 8	Shape 9		
Shape 2	Shape 4		
Shape 6	Shape 4		

Regions and pattern blocks

There are many interesting questions one can ask about pattern blocks and polygonal regions. The open-ended nature of these questions and the hands-on context can provide an engaging environment for students of all levels. In the following problems we will assume that the types of pattern blocks shown in Figure 52 [13] are available.
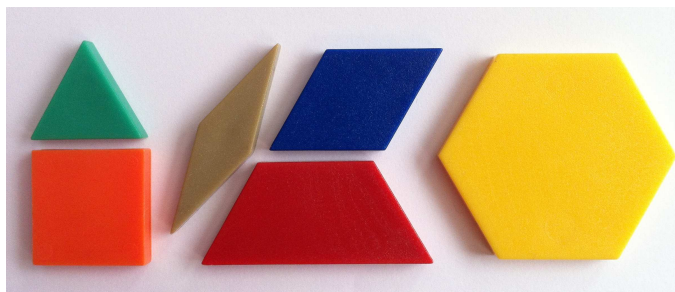
Figure 52. Types of pattern blocks

Alternatively, a free interactive applet allowing the user to play with pattern blocks can be found at the National Library of Virtual Manipulatives nlvm.usu.edu.

Problem 4.2.2. Using only pattern blocks, determine whether you can build each of the following regions. In each case, give a logical explanation of whether or not it is possible to build the region.

  (1) a polygonal region with 7 vertices?
  (2) polygonal region with 8 vertices?
  (3) a polygonal region with two reflex interior angles?
  (4) (Challenge) a convex polygonal region with 8 vertices?
  (5) (Challenge) a convex polygonal region with 7 vertices?

There are many perimeter and area questions that can be asked about polygonal regions created with pattern blocks; we will explore some of these in Chapter 5.

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[13]Photo credit I. Radu.

4.3. **Triangles.**

A *triangle* is a polygon with three edges, that is, a 3-gon.

There are different conventions on whether a triangle is allowed to be *degenerate*, that is, to have colinear vertices. We choose to not allow triangles to be degenerate.

A triangle is *isosceles* if it has (at least) two sides of equal length, *equilateral* if all sides have equal lengths, and *scalene* otherwise.
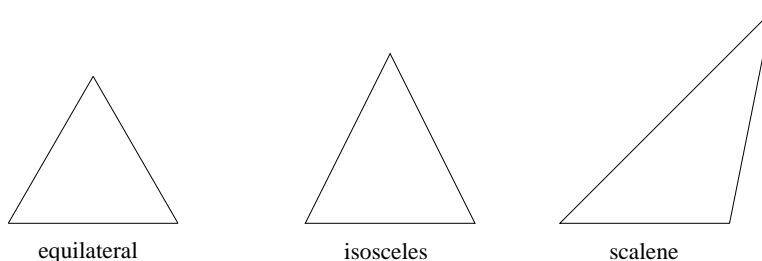
FIGURE 53. Types of triangles

In our conventions, an equilateral triangle is a special kind of isosceles triangle.

A *right triangle* has a right angle.

An *acute triangle* has only acute angles.

An *obtuse triangle* has one obtuse angle.

The Common Core State Standards are unclear about whether students are expected to know the terms *scalene, acute, obtuse*. Many state assessments have tested this vocabulary in the past.

For any right triangle, the *hypotenuse* is the longest side, or equivalently, the side opposite the right angle. (The equivalence of these two definitions follows from the Pythagorean theorem, which is discussed later.)

The *legs* of a right triangle are the two shorter sides, or equivalently, the sides adjacent to the right angle.

Problem 4.3.1. Indicate whether each statement is true or false. Explain your answers.

 (1) Any triangle has at most one right angle.
 (2) Any triangle has at most two acute angles.
 (3) Any isosceles triangle is equilateral.
 (4) Any equilateral triangle is isosceles.
 (5) Any scalene triangle has at most two acute angles.

4.4. **Quadrilaterals.**

The following problem is meant for discussion, to illustrate the wide range of definitions about quadrilaterals within any particular group of people, including children. This is especially true of any group with international backgrounds, since vocabulary for quadrilaterals varies from country to country.

Problem 4.4.1. Answer true/false, based on your understanding of types of quadrilaterals, but without looking up the precise definitions. Explain your reasoning.

(1) Any rectangle with three equal side lengths is a square.
(2) Any rectangle that is also a rhombus is a square.
(3) Any kite with three equal side lengths is a rhombus.
(4) In any rhombus, any two opposite angles are congruent.
(5) In any rhombus, the measures of any two consecutive angles sum to 180 degrees.
(6) In a trapezoid, any two opposite angles have the same measure.
(7) Any trapezoid with three right angles is a rectangle.
(8) Any quadrilateral with three right angles has two pairs of parallel sides.
(9) Every quadrilateral with at least two right angles is a trapezoid.
(10) Every polygon with four right angles is a rectangle.
(11) Every triangle with two congruent angles also has two equal side lengths.
(12) Every polygon has at least one acute angle.

Hint: (9) is tricky.

Vocabulary on quadrilaterals

We will use the following definitions for different types of quadrilaterals.

A *quadrilateral* is a 4-gon, meaning it is a polygon with 4 sides. Some books also call quadrilaterals *quadrangles*, meaning they have 4 angles.

A *square* is a quadrilateral with four congruent sides and all angles right angles.

A *parallelogram* is a quadrilateral with two sets of parallel sides.

A *trapezoid* is a quadrilateral with at least one pair of parallel sides. Some books require a trapezoid to have exactly one pair of parallel sides. In fact, the conventions in various countries are slightly different.

A *rhombus* is a quadrilateral with four congruent sides.

A *rectangle* is a quadrilateral with four right angles.

A *kite* is a quadrilateral with two pairs of equal-length sides such that the equal-length sides share a vertex. Kites are not necessarily convex.
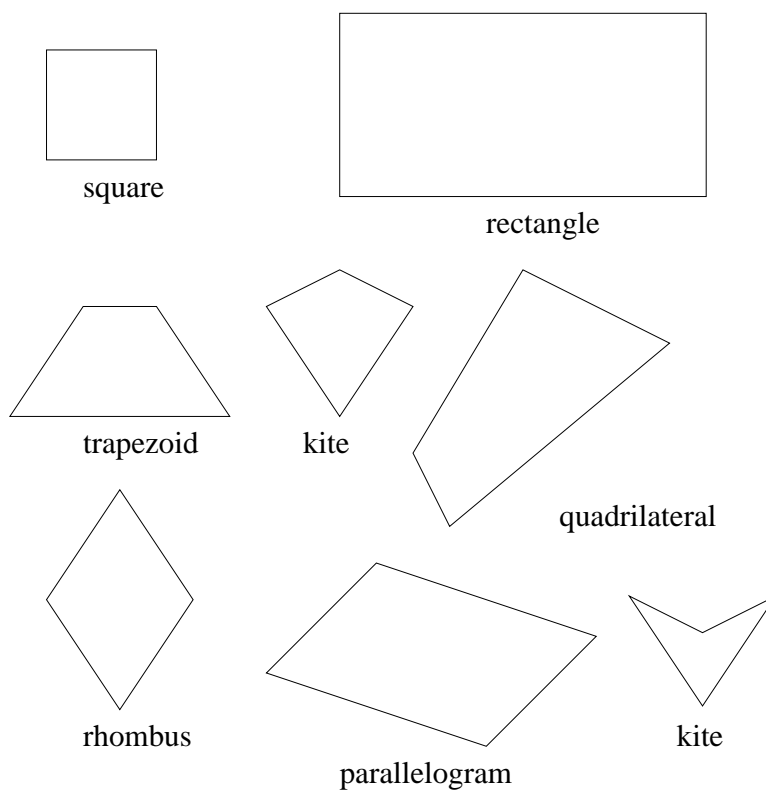
FIGURE 54. Types of quadrilaterals

Note that in the Common Core State Standards, a square is a special kind of rectangle. Continuing this logic, a parallelogram is a special kind of trapezoid, a

square is a special kind of rectangle, a rhombus is a special kind of kite, and these are all special kinds of quadrilaterals.

Other authors have different conventions. For example, for Euclid a trapezoid was anything that was not a rhombus. In mathematics there is no "right answer" on questions of terminology. Only once everyone agrees on conventions can every statement be true or false. In other words, the meaning of the word "trapezoid" can be decided by a vote (it is an issue of vocabulary) but the question of whether $2 + 2 = 4$ is true cannot be decided by a vote (we all agree what it means).

Problem 4.4.2. (Adapted from [MCAS]) Jason drew a closed shape with the following properties: It has exactly four angles; All angles are right angles. Opposite sides are congruent. Opposite sides are parallel.

(1) On graph paper, draw a shape that has the same properties as Jason's shape.
(2) On graph paper, draw a different shape that also has the same properties as Jason's shape.
(3) Jason also drew the trapezoid shown below.
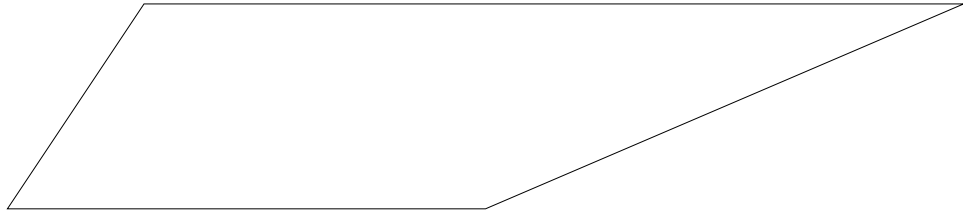
List 3 properties of the trapezoid.

GeoGebra as a manipulative

GeoGebra can be quite helpful in exploring whether universal statements such as those in Problem 4.4.1 are true. Universal statements are those that assert that something holds true for all objects of a certain type. A way to approach a universal statement is to use GeoGebra's point-dragging feature. This allows the user to consider a wide variety of examples quickly, in a dynamic manner, and note whether the property of interest appears to hold in each of those examples. This playful investigation is not meant to replace more formal arguments, but rather to give students an intuitive feel for the relationships among the geometrical objects in the problem; such dynamic exploration can later inspire a more formal argument or suggest a counterexample.

Constructing objects with certain built-in properties in GeoGebra requires a significant amount of geometric reasoning in itself. By "built-in" we are referring to properties that should hold even when certain points of the drawing are being dragged. Additionally, no other properties should be built-in: if you are asked to draw a rhombus, you should not draw a square, as it would have more built-in properties than required and would not provide the most general object with the given properties. For example, to investigate the statement

Any quadrilateral with two pairs of congruent opposite sides is a parallelogram

one needs to construct a quadrilateral with the specified built-in properties (i.e., two pairs of congruent opposite sides) and no additional built-in properties, and then investigate whether it appears to be a parallelogram no matter how much point dragging is done. How can such a quadrilateral be constructed? There are several ways to accomplish this, each requiring the user to think deeply about the nature and properties of basic geometric concepts such as circles, angle and segment bisectors, congruent polygons, and transformations, to name only a few.

Geoboards as manipulatives

A Geoboard is a manipulative that lends itself to a great variety of explorations on the topics of polygons and perimeter and area of polygons. It consists of a physical board with a certain number of nails half driven in it, around which one can wrap plastic rubbers to create representations of polygons. A popular version contains nails in a 5×5 formation, as shown in the picture below. For the remainder of this book we will refer to this type of Geoboard.
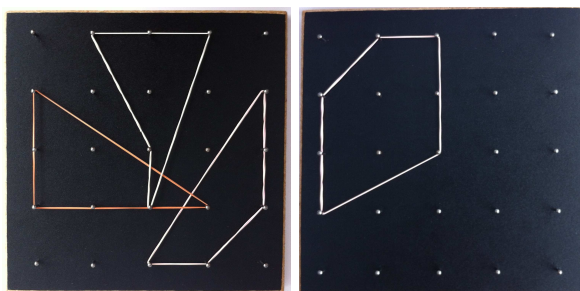
FIGURE 55. Geoboards

A few examples of Geoboard questions/tasks involving polygons are:

Problem 4.4.3.

(1) Make a few right triangles. What is the largest one you can make?
(2) What is the smallest right triangle you can make?
(3) Can you make an isosceles trapezoid with bases of length 3 units and 2 units?
(4) What is the longest rhombus you can make?

These questions can be attempted by students of all levels and at the same time can be taken into more challenging directions by either the instructor or students, as needed. Note the use of the terms "largest", "longest", and "smallest" when referring to polygons; they are not part of the conventional terminology on polygons, which creates an opportunity to ask students what they might mean and negotiate a unique meaning for each of them. Additionally, the questions implicitly invite a discussion of what a "unit" may be in the context of a Geoboard. Could it be 1 inch? Or perhaps the distance between two consecutive nails (along a horizontal or vertical line)? Does the problem solver have a choice, and if so, what should guide that choice?

A free easy-to-use interactive Geoboard can be found at:

Mathematics Learning Center

In Chapter 5 we will continue the discussion on Geoboards by exploring perimeter and area questions.

4.5. **Venn diagrams**[*].

In this chapter we discussed various types of polygons and the relationships among them. For example, a square is a special type of rhombus, etc.. We can use Venn diagrams to represent these relationships in a more visual manner. A *Venn diagram* shows the relationship between different groups. For example, if one group is contained in another, one draws a bigger circle used to represent the larger group around a smaller circle used to represent the smaller group. If two groups have members in common, one draws the two circles intersecting.

Every member of S is also a member of T

S and T have no members in common

S and T have some members in common but one is not contained in the other
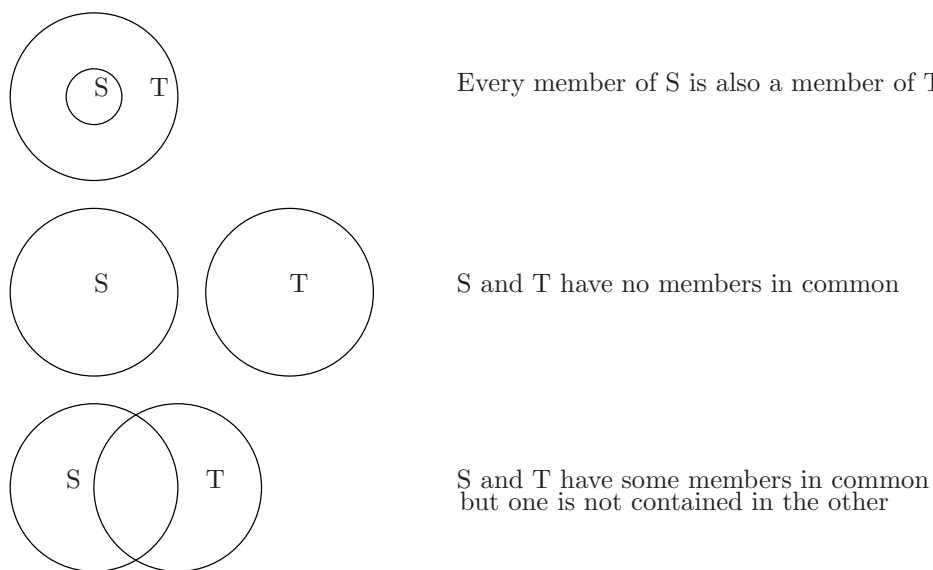
FIGURE 56. Relationships between groups

Here is a sample problem: Draw a Venn diagram for the following sets: 10th grade students, 11th grade students, high school students, female students.

A sample answer is as follows. First note that 10th grade students and 11th grade students do not have any students in common, so the picture looks like this:
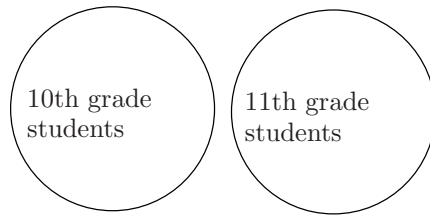
Now both of these groups are contained in high school students:
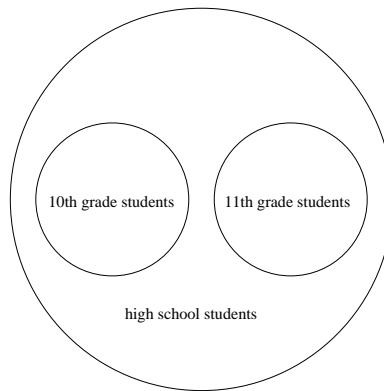
Finally female students intersect all of these groups, so the final picture looks like this:
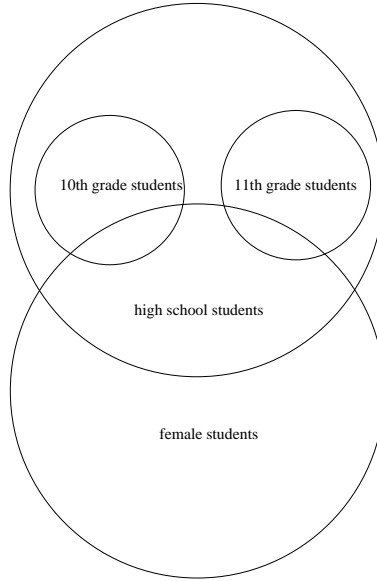
Problem 4.5.1. Draw a Venn diagram for the following sets: 10th grade students, players on the high school football team, players in the high school orchestra, and kindergartners.

Using Venn diagrams, squares, rectangles, rhombi, and parallelograms are related as follows.
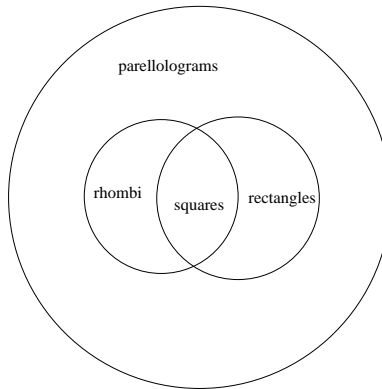
A common confusion is whether an equilateral triangle is isosceles. According to our conventions, the answer is yes: an equilateral triangle has three equal side lengths, so at least two equal side lengths, and so is isosceles. That is, our definition of isosceles triangle is the *inclusive* definition which includes equilaterals, as opposed to the *exclusive* definition which disallows them.

Problem 4.5.2. Draw a Venn Diagram representing the relationship among squares, rectangles, rhombi, and trapezoids.

4.6. **Interior angles of polygons.**

Cutting polygons into triangles

We already saw that the sum of the measures of the interior angles of a triangle is 180 degrees. What about the sum of angles of a quadrilateral? A special case is a rectangle. In this case, there are four right angles, so the sum of the angles is 360 degrees.

Problem 4.6.1. Provide an argument for the fact that any quadrilateral can be cut into two triangles. Deduce a formula for the sum of measures of interior angles of a quadrilateral.

The following theorem is often used to justify the formula for the sum of the interior angles of a polygon.

> Any polygon can be divided into triangles.

A viable argument for convex n-gons can be given as follows. Choose one vertex and connect it to all the other vertices. The result is a division of the polygon into $n - 2$ triangles. (Can you find a general argument for why this is true for any convex polygon?) Alternatively, put an extra point in the middle of the polygon and connect it to the outside vertices. The result is a division of the polygon into n triangles.
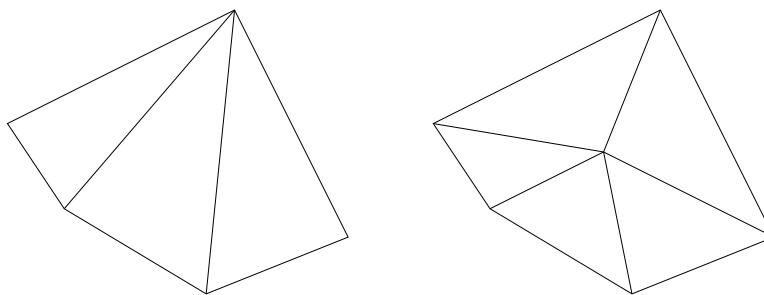
FIGURE 57. Ways of dividing a convex polygon into triangles

Problem 4.6.2. (Optional challenge problem) Show that any polygon can be divided into triangles. (Hint: There are always two vertices that can be joined by a line segment that does not pass outside the polygon. To see this, arrange the polygon so there is a unique "highest" vertex. Consider the triangle formed by that vertex and the two adjacent vertices. Either this triangle contains another vertex of the polygon, or it does not. In the first case, connect the two vertices adjacent to the highest vertex. In the second case, find the highest vertex inside the triangle and join it to the highest vertex in the polygon. Use this fact to show that any polygon can be divided into triangles.)

The sum of interior angles of a polygon

Problem 4.6.3. Cut the pentagon below (or a copy of it on paper) into triangles. Without measuring or using any formulas, can you relate the sum of the interior angles of the pentagon to the sum of the interior angles of the triangles? Explain your answer.
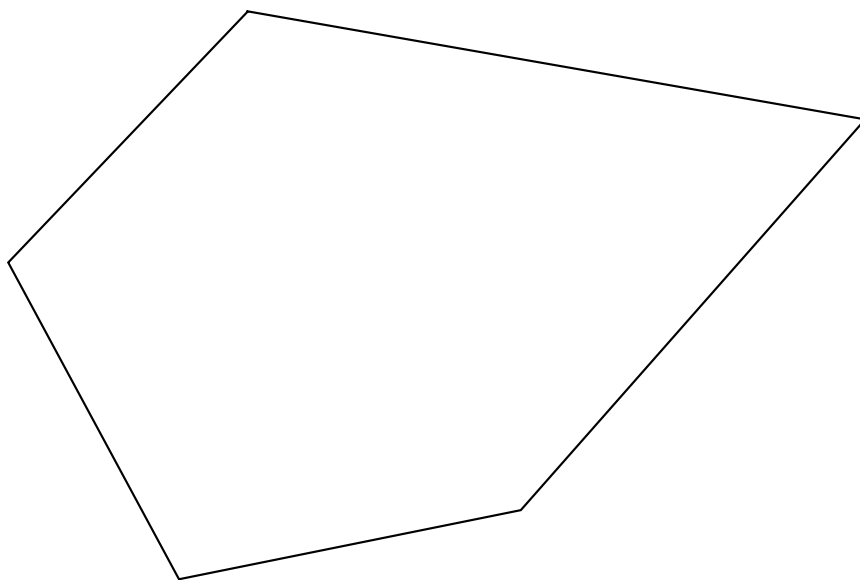
FIGURE 58. A convex polygon

The Problem 4.6.3 motivates the following theorem:

For any integer $n > 2$, the sum of the measures of the interior angles of an n-gon is $180(n - 2)$ degrees.

Students should be able to *construct a viable argument* for a such statement. One such argument reduces the problem to a *simpler form*, by dividing the n-gon into $n - 2$ triangles. Note that the sum of the angles in any triangle is 180 degrees, and the sum of the interior angles of the n-gon is the sum of the interior angles of the triangles. Hence, the sum of the measures of the interior angles is the number of triangles times 180 degrees, that is, $(n - 2)180°$.

A *regular polygon* is one for which each side has the same length and each interior angle has the same measure. For example, a regular quadrilateral is a *square*. Since a regular n-gon has n interior angles with the same measure, each angle measure is equal to $180(n - 2)/n$ degrees.

The sum of the exterior angles of a polygon

What can one say about the sum of the measures of the exterior angles of a polygon?

Problem 4.6.4. Draw a polygon and its exterior angles on a piece of paper. (An example is shown below.) Using scissors, cut out the exterior angles and arrange them so that they have a common vertex. What is the sum of the measures of these exterior angles?
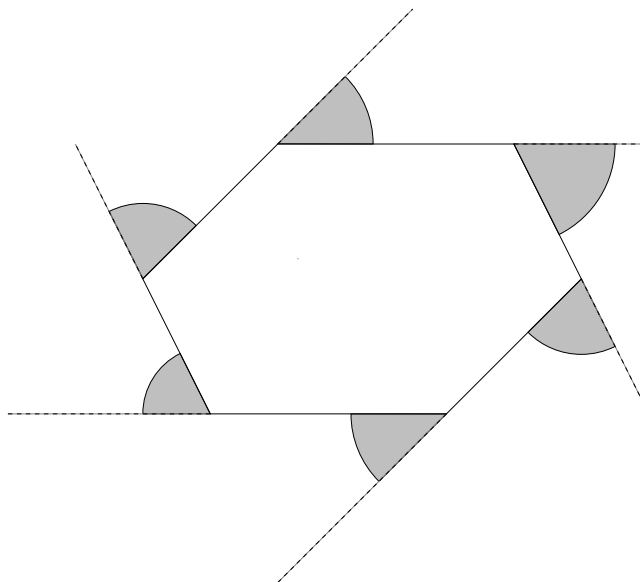
The activity in Problem 4.6.4 supports the following conjecture:

> For any integer $n > 2$, the sum of the measures of the exterior angles of a n-gon is 360 degrees.

Here is a *viable argument* for this fact about exterior angles. Think of the n-gon as a clockwise path. The total amount of turning is the sum of the measures of the exterior angles. Since the total amount of turning once around the path is 360 degrees, starting at and returning to the same point, the sum of the measures of the exterior angles is also 360 degrees.

If you have internet access, you can see an illustration of this in the interactive GeoGebra applet available by clicking on the following link:

<div align="center">

Exterior angles of a polygon

</div>

If the polygon is not convex, then the sum-of-angles formula above is still true if one takes the angles to be *directed* as on page 35, so that the exterior angles may have negative measure.

We can use the fact that the sum of the exterior angles of a polygon is 360 degrees to deduce the formula for the sum of the measures of the interior angles. Note that

each interior angle is equal to 180 degrees minus the corresponding exterior angle. Therefore, the sum of the interior angles is the sum of n 180-degree angles minus the sum of all exterior angles. So, the sum of the interior angles is equal to $n(180) - 360$ degrees, which is equal to $(n - 2)180$ degrees.

4.7. **Workshops.**

Workshop 4.7.1.

(1) (a) Give precise directions for someone who wants to travel along the pentagon shown below (including angle measures and distances).
(b) Find the sum of the measures of the interior angles by measuring.
(c) Explain how you can find the sum of interior angles without measuring.
(d) Explain how you can find the sum of exterior angles without measuring.
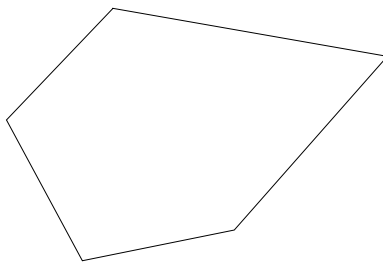
(2) (a) Give precise directions for someone who wants to travel along a pentagram shaped path as shown below, using only four turns.
(b) Is this path a polygon? Explain.
(c) Find the sum of the interior angles at the vertices (turning points, which are the shaded points) by measuring. Find the sum of the exterior angles at the vertices (including a final fifth turn necessary come back to the starting direction) by measuring.
(d) Can you find the sum of the measures of the five exterior angles without measuring? Explain why or why not.
(e) Can you find the sum of the measures of the interior angles without measuring? Explain why or why not.
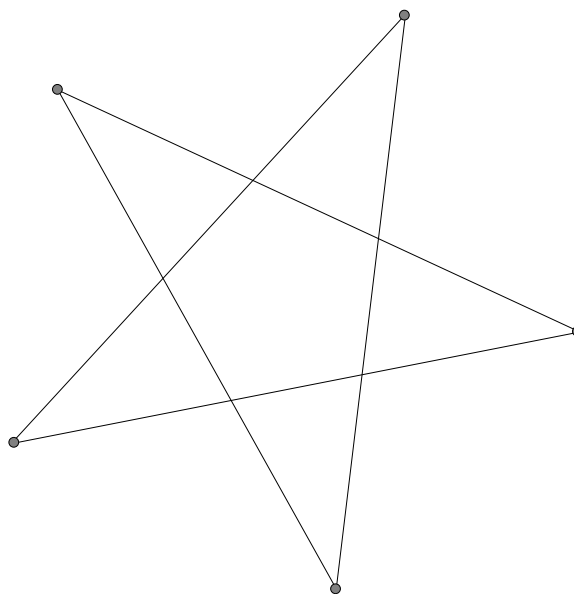
Workshop 4.7.2.

(1) For each labelled angle, use your protractor to bisect the angle. Then, use your ruler to bisect the side opposite the angle. In each case, say whether the ray bisecting the angle bisects the opposite side as well.
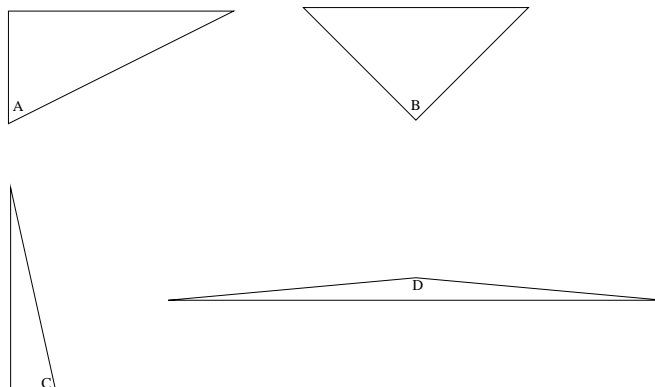
  Can you give a criterion characterizing the triangles that have an angle bisector that bisects the opposite side? Explain.

(2) Suppose that a kite $ABCD$ has equal side lengths $AB = BC$ and $CD = DA$.

  (a) Does $\overline{AC}$ bisect $\overline{BD}$ (for any such kite)? Why or why not? If not, give a counterexample. If so, give a logical argument which works for any kite. (You might want to consider, for example, what happens when you fold the kite along $\overline{BD}$; do the two halves exactly match? Why or why not.)

  (b) Does $\overline{BD}$ bisect $\overline{AC}$ (for any such kite)? If not, give a counterexample. If so, give a logical argument which works for any kite.

(3) Suppose that $ABCD$ is a rhombus.

  (a) Does $\overline{AC}$ bisect $\overline{BD}$? If not, give a counterexample. If so, give a logical argument which works for any rhombus. Are these segments perpendicular? Explain.

  (b) Does $\overline{BD}$ bisect $\overline{AC}$? If not, give a counterexample. If so, give a logical argument which works for any rhombus.

(4) Suppose that $ABCD$ is a rectangle.

  (a) Does $\overline{AC}$ bisect $\overline{BD}$? Why or why not? If so, give a logical argument which works for any rectangle.

  (b) Does $\overline{BD}$ bisect $\overline{AC}$? Why or why not? If so, give a logical argument which works for any rectangle.

  (c) Are the segments $\overline{AC}$ and $\overline{BD}$ congruent? Why or why not? If so, give a logical argument which works for any rectangle.

If your answer was "yes" for any of the "why" or "why not" questions above, make sure that you give a logical explanation which works for all of the given shapes.

Workshop 4.7.3.

The following workshop uses pattern blocks, which also can be found online at nlvm.usu.edu.

(1) Using only pattern blocks, can you build a polygonal region whose boundary has 11 vertices? 14 vertices? 18 vertices? If the answer is yes, show the shape.

(2) Is there any number of vertices for which one cannot build an associated polygonal region using only pattern blocks? Explain why or why not.

(3) Give an approximation for the interior angle measures of each type of block. Explain how you arrived at the approximation.

(4) Make a polygonal region with an interior angle of 330 degrees and an exterior angle of 30 degrees. Is it a convex region? If yes, explain why. If not, how many additional pattern blocks do you need to make it convex? (Possibly by changing the measures of the interior and exterior angles.) How many vertices does the new polygonal region have?

(5) Using only pattern blocks, can you make a polygonal region with an internal angle measuring more than 330 degrees?

Workshop 4.7.4.

For the following problem you may use a Geoboard or the online manipulative at

Mathematics Learning Center.

Your answer should explain how you interpret the informal terms "tallest" and "shortest".

(1) Make a few isosceles triangles. What is the tallest one you can make? Please be sure to consider triangles with tilted bases.

(2) What is the shortest isosceles triangle you can make? Please be sure to consider triangles with tilted bases.

(3) How many non-congruent isosceles triangles can you create on a Geoboard? Explain how you know that your triangles are isosceles.

(4) How many non-congruent squares can you make on a Geoboard? Be sure to consider "tilted squares". Explain how you know that the squares you constructed are isosceles.

(5) (Challenge) Can you make an equilateral triangle on a Geoboard? (Hint: You might want to consider slopes of the edges, or areas of the triangles that are possible to construct on a Geoboard.)

4.8. **Assessments and sample student work.**

The following is a sample assessment problem from fourth grade geometry.

Problem 4.8.1. (from [MCAS], grade four) For this question, you will draw three shapes on the grid below.

  (1) On a grid, draw a parallelogram. Label your parallelogram A.
  (2) On a grid, draw a triangle that has two sides that are perpendicular to each other. Label your triangle B. Put an X on each of the sides that are perpendicular. Explain how you know the sides you labeled X are perpendicular.
  (3) On a grid, draw a shape that has at least one pair of parallel sides, at least one pair of perpendicular sides, and only two right angles. Label your shape C.
  (4) What is the mathematical name of the shape you drew in part **??**?

The answer on the left below received one point out of four, while the answer on the right received two points.
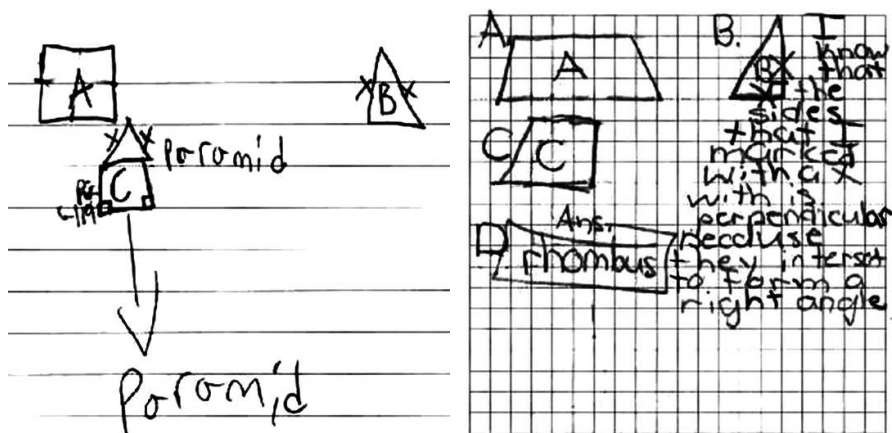
One interesting aspect of this problem is the existence of multiple correct answers to part (3). One correct answer is a right trapezoid. The reader is encouraged to find additional correct answers to this part.

5. Perimeter and area

There are several notions of how big a region is. One measure is the length of the boundary, which is called the *perimeter* of the region. Another measure of the bigness of a region is called the *area*. We will explain these concepts below.

The objectives of this chapter are: (i) to have a better understanding of the concepts of perimeter and area, especially of the fundamental properties governing the concept of area, and (ii) to use these properties to understand how the area formulas for rectangles, parallelograms, triangles, and trapezoids came about.

5.1. **Perimeter.**

Recall from the discussion around Figure 46 that the *boundary* of a region is its edge. The notion of a boundary is an idealized concept in which the boundary has no thickness, just as a line has no thickness. In real life, boundaries often have width; for example, the beach is the boundary between the ocean and land.

The terminology is a little vague as to whether the boundary is part of the region or not. For example, is the square part of the square region it encloses? We will formulate questions to avoid this problem. (The technical term for a region which contains its boundary is a *closed region*, while a region that does not contain its boundary is an *open region*.)

Some examples of regions and their boundaries are:

the perimeter of a playground is the length of fence that is needed to go around the boundary;

the perimeter of an ocean is the length of its shore;

the perimeter of a country is the length of its border.

The *perimeter of a region* is the length of its boundary. In English, perimeter is sometimes used as a synonym for boundary, especially for camps, as in the sentence *the soldiers guarded the perimeter of the camp.*

Perimeters of polygons

When we talk about perimeters, it is always in the context of regions. This makes it confusing when we talk about perimeters of polygons, since we really mean the perimeter of the *region inside* the polygon.

Here is an example involving rectangles. The perimeter of a rectangle with length l and height h really means the perimeter of the *region inside* the rectangle. Since there are two sides of length l and two sides of length h, the perimeter of the rectangle is $2l + 2h$.

To find the perimeter of a polygonal region, measure the length of each portion of the boundary and add. If the boundary of a region is a collection of segments, the

measuring can be done with a ruler. For curved regions, one can sometimes use a piece of string.

The following are past assessment problems involving perimeter.

Problem 5.1.1. (1) (Adapted from [MI]) Find the perimeter of the rectangle shown, in centimeters, by measuring.

(2) (Adapted from [NJ]) Bob has a rectangular garden that is 4 feet wide and 12 feet long. Find the perimeter.
(3) (Adapted from [CST]) An isosceles triangle has two sides with length y and one side with length $y/2$. What is its perimeter?
(4) (Adapted from [CST]) A rectangle with one side length equal to 15 inches has perimeter p inches. Which equation could be used to find the width of the rectangle?
   (a) $p = 15 + w/2$.
   (b) $p = 15 - w$.
   (c) $p = 30 + 2w$.
   (d) $p = 30 - 2w$.

Regions with no well-defined perimeter

Some regions are so complicated that their boundary does not have a well-defined length, that is, their perimeter is not easily defined. Figure 59 shows a *Mandelbrot set*. In real life, something like a coastline is so complicated that it does not have a well-defined length.[14]
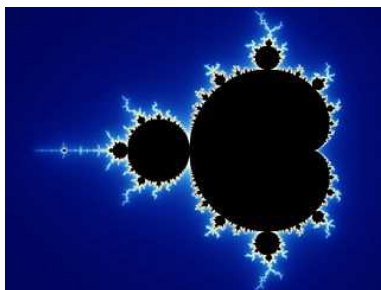

FIGURE 59. A region with a complicated boundary

The Common Core State Standards do not seem to have an explanation of perimeter of a region as the length of the boundary of a region. However, students are expected to understand perimeter.

Perimeters of regions with disconnected boundaries

The boundary of a region can have different parts not connected to each other, in which case the perimeter is the sum of the lengths of the parts. For example, suppose that a farm in the shape of a square with side lengths 2 mi is located in a forest, which is the shape of a square with side lengths 10 mi as in Figure 60.
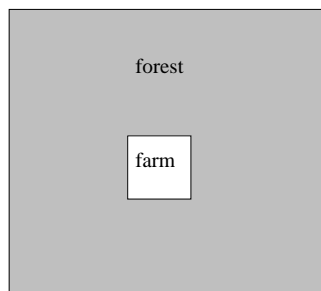

FIGURE 60. A region with a disconnected boundary

The boundary of the forest has two parts: the outer part with length 40 miles, and the boundary around the farm, which has length 8 miles. The perimeter of the forest is the total length of the boundary, that is, 48 mi.

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[14]Created by Wolfgang Beyer with the program Ultra Fractal 3. Used under the Creative Commons License.

Geoboard activities on perimeter

Geoboards provide a fun, engaging environment for exploring perimeter (see Chapter 4.4 for a description of a Geoboard). Questions can range from routine to challenging. A few non-trivial questions are:

Problem 5.1.2. For the following problem you may use a Geoboard or the online manipulative at

<div align="center">

Mathematics Learning Center.

</div>

(1) Approximate (without measuring) the perimeter of the polygon below.
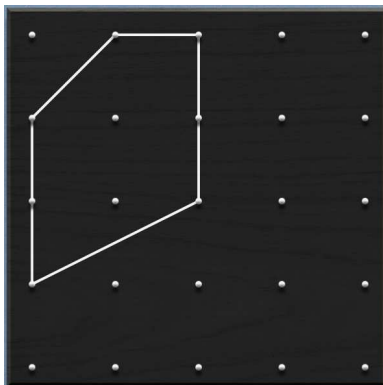
(2) What is the greatest possible perimeter for a Geoboard trapezoid? Explain your answer and give an approximation for its perimeter.
(3) Make a polygon of greater perimeter than the trapezoid from the previous question. How do you know it is greater? Use any tools you deem necessary.

These types of questions can lead to open discussion on: what the unit should be if one is not specified by the problem; how the distance between two consecutive "diagonal" nails compares to that between two consecutive nails along a horizontal or vertical line; and how to use available tools such as string or pencils to compare these two distances if the students are not familiar with the Pythagorean theorem.

The notion of perimeter can also be explored with pattern blocks, as in the Problem below.

Problem 5.1.3. If the side of the square block is 1 unit in length, what is the perimeter of the polygonal region in the Figure below?
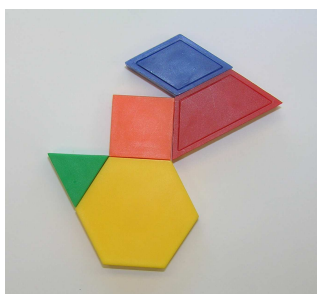
5.2. **Areas of regions.**

Children often divide themselves into groups in order to play games. Each game often takes place in some region of the playground. The number of children that can fit into each region is closely related to the *area* of each region. A *unit square* (also called a *square unit*) is a square with side lengths equal to one unit. The area of a region is *defined* as follows:

> The area of a region is the number of unit squares (or parts of unit squares) needed to cover the region without overlap. Conventionally, we add the phrase *square units* (or *square inches*, *square meters*) after the number to indicate what type of area unit the number refers to.

"Without overlap" means that the insides of the unit squares should not overlap; the unit squares are still allowed to overlap at vertices and edges.

What this definition means practically is that to find the area of a region using the definition of area, one can cover the region with non-overlapping square units (or parts of square units) and count them. For many regions, there may be no good way of breaking up the region into fractions of unit squares, and a person trying to find an area may have to resort to approximation techniques; that is, approximating the region by one that can be broken up into fractions of unit squares.

Activities involving square manipulatives can be used to illustrate the meaning of area for children. For example, Figure 61 shows a region made of seven squares.[15]
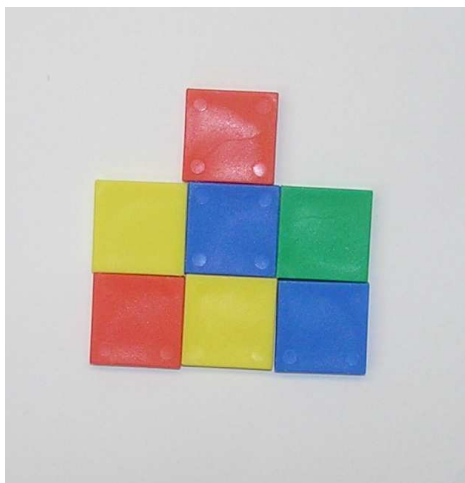

Figure 61. A polygonal region built from manipulatives

If each square is a unit square, then the region has area seven square units. Children can be asked to find several different polygonal regions with the same (given) area. An issue that arises in this activity is whether the corners of the squares need

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[15]Photo credit I. Radu

to match up. In other words, does sliding one of the squares slightly change the area? This is related to the Moving Principle for Area discussed on page 103. This activity provides a good context for discussing the importance of being explicit about additional assumptions one may be making about a given problem if certain aspects of it are not clear, and the fact that different assumptions may lead to different answers.

Area can be explored using pattern blocks involving shapes other than squares, as in the following problem.

Problem 5.2.1. Assuming that the area of the triangle in the figure below is 1 square unit, find an approximation for the area of the region in the figure. [16]
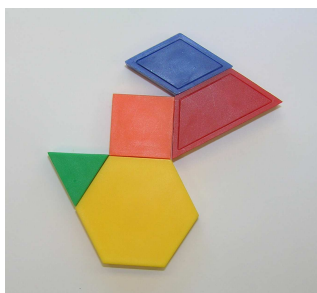
Sometimes the area of a region cannot be expressed as an integer or even fractional numbers of the chosen type of square unit. In Figure 62, the region on the left can be covered by three unit square inches, so the area is 3 square inches. In Figure 62, the region on the right can be covered by one square inch and two rectangles each with side lengths $\sqrt{2} - 1$ inches and 1 inch. So the total area is

$$1 \text{ in}^2 + (\sqrt{2} - 1) \text{ in} \times 1 \text{ in} + 1 \text{ in} \times (\sqrt{2} - 1) \text{ in} = (2\sqrt{2} - 1) \text{ in}^2.$$
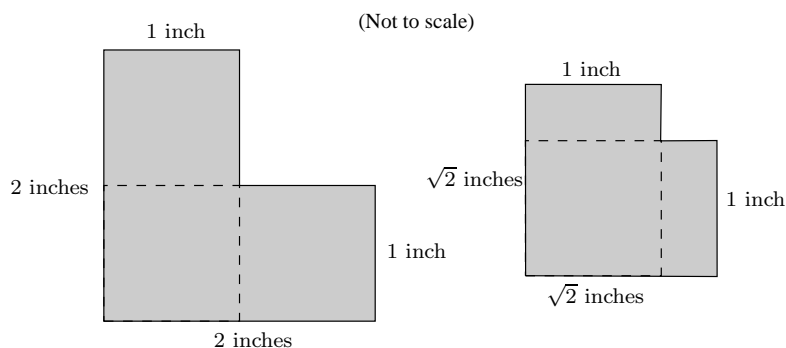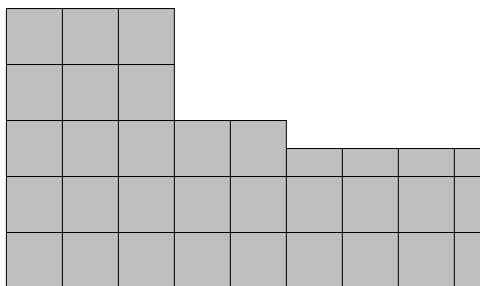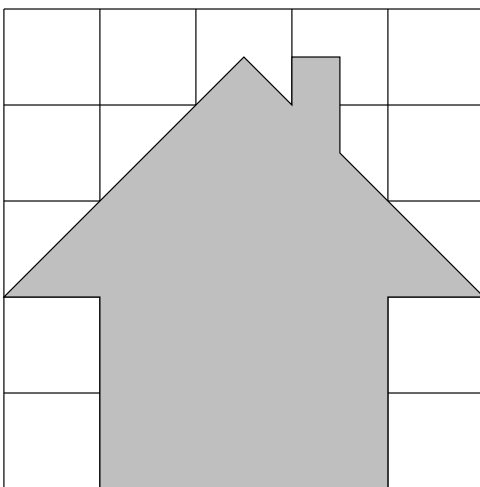

FIGURE 62. Two regions

Problem 5.2.2. Find the area of the following region.

Here is a sample answer: The given region is covered by 27 full units, 5 half units, and 1 quarter unit, for a total of 29.75 square units.

Problem 5.2.3. (Adapted from [NJ]) Find the area of the region below, where each little square is 1/4 of a square inch.

Note that in the case of published assessment problems such as this one, we have deliberately avoided changing the wording or adding clarification; the wording of this question and others like it can be an interesting topic of discussion. An obvious question is whether the test writers mean the students to round off the lengths to the nearest half-centimeter or not, since the vertices do not seem to fit well with the lattice points.

Problem 5.2.4. Using only pattern blocks (and assuming that the side of the square is 1 unit):

(1) Make a polygon of perimeter 8 units
(2) Can you make a polygon of perimeter 8 units whose area is greater than that of the polygon you created for part (1)?
(3) Can you make a polygon of perimeter 8 units whose area is smaller than that of the polygon you created for part (1)?

Properties of Area

Often a playground will be divided into different regions devoted to different games. The area of the playground is the sum of the areas of the regions for each game. This is an example of the following principle.

Additivity principle for area: If two regions do not overlap (except at the boundaries), then the area of the combined region is the sum of the areas of the two regions.
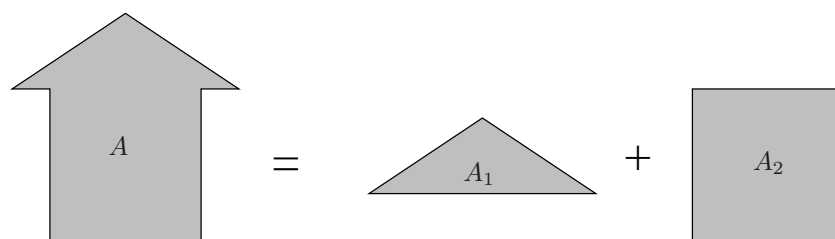
FIGURE 63. Additivity principle for area

There is also a slightly different principle for the area of a *difference* of two regions:

Difference principle for area: If one region is inside another region, then the area of the region outside the first but inside the second of the two original regions is the difference of the areas.

FIGURE 64. Difference principle for area

Moving principle for area: If one region can be moved to perfectly overlap with another without stretching, then the areas of the two regions are equal.

5.3. Areas of rectangles.

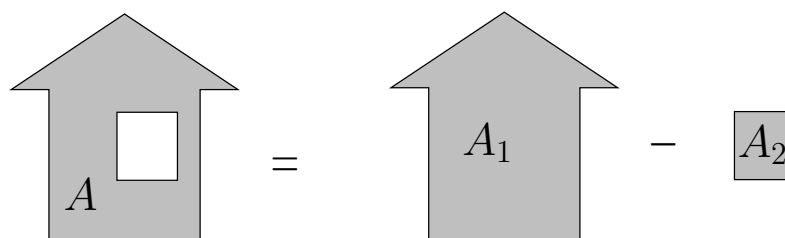Children should understand the meaning of quantities in geometry, not just how to compute them. This is especially true for areas, where an overemphasis on "plug-and-chug" formulas can lead to an inability to investigate the area of a region of a shape for which there is no formula.

Problem 5.3.1.

For this problem, assume you do not know the formula for the area of a rectangle.

(1) Find the area of the rectangle below in several ways.
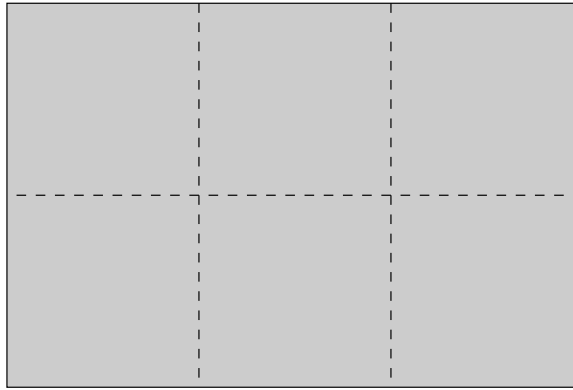
1in

(2) Explain in several ways why the area of the rectangle below is $3\frac{1}{9}$ square inches, assuming that the area of each big square is 1 square inch. On what assumptions are you basing your answer?

Areas of rectangles

A standard shape for playgrounds, playing fields, rooms etc. is a rectangle. The area of a rectangle with *integer side lengths* can be found using the *additivity principle* for area: A rectangle with base b and height h can be broken up into $b \times h$ unit squares and so has area

$$A_{\text{rect}} = b \times h.$$

The above statement is not the *definition* of area, but something that follows from the definition. We also sometimes refer to the dimensions of a rectangle as *length* and *width* instead of base and height, in which case the area is the product of the length and width.

Here is a sample problem that uses the formula: If a playground is a rectangle of width 30 feet and length 40 feet, what is the area of the playground?

$$30 \text{ feet} \times 40 \text{ feet} = (30 \times 40) \times (\text{ feet} \times \text{ feet}) = 1200 \text{ feet}^2.$$

Note the correct use of units.

The area formula for a rectangle above holds *even if the side lengths are not integers*. To see this we consider a sequence of cases. First, suppose that the base b and height h are fractions of the form $b = 1/n$ of a unit of length, and $h = 1/m$ of a unit of length where n and m are whole numbers. We can make a unit square by taking $n \times m$ copies of the original rectangle and arranging them in m rows and n columns. The original rectangle has $1/nm$-th of the area of the unit square, that is, $1/nm$ square units. Note that

$$\frac{1}{nm} = \frac{1}{n} \times \frac{1}{m} = b \times h$$

units of area. So the formula holds in the particular case of the base and height being unit fractions of one unit of length. See Figure 65 for an illustration of an example. An exploration of why the formula holds for non-unit fractions is given in Problem 10.5.2.
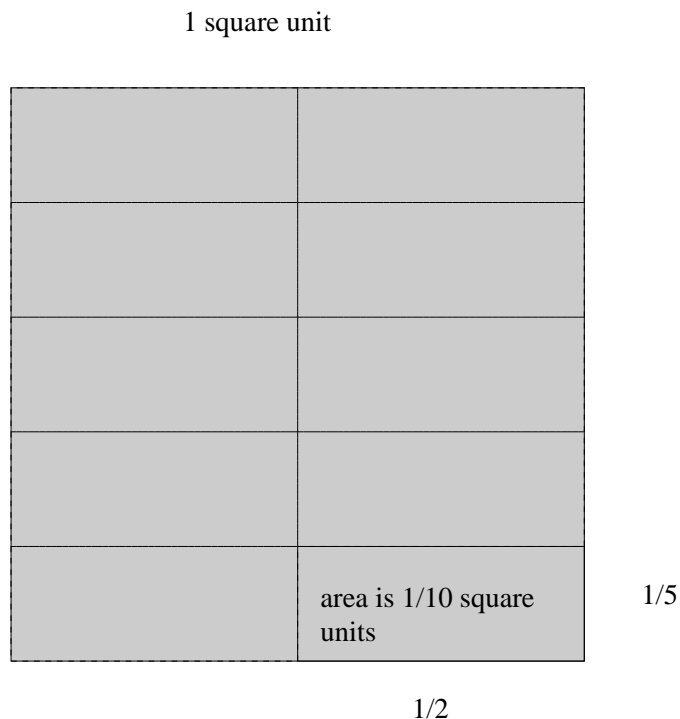
1 square unit

area is 1/10 square units

1/5

1/2

FIGURE 65. Explaining the area formula in the case of fractional units

In the case that the base b and height h are possibly irrational, we approximate b and h by rational numbers, say b' and h'. We can apply the area formula for this rectangle since it has rational lengths. So the area of rectangle with base b' and height h' is $b' \times h'$. By taking b' and h' closer and closer to b and h, respectively, one gets that the area of the original rectangle is $b \times h$.

The following problem explores the formula further in the case of rational side lengths, assuming that the formula is known for integer side lengths and unit fraction side lengths.

Problem 5.3.2. Given a rectangle of side lengths 3/4 units and 2/5 units, find its area in two different ways as follows:

(1) by breaking it up into smaller rectangles;
(2) by forming a larger rectangle using a number of copies of the original.

How does this support the formula for area of a rectangle in the case of rational side lengths?

Problem 5.3.3. (Adapted from [PARCC]) Small squares with edge lengths of 1/4 inch will be packed into a rectangle with length $4\frac{1}{2}$ in and width $3\frac{3}{4}$ in. How many small squares are needed to completely fill the rectangle?

Problem 5.3.4. If a playground is a rectangle of width 30 feet and length 40 feet, and each child needs roughly 25 square feet of play space, then how many children roughly fit on the playground?

Problem 5.3.5.

(1) (Adapted from [NJPEMSM]) The Johnsons are planning to build a 5-foot wide brick walkway around their rectangular garden, which is 20 feet wide and 30 feet long. Find the area of the walkway in two ways.
(2) Find the perimeter of the walkway from (a). (The boundary of the region has two parts.)
(3) (Adapted from [NJ]) : Mrs. Rodriguez is planning to build a new rectangular brick patio. She plans on using bricks that are 1 foot wide by 2 feet long. She would like the new patio to be 12 feet by 14 feet. She expects that no bricks will need to be cut. Mrs. Rodriguez uses only whole 1-foot-by-2-foot bricks. State how many bricks should be used. Justify your answer.
(4) Determine the total area and the perimeter of the new patio from part (3). Show all your work.

The following problem concerns the relationship between area and perimeter.

**Problem 5.3.6.** A rectangle with height 4 and base 6 has area 24 and perimeter 20.

(i) Find a rectangle, also with perimeter 20, that has more area than the given rectangle.

(ii) Find a rectangle, also with perimeter 20, that has less area than the given rectangle.

Problem 5.3.7.

(1) Use the moving and additivity principles about area to determine the area, in square inches, of the shaded region in the figure below. The shape is a 2-inch by 2-inch square, with a square placed diagonally inside, removed from the middle. The lengths of the diagonals of the small square are 1 inch; note that these are oriented horizontally and vertically. In determining the area of the shape, use no formulas other than the one for area of rectangles. Explain your method clearly.

2 inches

1 inch
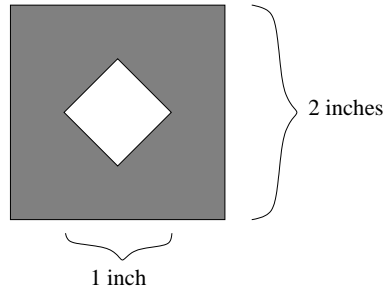
(2) Find the area of the following regions without using the formula for the area of a triangle. Each grid square is a unit square. (Hint: how much area do the "white regions" have?)
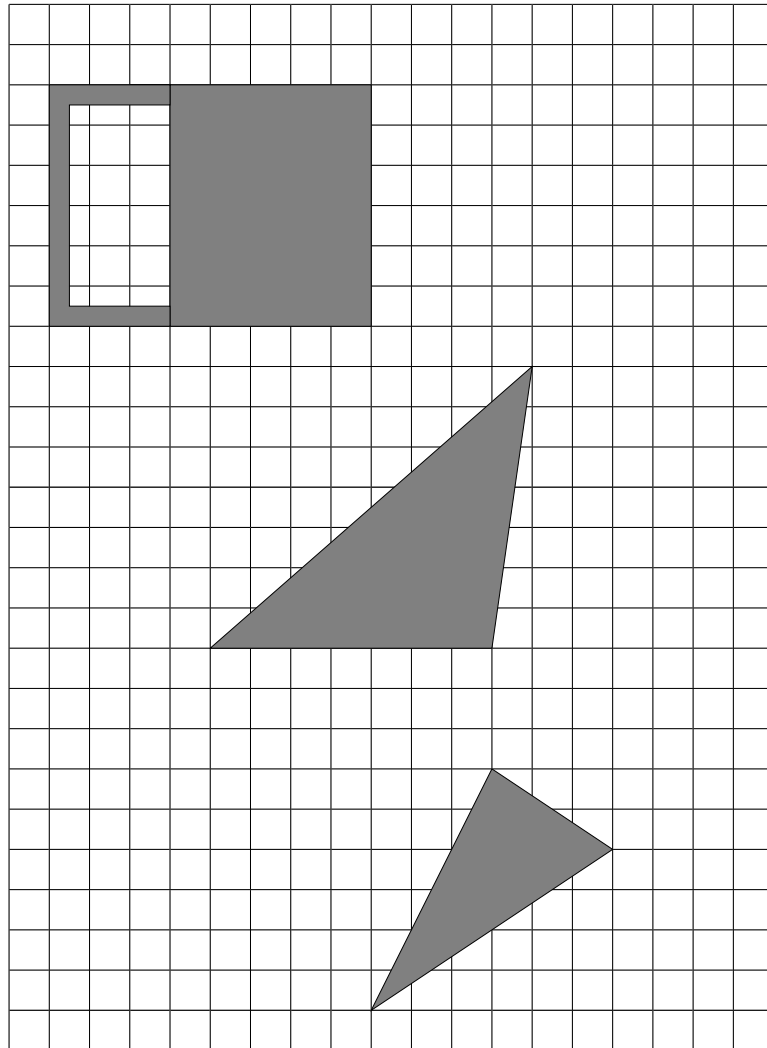
(3) Find the areas of the figures below by fitting each in a big rectangle and subtracting off areas of simple shapes. Each grid square is a unit square.
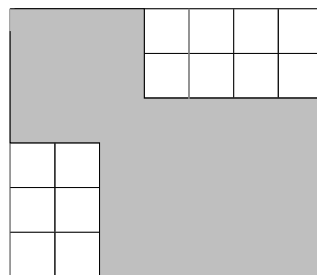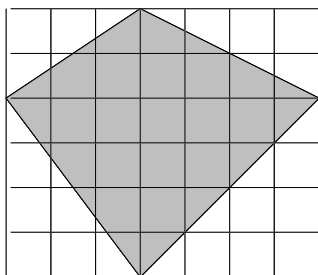
5.4. **Areas of parallelograms.**

Our strategy for finding the areas of shapes more complicated than rectangles is to reduce to the case of a rectangle by using the moving and additivity principles of area. We start with parallelograms.

Problem 5.4.1. Indicate how you might cut and re-arrange each parallelogram to make it into a rectangle. In each case, how do you know that the new quadrilateral is a rectangle?
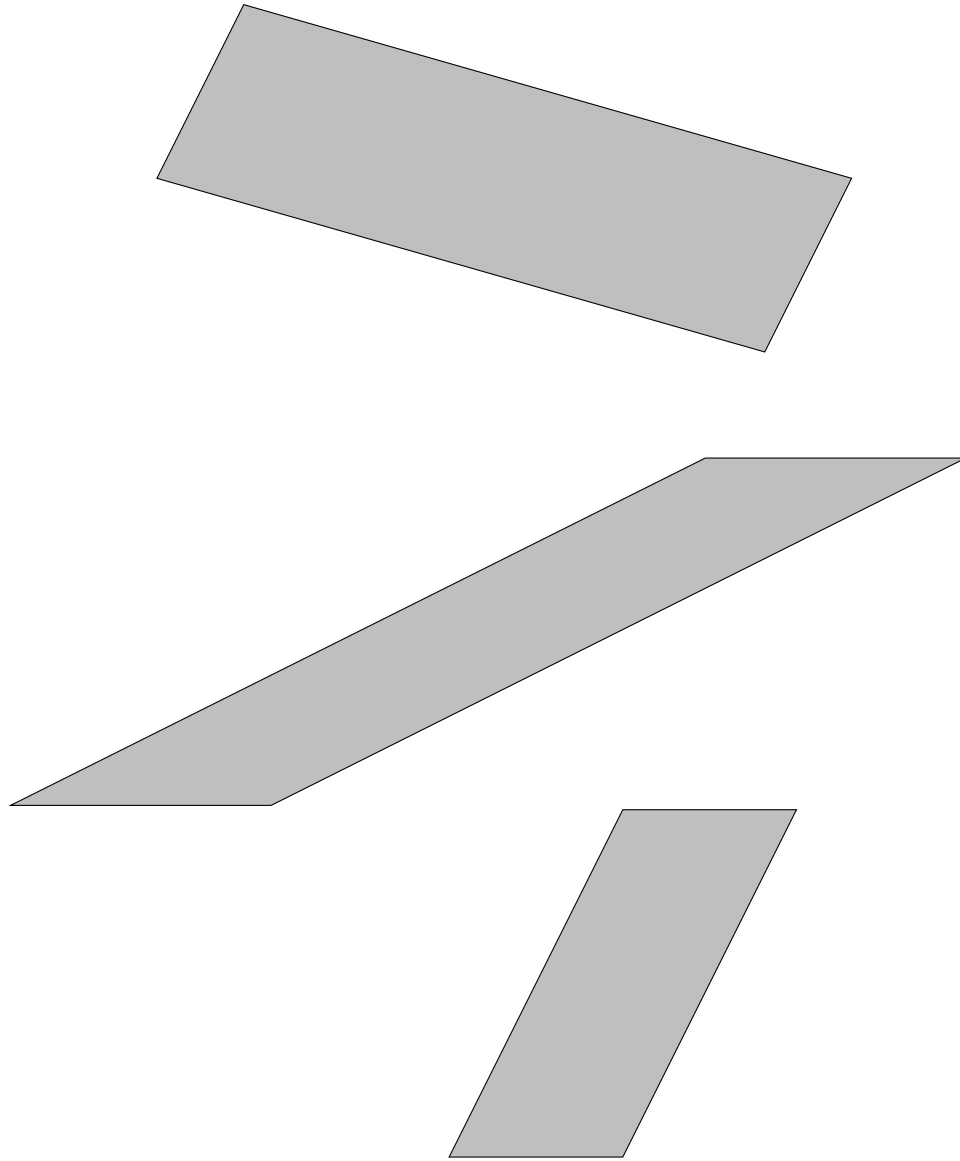
For middle school classroom activities of this type, we recommend the use of colored stock paper to show the regions that have been cut and pasted.

Any parallelogram can be changed into a rectangle by the following steps: First, choose a base of the parallelogram. Second, cut off a part of the parallelogram by making a perpendicular line at one end of the base. Third, move the part of the parallelogram that is not over the base to the other side. Repeat this procedure to make a rectangle with the same base and height as the original parallelogram.

move the triangle to make a rectangle
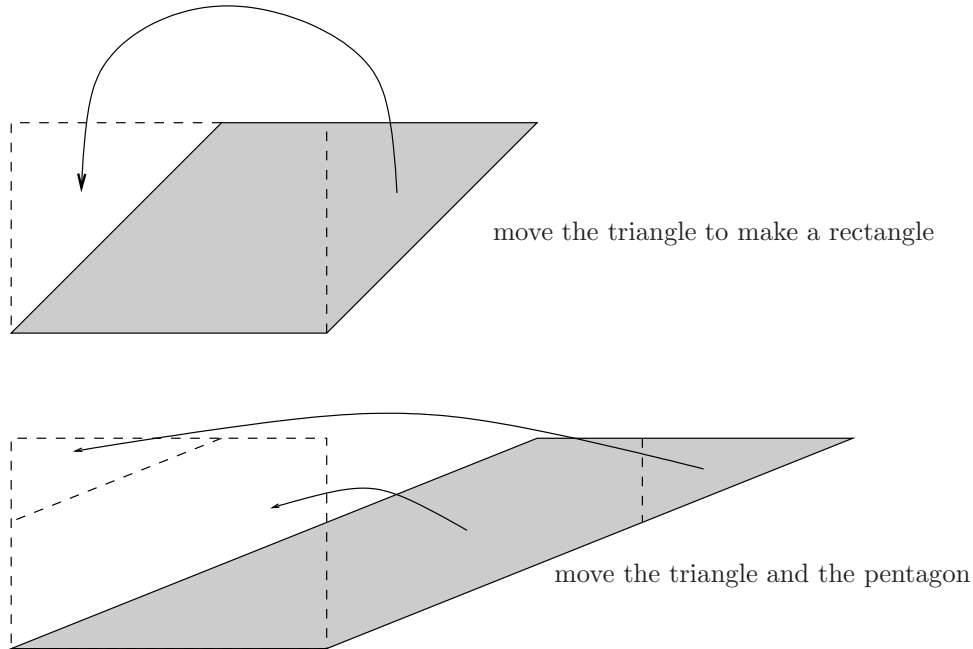

move the triangle and the pentagon

FIGURE 66. Cutting and pasting to change a parallelogram into a rectangle

Since cutting and pasting (without overlap) does not change area, the area of a parallelogram with base b and height h is the same as that of the rectangle,

$$A_{\mathrm{par}} = b \times h.$$

Note that there is some freedom with the choice of base. In the second region in Figure 66, if one chooses the long side as the base then only one cut and paste is necessary. In this case the base and height are different than for the original choice, so one obtains a different expression for the area.

Another method for re-arranging, which leads to the same result, is shown in the following GeoGebra applet, if you have internet access:

Rearranging a parallelogram into a rectangle

The area formula for the parallelogram can also be explained using the following *approximation method*. Suppose we cut the parallelogram into layers and slide them over (that is, *shear*) to make an approximate rectangle, as in Figure 67. [17]
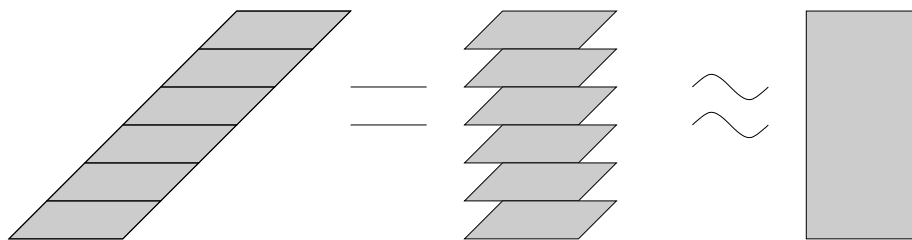
[17]Photo credit I. Radu

FIGURE 67. Shearing a rectangle does not change the area

The approximate rectangle has the same area as the parallelogram, by the moving and additivity principles for area. The difference between the approximate rectangle and the rectangle becomes smaller and smaller as we take smaller and smaller layers, so we may pretend that the approximate rectangle is in fact a rectangle.

The rectangle has the same height and base as the parallelogram, so the area of the parallelogram is

$$A_{\text{par}} = A_{\text{approximate rectangle}} \approx A_{\text{rect}} = b \times h.$$

A GeoGebra demonstration of this shearing argument is given, , if you have internet access, at

<p align="center" style="color:darkred">Finding the area of parallelograms by shearing</p>

The shearing argument for the area of a parallelogram formula can also be explored using a set of index cards, as shown in Figure 68. There are good opportunities for engaging middle school students in conceptually rich discussion around this activity.


FIGURE 68. Shearing a stack of paper

Problem 5.4.2. (1) Where do you see a parallelogram in Figure 68?
(2) In Figure 68, how do the dimensions of the rectangle compare to those of the parallelogram?
(3) What happens to the area of the "parallelogram side" of the stack when we shear?

5.5. **Areas of triangles.**

Problem 5.5.1. (1) Examine the triangles in the figure below. In what ways are the triangles similar? In what ways are the triangles different?
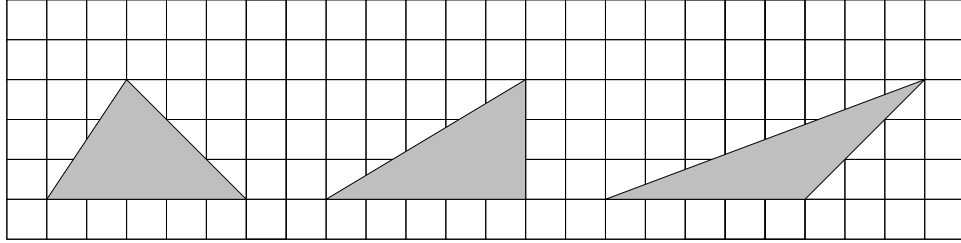
(2) Using the techniques we have considered so far, find the area of each triangle. (Do not use any area formula for the triangle.) Assume that a grid square is a unit square.
(3) How do the areas of the triangles relate to those aspects that the triangles have in common?
(4) Conjecture a formula for the area of a triangle.

The formula for the area of the triangle is one of the most well-known formulas in elementary geometry. This formula is actually *three* formulas, since there are three different possible choices of base.

A *base* of a triangle is a side, thinking of the triangle as placed on that side. There are three choices of which side to choose as base. We also use the term *base* to refer to the *length* of the side chosen as base. Given a base, the *height* is the distance of the vertex not on the base to the line containing the base, that is, the length of the perpendicular segment from that vertex to the line.
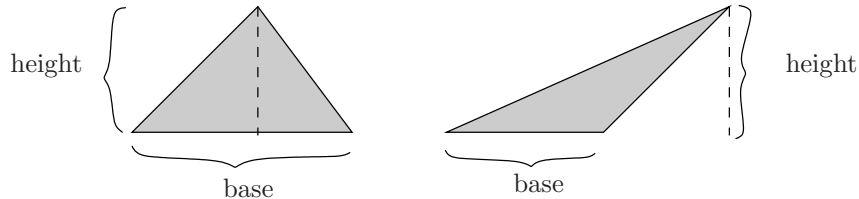

FIGURE 69. Base and height

One can find the height of a given triangle with respect to a given base as follows. If the vertex is not over the base (see the figure above), extend the line segment containing the base using your ruler. Using a right angle tool, find the line segment perpendicular to the base that passes through the vertex opposite the base. The height is the length of this segment. See Figure 70.

The distance from the vertex to the base is *not* the height of the triangle. This distance is called the *slant height*; note that it plays no role in the area formula.
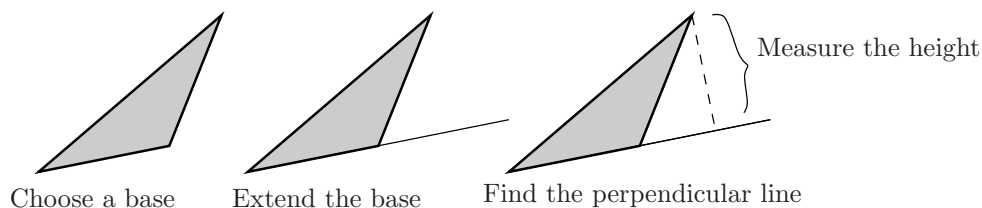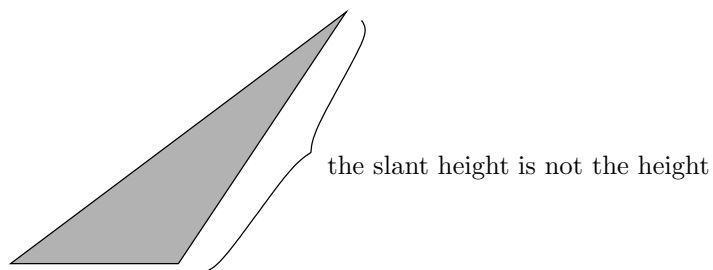
FIGURE 70. Finding the height

FIGURE 71. The slant height

The area of a triangle (that is, the area of the region inside the triangle) with base b and height h is

$$A_{\text{tri}} = b \times h/2.$$

A justification of the formula for the area of a triangle is given in Workshop 5.7.3. If you have internet access, you can see a GeoGebra applet illustrating the principle in the following link:

Area of a triangle as half the area of a parallelogram

Problem 5.5.2. Find the area for the following triangle in two ways:

(1) by choosing a base side and using a ruler to find the approximate base and height of the triangle; and
(2) by finding the area of the big rectangle in which the triangle "sits" and then subtracting off areas of right triangles that have a horizontal base.

How do the area values found in the two ways compare? Discuss in which situations each method is a better method for finding the area of a triangle.
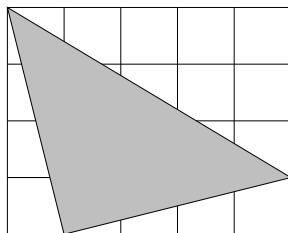
Problem 5.5.3. Activity from [SM]: The figure below is made up of a rectangle and a triangle. The area of the triangle is 72in^2. Find the area of the figure. (Figure is not drawn to scale.)

8 in.

6 in.
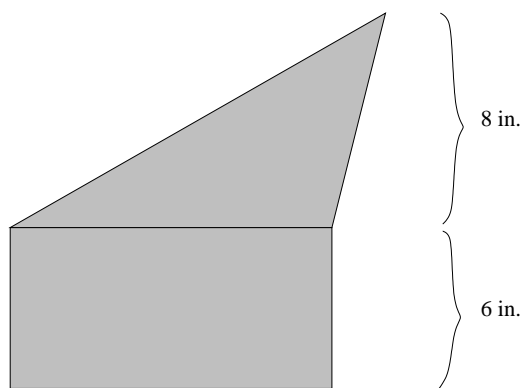
Problem 5.5.4. (Challenge) Show that the area of the triangle below is $(ad-bc)/2$. (Do not assume any numerical values for a, b, c, d.)

c

d

b

a
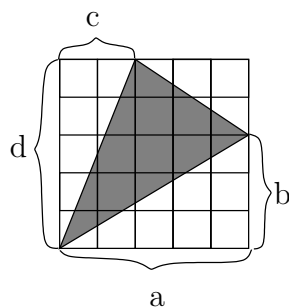
Problem 5.5.5. Find the areas of the regions shown in the figure.
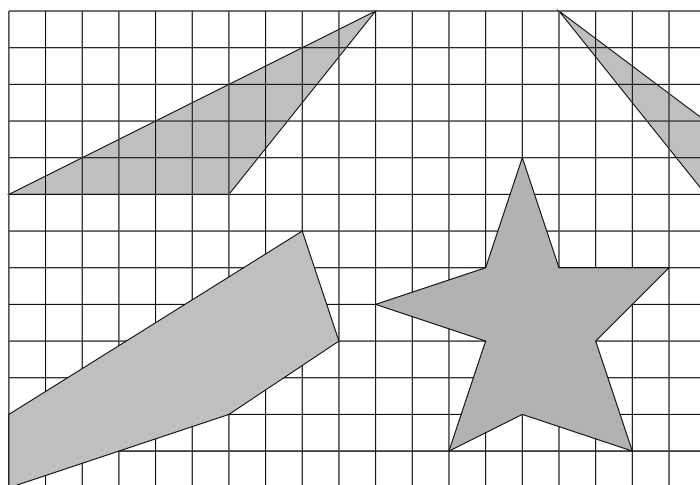
5.6. **Areas of trapezoids.**

Our strategy for finding the areas of trapezoids is similar to that for triangles. We put two congruent trapezoids together to make a region whose area we already know.

Problem 5.6.1. Show that a parallelogram can be formed from two congruent trapezoids. Explain how you know that the new figure is a parallelogram.

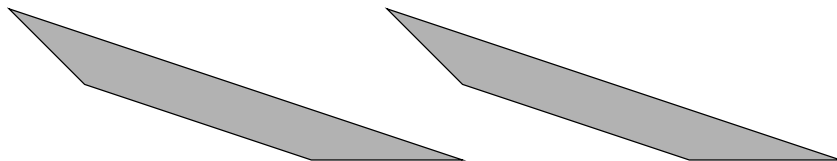The problem above shows that given a trapezoid with height h and bases b_1, b_2, we can take a copy of the same trapezoid and put it together with the original to make a parallelogram with base $b_1 + b_2$ and height h, as in Figure 72. Since the area of the trapezoid is half the area of the parallelogram, the *area of a trapezoid* with parallel side lengths b_1 and b_2 and height h is

$$A_{\text{trap}} = \frac{(b_1 + b_2)}{2} \times h$$

that is, the average of the lengths of the parallel sides, times the height.
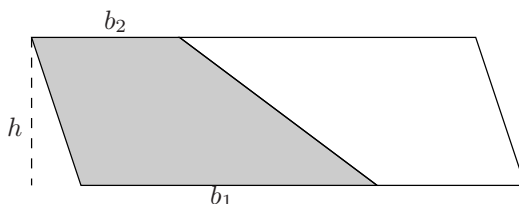

FIGURE 72. Making a parallelogram from two copies of a trapezoid

Problem 5.6.2. Activity from [MCAS]: The rear window of Alex's van is shaped like a trapezoid with an upper base measuring 36 inches, a lower base measuring 48 inches, and a height of 21 inches.

What is the area, in square inches, of the entire trapezoidal rear window? Show or explain how you got your answer.

5.7. **Workshops.**

Workshop 5.7.1.

Consider the rectangle below: (not to scale)

Assume the rectangle has a width measuring 4 cm and a length measuring 9 cm. Suppose further that the rectangle is cut in half horizontally and one piece is attached to the other, without overlap, to form an L-shaped polygon shown below.
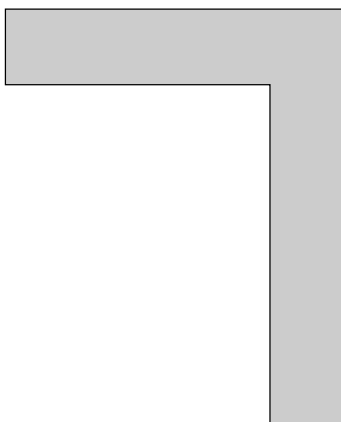
(1) How does the area of the new figure compare to the area of the original rectangle? Explain.
(2) How does the perimeter of the new figure compare to the perimeter of the original rectangle? Explain.
(3) Make a conjecture about how you can cut and rearrange the original rectangle (with one cut) so that the resulting quadrilateral has the smallest possible perimeter. Provide supporting evidence.
(4) Make a conjecture about how you can cut and rearrange the original rectangle (allowing multiple cuts) so that the resulting quadrilateral has the smallest possible perimeter. Provide supporting evidence.

Workshop 5.7.2. from [NJ]:

Veronica is making a rectangular garden. She plans to put a fence around the garden using 28 feet of fencing, and she wants the garden to be 8 feet long.

(1) How wide will Veronica's garden be? Show how you got your answer.
(2) If Veronica is going to put fence posts two feet apart around the outside of the garden, how many fence posts will she need? Show all of your work to explain your answer.
(3) By what factor would the area of her garden change if the length and width are scaled (multiplied) by a factor of two?
(4) How many posts would she need for the new garden (with twice the length and width) if they are still spaced two feet apart?
(5) Suppose that, instead of an 8 foot long garden, she wants a garden of 45 square feet (still using 28 feet of fencing). What should the dimensions of her garden be?
(6) Is there a non-rectangular shape that would give the same perimeter and area as her garden in (5)?

Workshop 5.7.3.

Explain the area formula for triangles.

(1) Draw a right triangle. Find a way to make it into a rectangle by using copies or cutting and pasting. (It may be helpful to use colored paper and scissors.) Explain the logic of why the figure you created is a rectangle. Is there a second way of creating a rectangle from two copies of the given right triangle? Explain why or why not.

(2) Draw a triangle that is not a right triangle. Show that you can make a parallelogram using two copies of the triangle. (It may be helpful to use colored paper and scissors.) Explain the logic of why the figure you created is a parallelogram. Is there a second way of creating a parallelogram from two copies of the given triangle? (It may be helpful to use colored paper and scissors, or GeoGebra.)

(3) Draw a triangle that is an obtuse triangle. Show that there is a base so that using two congruent copies of the triangle, one can create a rectangle with the base and height of the original triangle by cutting and rearranging.

(4) Explain the area formula for a triangle, using additivity of area and one of your constructions from (2).

(5) Consider the triangle below.
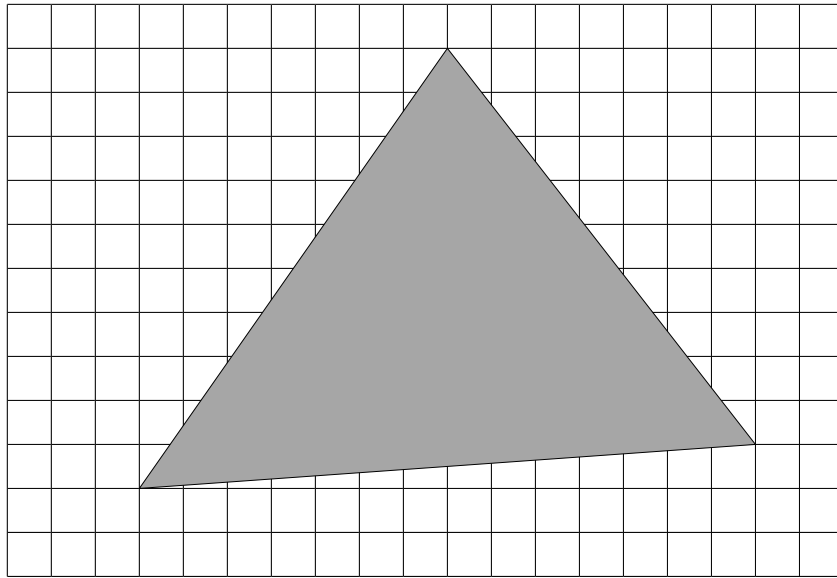
What is the best (most precise) way of finding the area of the triangle? Is there a method that gives the area exactly? Carry out the best method you can find.

(6) Explain the area formula for a trapezoid, using additivity of area. Make sure to explain why a parallelogram can always be created using two congruent copies of the trapezoid. Explain carefully why the resulting shape is a parallelogram, in particular why opposing sides are parallel.

Workshop 5.7.4.

All of the problems below should be done *without measuring*.

(a) Find the area of the triangle ABC below. Explain your logic.
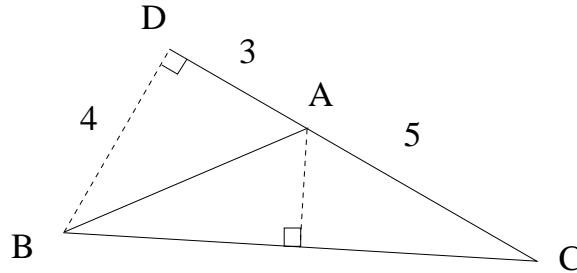
(b) Find the length CD in the figure below. Explain your logic.
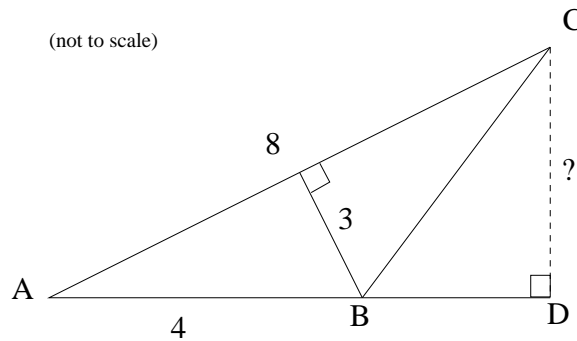
(c) Can you find the area of the parallelogram below without measuring? If yes, find it. If not, can you find a range of possible values for its area? In each case, explain your answer.
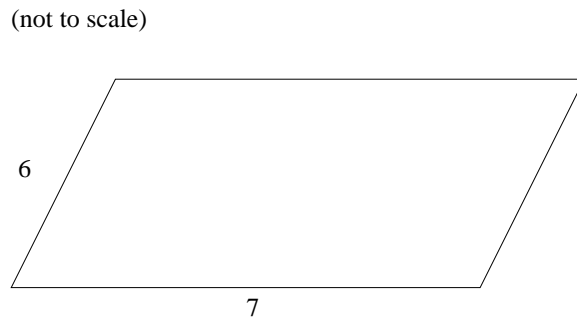
(d) In the diagram below, $\overline{AD}$ is perpendicular to $\overline{AC}$ and the area of the parallelogram $ABCD$ is 35 square units. Find the distance from B to $\overleftrightarrow{AC}$ (that is, the distance along the shortest path). Explain your logic.

(not to scale)
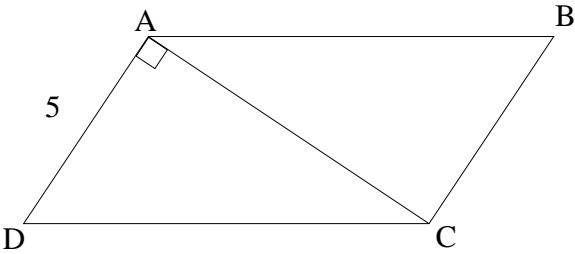
Workshop 5.7.5.

You may wish to use a Geoboard, or the online version at

<p style="text-align:center">Mathematics Learning Center</p>

to assist with this problem.

A *lattice polygon* is a polygon in the coordinate plane whose vertices have integer coordinates. (On a geoboard, we assume that the pegs are at the points with integer coordinates.)

(1) (a) Create or draw a lattice triangle with area 3. Explain your answer.
     (b) Create or draw a lattice square with area 5. Explain your answer.
(2) There is a relationship between the number of lattice points inside a polygon, the number of lattice points on the boundary, and the area of the polygon. In order to find this relationship, perform the following steps:
    (a) Draw several lattice polygons, and record the areas, number of lattice points inside, and number of lattice points on the boundary for each. What kind of formula might you expect? Discuss.
    (b) Possibly by drawing more polygons and collecting more data (you may need to reorganize the data into a table), find the formula. Check that your formula works for a type of polygon you have not already considered.
    (c) Justify the formula for rectilinear polygons (polygons with only right angles) by drawing unit squares centered at each lattice point; how many of these square are there? Explain. (Challenge: Can you think of a way of justifying the formula for any kind of polygon?)

5.8. **Assessments and sample student work.**

Problem 5.8.1. (Adapted from [MCAS]) Five geometric terms are given below: acute, obtuse, equilateral, right, isosceles.

David drew the six triangles shown below.
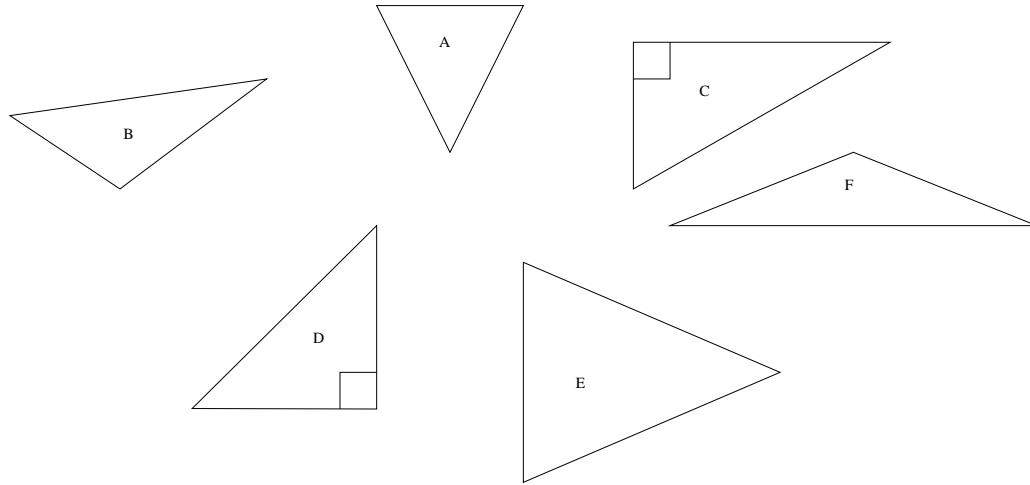
(1) Identify one of the geometric terms listed that can be used to describe triangle A. Explain your reasoning.
(2) Which two of the geometric terms listed can be used to describe triangle B? Explain your reasoning.

David grouped his triangles as shown below.
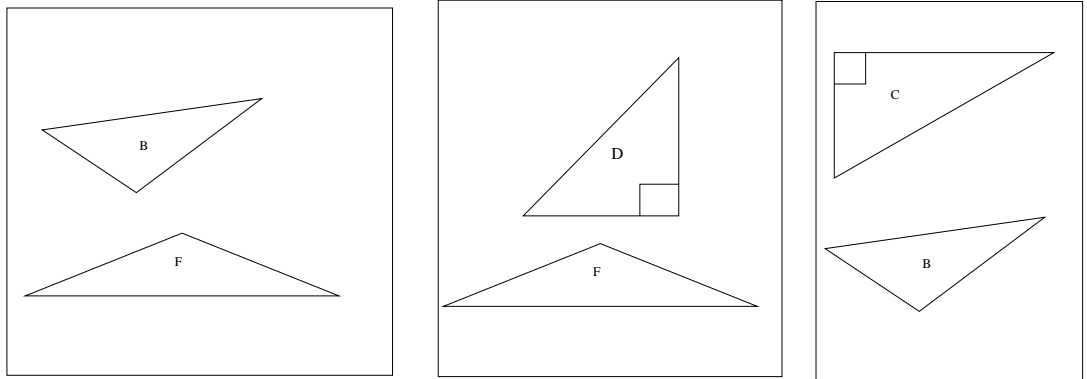
(3) Using geometric terms, explain what the two triangles in each group have in common.

Problem 5.8.2. Activity adapted from [MCAS]: Danielle measured two of the computer screens in her school's computer lab.

Screen 1 is in the shape of a rectangle with a width of 12 inches and height of 9 inches and a diagonal length of x. Screen 2 is in the shape of a rectangle with a width of y and a height of 8.2 inches and diagonal length of 15.4 inches.

1. What is the area, in square inches, of Screen 1? Show or explain how you got your answer.

2. What is x, the diagonal length in inches of Screen 1? Show or explain how you got your answer.

3. Which computer screen, Screen 1 or Screen 2, has the greater area? Show your work or explain how you got your answer.

Sample student answers are below. Note the incorrect use of units in most of the answers.

Score 4/4 (left) and 3/4 (right)

Score 2/4 (left) and 1/4 (right).

a) the area is: 108in.   12×9=108

b) the (x) diagnol length of screen 1 is: 15in

C) the areas are the same.

A. 12·9 = 108 square inches.

B. 108÷2= 54 inches   X=54 inches.

C. 15.4×2=30.8   30.8÷8.2=3.75   Y=3.75 in.

Score 0/4:

A) The area is 9.

12-9=3X3=9

B) 12-9=3   X=3

c) Screen 2

Screen 2   Screen 1
15.4-8.2=7.2×7.2=51.84   12-9 = 3×3 =9

Problem 5.8.3. (From [MCAS]) Carrie's garden is rectangular in shape and measures 10 feet in width and 14 feet in length.

1. What is the area, in square feet, of Carrie's garden? Show or explain how you got your answer.

Carrie wants to put a fence along the perimeter of her garden. She will pay $15 per foot of fence that she uses.

2. What is the amount of money, in dollars, that Carrie will pay for the fence? Show or explain how you got your answer.

Roberto has a garden in the shape of a square. The perimeter of his garden is equal to the perimeter of Carrie's garden.

3. What is the area, in square feet, of Roberto's garden? Show or explain how you got your answer.

Sample student answers showed confusion between area and perimeter. The fact that the second part of the problem required finding the perimeter first caused problems. Many of the answers showed incorrect work with units.

Problem 5.8.4. An assessment problem adapted from New York [NY] concerning area is given below:

Jeremy wants to determine the area of his school's library. A diagram of the library is given below. What is the area, in square feet, of the library?
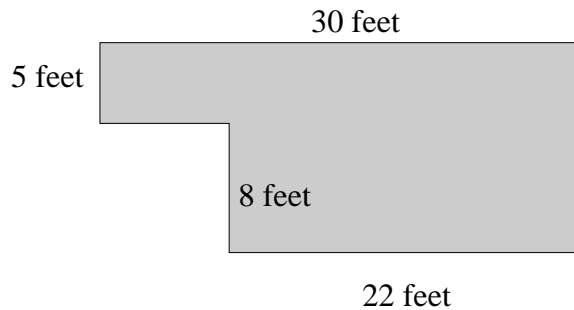
Sample answer for 1/3 points: "To find the area, I need to do length times width. The length is 30 feet. The width is $5 + 8 = 13$ feet. So the area is 30 times $13 = 390$ square feet."

6. CIRCLES

In this chapter we introduce circles, investigate their area and perimeter, and introduce the irrational number π. The number π can be defined either as

(1) the ratio of the circumference to the diameter of *any* circle, *or*
(2) the ratio between the area and the square of the radius of *any* circle.

The objectives of this chapter are: (i) to introduce circles and arcs, and investigate how they occur in real life; (ii) to describe how to draw circles using compasses; (iii) to study the circumference of circles, and introduce the number π; (iv) to study the area of circles, and the formula for the area of the circle; and (v) to study the lengths of arcs and areas of sectors.

6.1. Circles and compasses.

Children should try to use clear definitions in mathematics in discussion with others and in their own reasoning. It is quite difficult to state a clear definition of a circle, as the following problem illustrates.

Problem 6.1.1. Which of the following is a good definition of the circle and why?

(1) A circle consists of points that are the same distance from another point.
(2) A circle consists of points that are a given distance from a given point.
(3) A circle consists of points in the plane that are the same distance from a given point.
(4) A circle consists of all points in a plane that are a given distance from a given point in that plane.

Problem 6.1.2. Suppose a cell phone tower is located at a position P and the transmission has range r equal to 10 km. Suppose that the surface of the earth nearby is well approximated by a plane.

(1) Describe the set of points Q in the plane whose distance from P is less than r, both mathematically and practically in terms of the cell phone reception.
(2) Describe the set of points Q in the plane whose distance from P is more than r, both mathematically and practically in terms of the cell phone reception.
(3) (Challenge) In what pattern should towers be placed so that everyone has cell phone reception, and the smallest number of towers is used? (Assume that every tower has the same transmission radius.)

The following vocabulary relates to circles and regions associated to them:

A circle of *radius* of length r with center P is the set of points in a plane containing P whose distance to P is r.

The *region inside of the circle* is the set of points in the plane whose distance from the center is *less than* r. This is also called the *disk* of radius r with the given center.

The *region outside the circle* is the set of points in the plane whose distance from the center is *greater than r*.

A *radius* of a circle is any line segment connecting the center of the circle with a point on the circle, or the length of any such segment.

A *diameter* of a circle is a line segment with endpoints on the circle and passing through the center, or the length of any such segment.
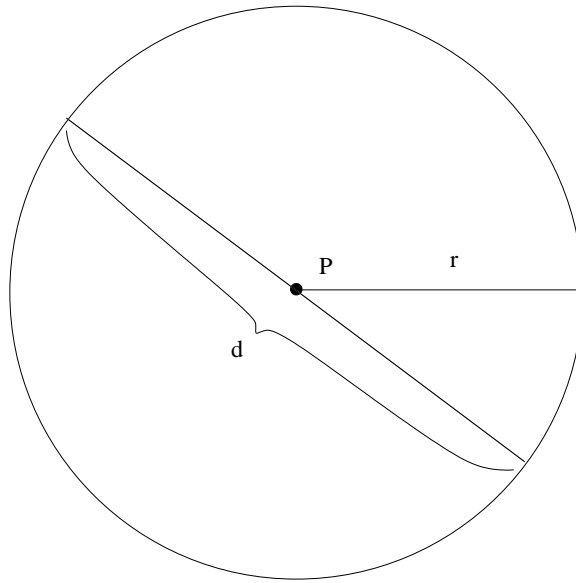
FIGURE 73. A circle with center P, radius r and diameter d

Drawing circles

Circles can be drawn using a *compass*. Traditionally, a compass is a device consisting of two arms joined at a vertex. More recent versions, such as a *safety compass*, simply have an arm of a fixed length which may be placed onto the paper and a sliding piece that allows the user to draw circles of any radius. See Figure 74 and Figure 75 for examples.

FIGURE 74. A Sparco compass. Probably not allowed in today's classrooms
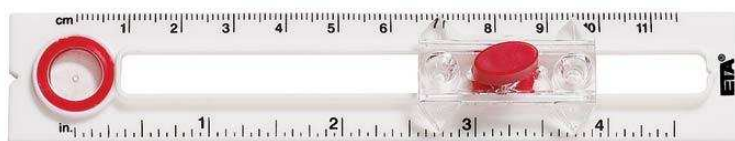
FIGURE 75. An ETA safety compass with no sharp points

To draw a circle of a given radius r with a given center P, first find a point Q with the distance r to the center, using a ruler. Then place the point of the sharp arm of the compass on P, adjust the compass so that the second arm ends on the point Q, and rotate the second arm to trace out the circle. See Figure 6.1.
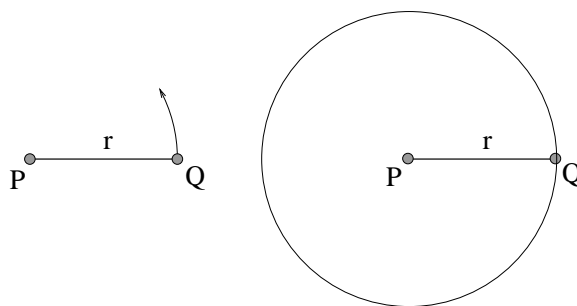
FIGURE 76. Drawing a circle

Some state assessments use vocabulary involving arcs, such as:

> A *circular arc* is a portion of a circle.

> The *central angle* of an arc is the angle formed by the rays starting at the center of the circle and passing through the endpoints of the arc.

> A *semicircle* is half of a circle.

Problem 6.1.3. Compare and contrast the designated shapes. For each pair of shapes, find one similarity and one difference.
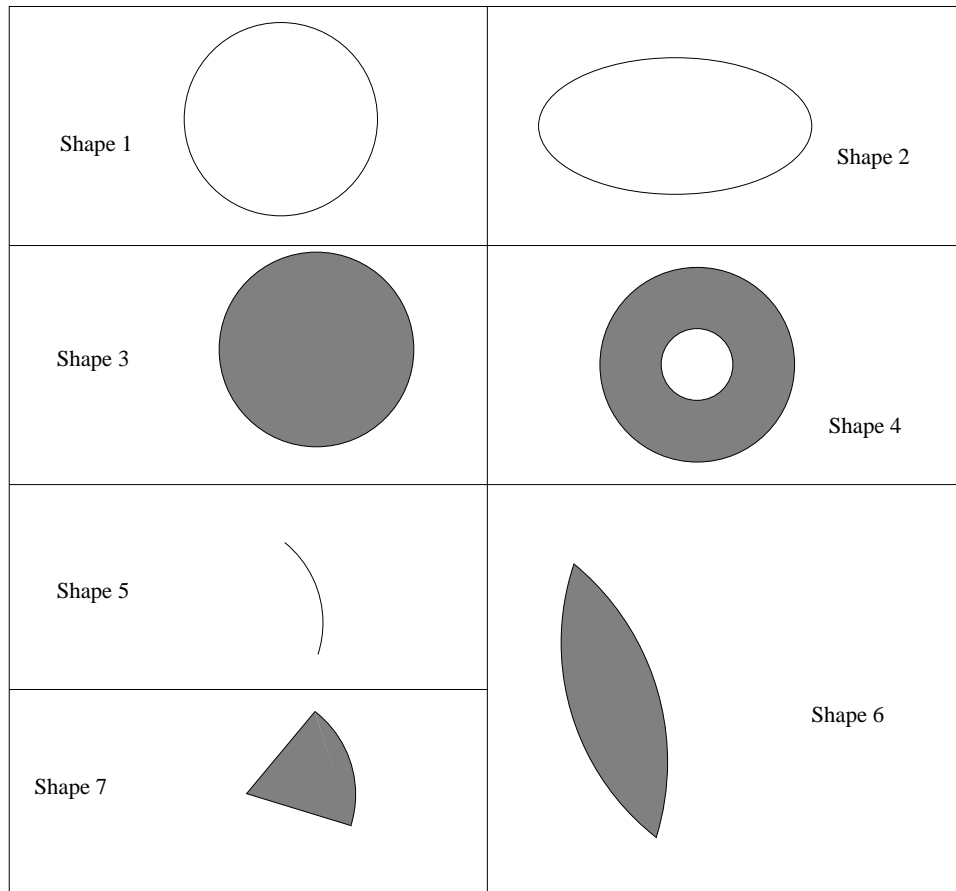
Shape	Shape	Ways shapes are alike	Ways shapes are not alike
Shape 1	Shape 3		
Shape 1	Shape 2		
Shape 3	Shape 4		
Shape 5	Shape 7		
Shape 3	Shape 6		
Shape 6	Shape 7		

Most circles appearing in real life do not come "with centers marked" and so to find areas one has to find the center, as in the following problem.

Problem 6.1.4. (a) Find the center of the circle below by folding.
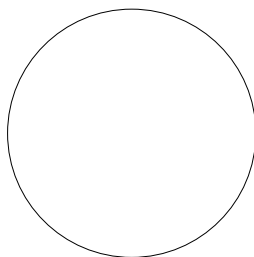
(b) Find the center of the circle by drawing at least one *tangent line* (line that meets the circle exactly once)[18] and drawing a perpendicular through the intersection point. Where is the center? Give a brief justification of your answer.
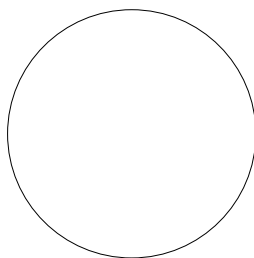
(c) Find the center of the circle by drawing at least one *chord* (line segment with endpoints on the circle) and drawing a perpendicular through its midpoint. Where is the center? Give a brief justification of your answer.
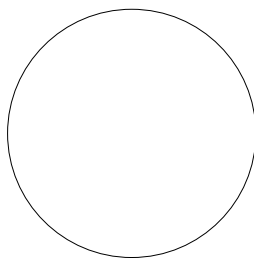
(d) Explain why any pair of points lies on infinitely many circles.

(e) (Challenge) Show that each triple of distinct points either lies on exactly one line or exactly one circle.

---

[18]In general, the tangent line to a curved path at a point is the line passing through the point with the same "direction" as the curve at that point.

Problem 6.1.5. (1) Draw a circle of radius 1 inch around the point A in the figure below.

○
A

○
B
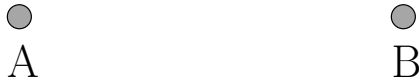
(2) Draw a circle of radius 1.5 inches around the point B.
(3) Using one color, shade the region consisting of points that are less than 1 inch from the point A.
(4) Using another color, shade the region consisting of points that are more than 1 inch from the point A, and less than 1.5 inches from the second point B.
(5) Using a third color, shade the region that is more than 1 inch from the point A, and more than 1.5 inches from the second point B.
(6) Choose an intersection point of the two circles and label it C. What are the distances AC and BC?
(7) What is the relationship between AB, AC and BC? Explain your reasoning.

Constructions with compass and straightedge

Circles and their properties can be used to construct many other geometrical figures. By a "construction with compass and straightedge" we mean a construction that can be carried out only by drawing circles of arbitrary radius and drawing lines of arbitrary length; measuring distances, angles, or using a right angle are not allowed.

Problem 6.1.6. (From [NJPEMSM])

(1) Draw two points A, B and a line segment between them. Using your compass find two points C, D such that A, B, C, D are the vertices of a rhombus. Describe your steps and explain why your figure is a rhombus.

(2) Use your construction from part (1) to construct the midpoint of the segment $\overline{AB}$. Explain your answer.

(3) Use a straightedge and compass to construct a regular hexagon. Describe your steps and explain why your figure is a hexagon.

The following problem illustrates the differences between compass-straightedge constructions and constructions using ruler and protractor as well.

Problem 6.1.7. (From [NJPEMSM])

(1) (a) Use a ruler to draw an isosceles triangle. Describe your steps.

(b) Use a straightedge and compass to help you draw an isosceles triangle. Explain why the triangle you drew must be isosceles.

(c) With only a ruler and pencil, try to draw an equilateral triangle. Why is this difficult?

(d) Afterwards, use a straightedge and compass (and no ruler) to help you draw an equilateral triangle and explain why the triangle you drew must be equilateral.

(2) (a) Using a ruler and compass, draw a triangle with one side of length 5 inches and the angles at the vertices on that side measuring 30 degrees and 45 degrees, respectively. (You may use a protractor to measure angles.)

(b) Can you construct any other triangle(s) with the same conditions?

(c) Given the measures of two angles of a triangle and the length of a side between them, how many different-looking triangles can be drawn?

6.2. **Circumference.**

The *circumference* of a circle is the perimeter of the region inside it.

You can measure the perimeter of a circle approximately by placing a string around the circle and then straightening it out and measuring the length.

If you have internet access, clicking on the following link will display an applet. Drag the point to straighten out the perimeter:

<p style="text-align:center; color:darkred;">Straightening out the perimeter of a circle.</p>

Problem 6.2.1. Measure the perimeter of a variety of circular objects (say a glass, a yogurt lid, ...) by using a string, and the diameter using a ruler. Make a table showing the diameter and circumference. What do you notice?

Object	Diameter	Circumf. of Circle	Ratio of Circumf. to Diameter
A			
B			
C			

The relationship between circumference, diameter, and radius

Problem 6.2.2.

  (1) How many ways can you think of to estimate the circumference of a circle with radius one inch?
  (2) Draw such a circle; try at least two different methods and compare them.
  (3) Try the same with a circle of radius two inches. How do your answers for the two circles relate? Can you explain the relationship?

As the above activity suggests, doubling the radius of a circle also doubles the circumference. More generally, rescaling a circle by any factor rescales the circumference by the same factor. So the ratio of circumference to diameter is the same for any circle. This is the basis for the definition of the number π:

> The ratio of circumference C to diameter d is the same for any circle, and is denoted π. That is, $\pi = C/d$.

The first few digits of π are 3.1415926.... The number π is *irrational*: it cannot be represented as a fraction of integers, or equivalently, its decimal expansion cannot be represented with a repeating pattern. This means that in practice, approximations such as $\pi \approx 3.14$ are used.[19]

Because diameter is twice the radius, the circumference is also related to the radius of the circle. Using the definition of π and the relationship between diameter and radius, we have

$$\pi = C/d = C/(2r),$$

and multiplying both sides by $2r$ gives

$$2r\pi = 2r(C/2r) = C.$$

In summary:

> The circumference C of a circle of radius r is given by $C = 2\pi r$.

Problem 6.2.3. Suppose a string is wrapped tightly around the equator of the earth. How much extra string is needed to be able to lift the string one foot above the ground along the entire equator? (Suppose the earth is uniformly spherical for simplicity.)

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[19]A *viable argument* for the irrationality of π is outside the scope of this book, although you might become somewhat convinced of this irrationality by looking at the decimal expansion and seeing there are no patterns.

Lengths of Circular Arcs

Problem 6.2.4. Find the lengths of the circular arcs using proportionality. (What part of each circle does each arc represent?) Use any tools you deem necessary.

5 cm   4 cm   5 cm
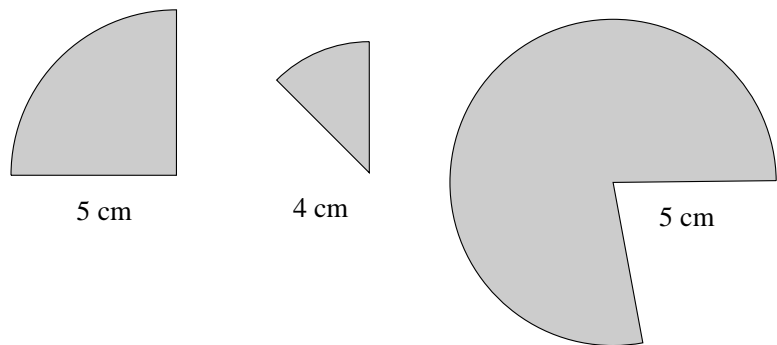
Arc	Circumference	Measure of central angle	Length of Arc
First			
Second			
Third			

As the problem above suggests, an approach to finding the length of circular arcs that works for any arc involves finding the fraction of the circle that the arc represents. If the measure of the central angle is a (in degrees) then the fraction of the whole circle represented by the arc is $a/360°$. Given that the circumference of a circle of radius r is $2\pi r$, this shows the following:

The length of the arc with central angle of measure a is

$$l_{\text{arc}} = 2\pi r(a/360°).$$

Problem 6.2.5. A pizza is 1.5ft in radius and is cut into 12 congruent slices. Find the perimeter of each slice.

Problem 6.2.6. New York and Philadelphia are approximately 100 miles apart. Suppose that a town is located within 60 miles of New York and 80 miles of Philadelphia.

(a) Sketch, by shading, the possible locations of the town, using a scale of 1cm : 20mi.

(b) Find the perimeter of the shaded region from (a). You can measure angles and (straight?) distances using any tools you deem necessary.

6.3. **Areas of circles.**

Problem 6.3.1. Cut up the circle below into thin sectors ("pizza slices") and arrange them to make an approximate rectangle.

(a) What are the dimensions of the rectangle, in terms of the dimensions of the original circle?

(b) What is the area of this rectangle, which is approximately the area of the circle?

(c) Use this to express the area of a circle in terms of the radius.
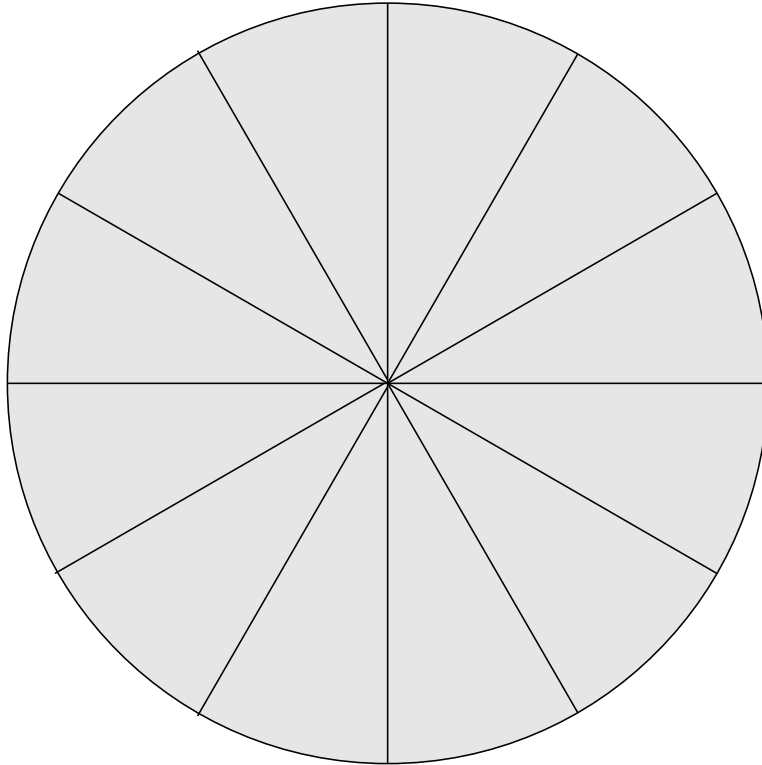
The formula for the area of a circle

The area of a circle (that is, of the region inside it) can be found using the additivity principle for area by dividing a circle up into congruent sectors (pizza slices) as follows. Consider a circle of circumference C and radius r. Divide the region inside the circle into congruent sectors. These sectors can be re-arranged to make an approximate rectangle with length $C/2$ and width r, as shown in Activity 6.3.1. If you have internet access, you can click here to bring up a GeoGebra applet that illustrates the principle. (Click on the circumference box that appears to the right of the drawing after straightening.)

<p style="color:maroon; text-align:center">Straightening out the perimeter of a circle.</p>

The above activity and applet demonstrate that the area of a circle with circumference C and radius r is the same as the area of a rectangle with length $C/2$ and width r:

$$A_{\mathrm{circ}} = (C/2) \times r.$$

Since we already showed that $C = 2\pi r$, the area is

$$A_{\mathrm{circ}} = (C/2) \times r = (2\pi r/2) \times r = \pi r^2.$$

That is, the area A of a circle with radius r is

$$A_{\mathrm{circ}} = \pi r^2.$$

For example, (note the correct use of units) the area of a circle with radius 2 feet is

$$A = \pi(2 \text{ ft})^2 = \pi\ 4 \text{ ft}^2 = 4\pi \text{ ft}^2.$$

Working correctly with units will prevent students from confusing the area formula with the perimeter formula: only with the area formula is it possible to get an answer in *square* units.

Another definition of the number π

The area formula shows that the ratio of the area of a circle to the square of the radius is the same for any circle. We could have defined π this way, instead of as the ratio between the circumference and diameter.

Three amazing facts about circles

To summarize, in this chapter we've covered the following three *amazing facts*:

(1) The ratio between the circumference and diameter of a circle is the same for all circles!

(2) The ratio between the area and radius squared of a circle is the same for all circles!

(3) The two ratios are the same!

Annuli

An *annulus* is the region between two circles with the same center in a plane. The plural form of annulus is usually *annuli*. For children the most familiar example of an annulus might be the shape of a digital video disk (dvd). The area of the dvd is related to how much data can be recorded on it. See Figure 77. [20]

FIGURE 77. An annulus

Problem 6.3.2. Suppose there is a circular park with radius 200 feet with a circular reservoir in the center of radius 10 feet. The reservoir is off limits (and in fact is fenced off) and not considered part of the park.

  (1) What is the perimeter of the park?
  (2) What is the area of the park?

Your answers should use properties of area and perimeter rather than any formulas you might be familiar with.

A common mistake in solving the first part of problems such as Problem 6.3.2 is to subtract the radii to find the perimeter, rather than add them. Here is an example of how this confusion arises in problems. The answer depends on whether the fountain is part of the park or not; if it is not considered part of the park, then the perimeter of the park is $2\pi(210 \text{ feet})$. If it is considered part of the park, the perimeter is $2\pi(200 \text{ feet})$. Questions should be worded to avoid this confusion.

A common mistake in solving the second part of problems such as Problem 6.3.2 is to subtract radii rather than areas. For example, suppose the problem is to find the area of the annulus with inner radius 5 cm and outer radius 9 cm. Here is an incorrect answer: $A = \pi(9-5)^2 = 16\pi \text{ cm}^2$. A correct answer is

$$
\begin{aligned}
A &= \pi((9 \text{ cm})^2 - (5 \text{ cm})^2) = \pi(81 \text{ cm}^2 - 25 \text{ cm}^2) \\
&= \pi(81 - 25) \text{ cm}^2 = 56\pi \text{ cm}^2.
\end{aligned}
$$

The difference property of area may be used to give a formula for the area of an annulus. If the radius of the inner circle of an annulus is r_1 and the radius of the

outer circle is r_2 then the area A_{ann} of the annulus is

(1) $$A_{\text{ann}} = \pi r_2^2 - \pi r_1^2 = \pi(r_2^2 - r_1^2).$$

For example, the area of an annulus of inner radius 1 foot and outer radius 2 feet (note the correct use of units!) is

$$A = \pi((2 \text{ ft})^2 - (1 \text{ ft})^2) = \pi(4\text{ft}^2 - 1\text{ft}^2) = \pi(4 - 1)\text{ft}^2 = 3\pi\text{ft}^2.$$

Similarly there is a formula for perimeter: The perimeter of the annulus with inner radius r_1 and outer radius r_2 is the *sum* of the lengths of the inner and outer boundaries:

$$P_{\text{ann}} = 2\pi r_1 + 2\pi r_2 = 2\pi(r_1 + r_2).$$

Sectors

A *sector* of a circle is a region formed by two radii and the circular arc between them. In other words, a sector is the shape of a *pizza slice*. At a pizza party, the area of the pizza slice is related (in an inverse way) to how many pieces each child would eat.
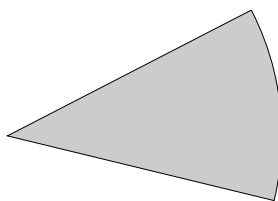
FIGURE 78. A sector

Problem 6.3.3. Find the areas of the following sectors using your compass, by viewing each sector as a portion of the region inside a circle. For each sector find what proportion of the circle it represents and then find its area.
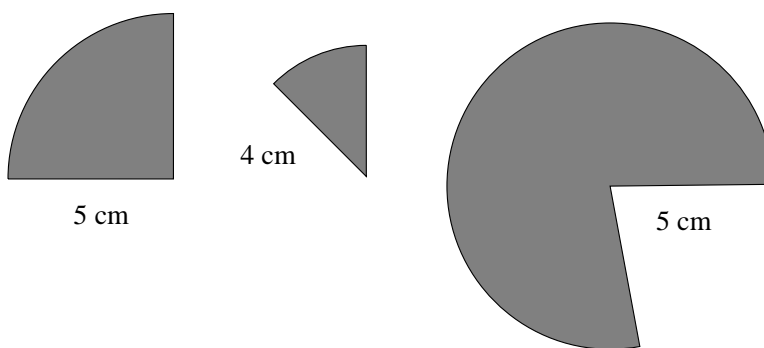
As the problem above suggests, an approach to finding the area of sectors that works for any sector involves finding the fraction of the disk that the sector represents.

If the measure of the central angle of the sector is a then the sector is $(a/360°)$-th of the circle with the same radius and center. Given that the area of a circle with radius r is πr^2, the area of the sector with central angle of measure a is

$$A_{\text{sector}} = \pi r^2 (a/360°).$$

Problem 6.3.4. A pizza is 1ft in radius and is cut into pieces so that the angle of each slice at the vertex is 30 degrees. Find the area of each slice.

We conclude the section on circles with a fun (maybe) problem involving rainbows and a few scientific facts about them.

Problem 6.3.5. Images of rainbows are always part of circles.

(1) Find the center of the circle of which the rainbow is part in the picture below by drawing several tangent lines and drawing the perpendicular lines at the tangent points.

(2) Using your compass, extend the picture of the rainbow outside of the photograph so that it forms a circle. (Not all will fit on the page.)
(3) What is the length of the portion of the photo of the rainbow shown? (The rainbow itself has no length; it is an optical effect.)
(4) What is the area of the sector whose boundary is the rainbow shown?
(5) (Challenge) Using the fact that the center of the circle, the viewer, and the sun are colinear, approximately what time was it when the picture above was taken? (Suppose the sun rises at 7am and sets at 7pm, to make things easier.)
(6) (Challenge) The sunlight reflects off the raindrops at maximum intensity at an angle of 42 degrees. (That is, the angle with vertex at the raindrop and rays given by the incoming and outgoing ray of light has angle measure 42 degrees.) Can there be a rainbow if the sun is directly overhead?

   If you have internet access, a GeoGebra demonstration of the maximum intensity at 42 degrees can be found here. See also [CA], available at this link.
(7) (Challenge) In what situation might you see a rainbow that is a full circle?

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[21]Retrieved under public domain license at www.publicdomainpictures.net on September 3, 2015. Credit: ALangova

6.4. **Ellipses, parabolas, and hyperbolas***.

Ellipses, parabolas, and hyperbolas can be thought of as shadows of circles. They play an important role in mathematics and physical science. For example, the earth moves around the sun in an *ellipse*. The orbit of the earth is almost circular; however for comets such as *Haley's comet* the orbit is much more elliptical. These concepts sometimes appear in science standards.

Ellipses

The following is an informal definition of an *ellipse*:

> An ellipse is obtained from a circle by stretching in one direction only, or by viewing the circle from an angle.

A common misconception about ellipses is that they have straight portions. That is true of, say, the shape of a track around a soccer field, but that shape is not an ellipse.

A more precise definition of an ellipse is the following: An ellipse with *focal points* P, Q (also known as *foci*) and *transverse diameter* d is the set of points whose distance to P plus the distance to Q is equal to d.

An ellipse can be drawn roughly by drawing a rectangle and then drawing the ellipse so that it is tangent to each edge of the rectangle as shown in Figure 79.
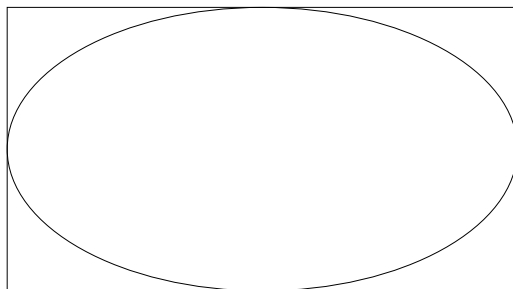
FIGURE 79. An ellipse inscribed in a rectangle

If a circle is tilted towards or away from the viewer, then it appears as an ellipse. For example, in most drawings of a cylinder the top and bottom appear as ellipses.

Here is a sample problem: Draw an ellipse whose foci are 4 units apart and whose points have the property that the sum of distances to the foci is 10 units.

A sample answer is as follows. Draw circles with radii $1, 2, 3, \ldots$ units centered on the first focus, and similarly for the second focus. For each pair of circles whose radii sum to 10 units, mark the two intersection points of the two circles. These points can be found by finding one point (say the point that is 3 units to the right of the right focus) and moving to other points with the same sum of differences by "following the diamonds" as in the figure below;
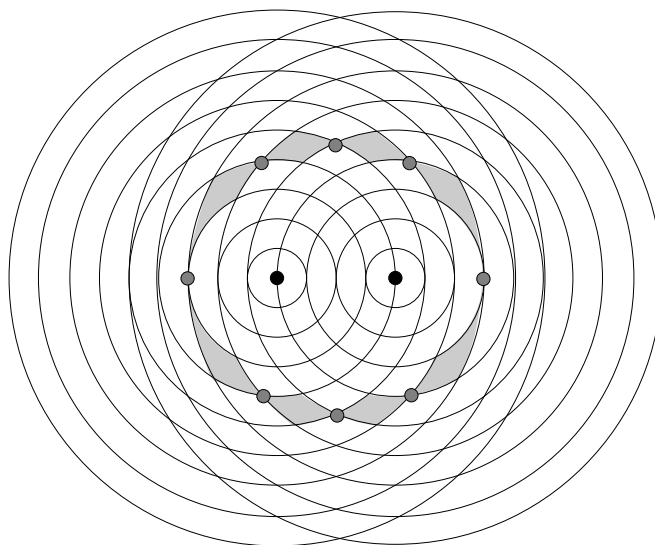
Because moving to the opposite vertex in a diamond raises one distance by one unit and lowers the other by unit, the sum of distances remains the same. A rough drawing of the ellipse can be obtained by connecting these points.
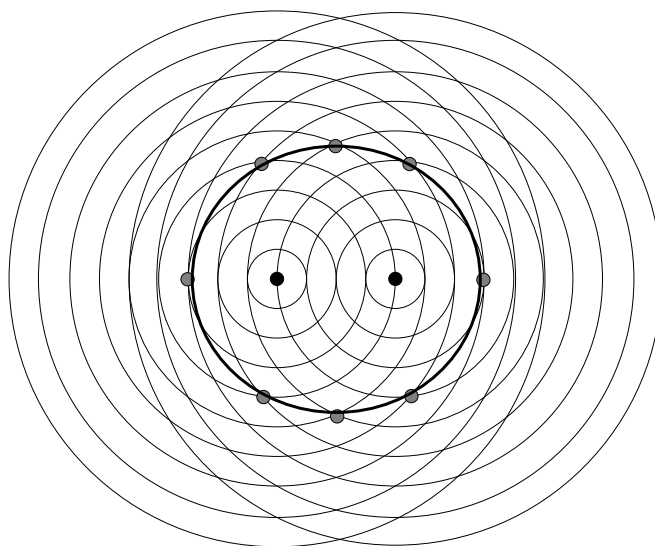
Problem 6.4.1. Draw the ellipse whose foci are shown that consists of points whose distances to the foci sum to 14 units.
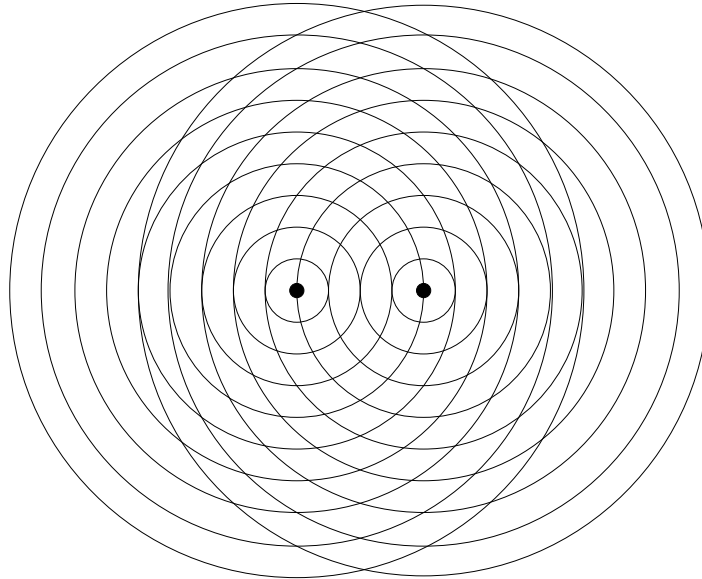
Problem 6.4.2. Draw an ellipse with transverse diameter 10 units and whose foci are 4 units by first drawing the rectangle that the ellipse is inscribed in. (Hint: to find the dimensions of the rectangle, use the fact that the intersections of the ellipse with the rectangle are points of the ellipse, and therefore meet the conditions in the definition of the ellipse.)

Problem 6.4.3. This problem, which requires internet access, explores how the transverse diameter (sum of distances from point on ellipse to the foci) and the distance between the foci affect the shape of the ellipse. Click here for the applet if you have internet access.

Parabolas

Children are familiar with *parabolas* as the shape of paths of thrown objects. For example, if a child throws a ball to another then the path taken is approximately a parabola.

More precisely, the parabola with focus point P and axis L is the set of points such that the distance to P is equal to the distance to L.

Here is a sample problem: Give a rough drawing of the parabola whose focus and axis are shown.

A sample answer is as follows: Draw circles with radii $1, 2, 3, \ldots$ units centered on the focus, and lines at unit intervals parallel to the axis. The parabola then passes through those intersection points corresponding to equal distance to the focus and axis.
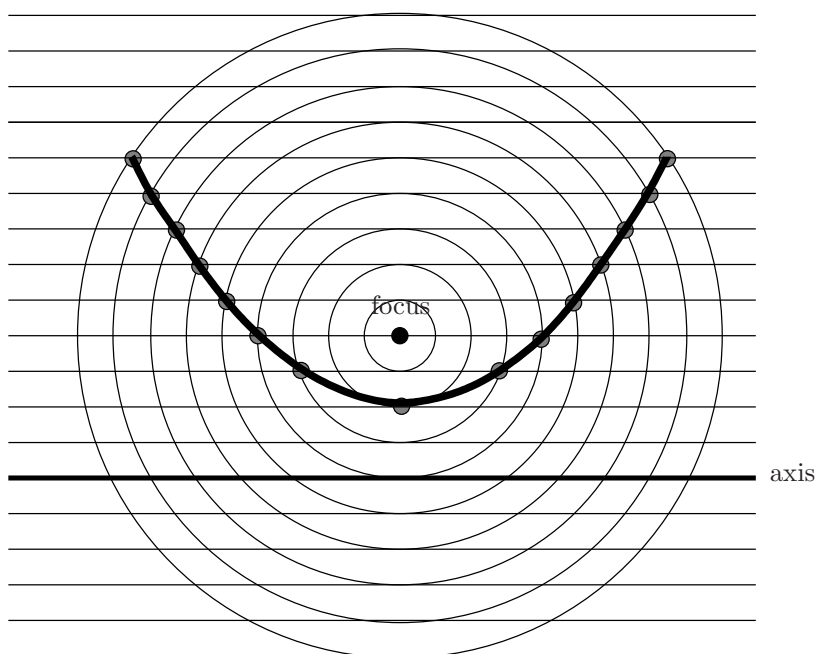
Problem 6.4.4. Give a rough drawing of the parabola whose focus and axis are shown.
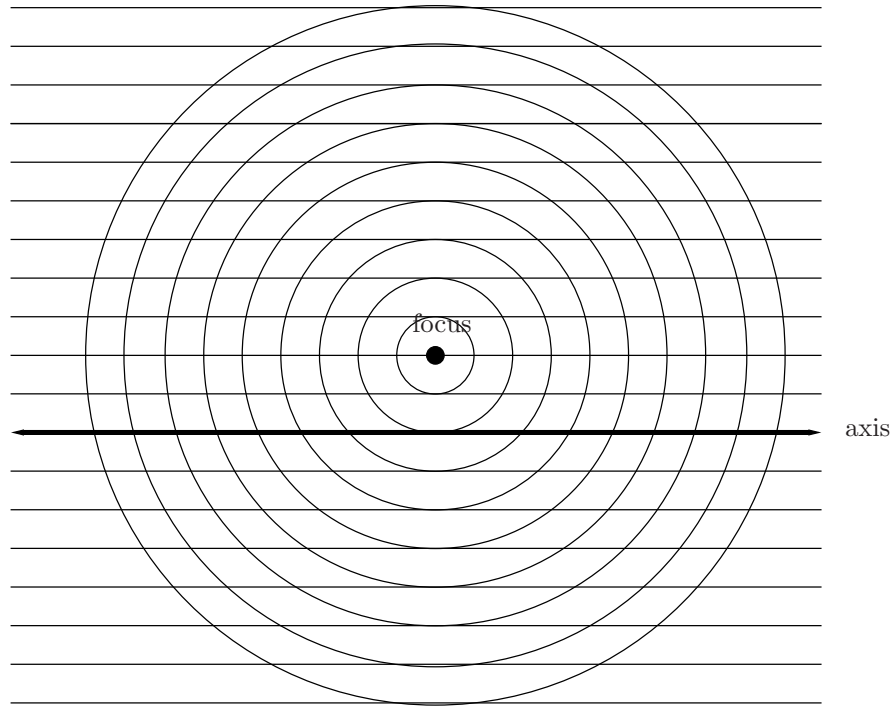
Problem 6.4.5. The following GeoGebra activity explores how the relationship between the focus and the directrix affect the shape of a parabola. It requires internet access and can be accessed here.

Hyperbolas

The shape of a *hyperbola* is less familiar to children than that of ellipses and parabolas. Cutting a cone vertically along a plane that does not contain the vertex of the cone produces a hyperbola. What might be more familiar to a child is the part of a wall illuminated by a flashlight: as the flashlight is angled to become parallel to the wall, the illuminated part changes from an ellipse to a hyperbola. In nature, hyperbolas occur as paths of objects in space that are not trapped into orbit.

FIGURE 80. A flashlight hyperbola
[22]

More precisely, a hyperbola with focal points C_1, C_2 and focal difference d is the set of points P such that the difference between the distances PC_1 and PC_2 is d. See Figure 81.

There are always two component curves to a hyperbola. To roughly draw a hyperbola, draw two bisecting lines (to be the *asymptotes* of the hyperbola) then the hyperbola, which gets closer to the lines "further out".
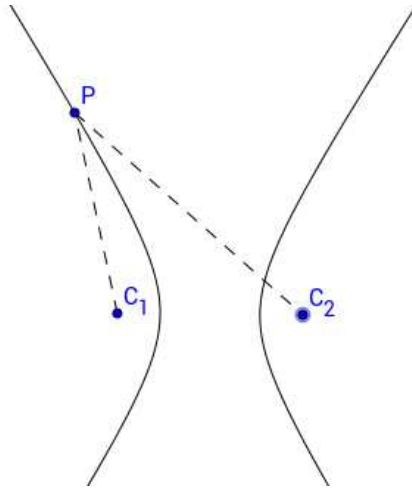
Figure 81. Part of a hyperbola

Here is a sample problem: Give a drawing of the hyperbola whose foci are 4 units apart, such that the difference of distances to the foci is 2 units.

A sample answer is as follows. Draw circles with radii $1, 2, 3$ units centered on the first focus and second focus. The intersection of the third circle around the first focus, and the first circle around the second focus, has distance 3 to the first focus and 1 to the second focus. The difference between the distances is 2, so this point lies on the hyperbola. Similarly, the intersection of the 4th circle around the first focus and the 2nd circle around the second, the intersection of the 5th circle around the first and the 3rd around the second are all on the hyperbola. You can make the picture more accurate by drawing circles with radii $1, 1.5, 2, 2.5$ units if necessary.
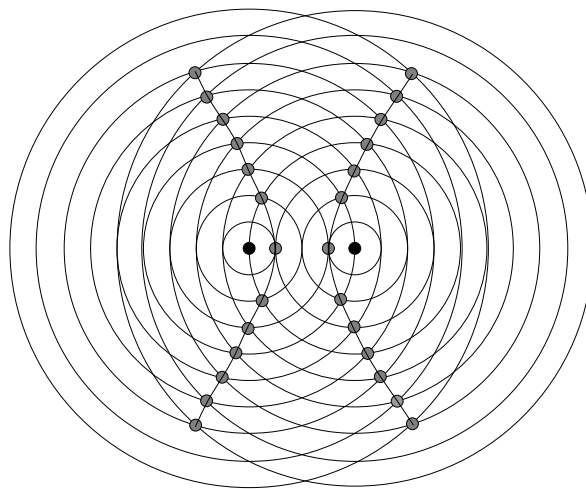
Problem 6.4.6. Give a rough drawing of the hyperbola whose foci are 4 units apart consisting of points whose distances to the foci have difference 1 unit.
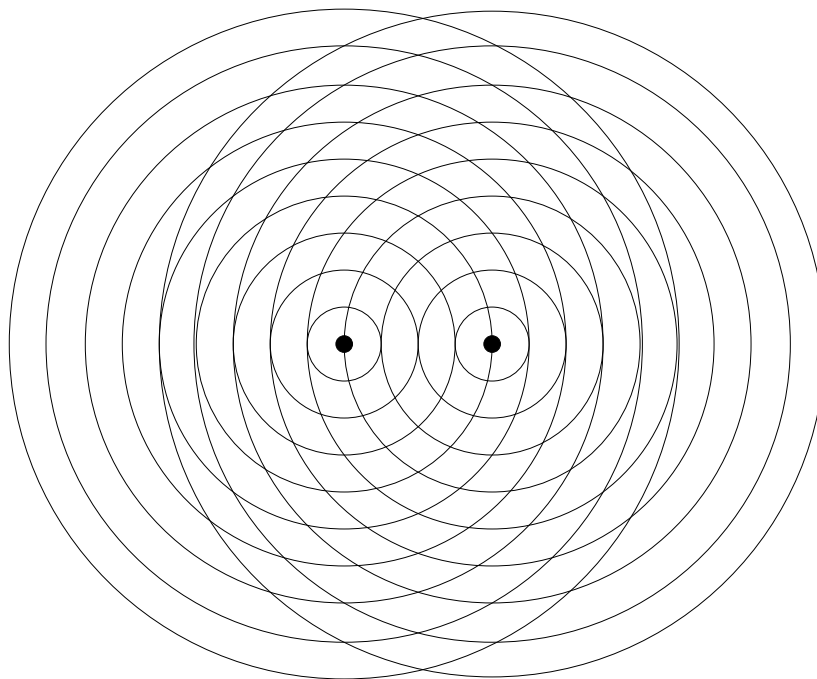
Problem 6.4.7. The following GeoGebra applet allows the user to investigate how the focal distance and the distance between the focal points affect the shape of a hyperbola. It requires internet access and can be accessed here.

The orbits of planets, asteroids, spacecraft and comets in our solar system that repeat are all elliptical, with one of the foci of the ellipse at the position of the sun. However, a hyperbolic path is also possible, if the asteroid or spacecraft goes back out into space. In fact, hyperbolic orbits are often used by NASA spacecraft to "sling themselves" around planets.

[23]

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[23]Retrieved under creative commons license from https://en.wikipedia.org/wiki/Conic_section#/media/File:Conic_Sections.svg March 2016.
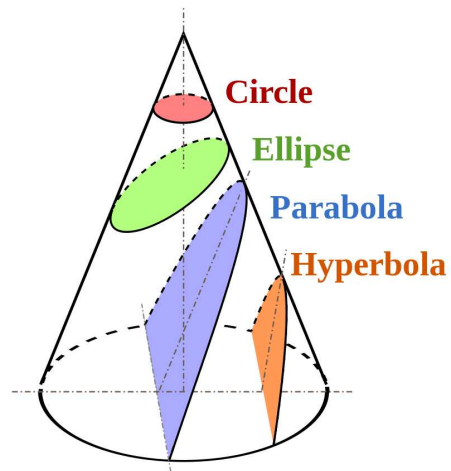
FIGURE 82. Conic sections

Circles, ellipses, parabolas, and hyperbolas can also be visualized as *conic sections*, that is, the set of intersection points of a cone with circular base and a plane. Figure 82 shows conic sections.

6.5. **Workshops.**

Workshop 6.5.1. (from [NJPEMSM])

(1) Smallville is 7 miles south of Gotham. Will is 8 miles from Gotham and 6 miles from Smallville. Draw a map showing where Will could be. (Try to make the figure colorful, sufficiently large, drawn precisely, and with suitable notation and labels.) Be sure to show a scale for your map. Explain your reasoning. How many possible locations are there for Will?

(2) What if Will is at most 8 miles away from Gotham and at most 6 miles away from Smallville?

(3) A new Giant Superstore is being planned somewhere in the vicinity of town A and town B, towns that are 10 miles apart. The developers will say only that all the locations they are considering are more than 7 miles from town A and more than 5 miles from town B. Draw a map showing town A and town B and all possible locations of the Giant Superstore. Be sure to show the scale of your map. Explain how you determined all possible locations for the Superstore.

Workshop 6.5.2.

(1) A reflecting pool will be made in the shape of the shaded region shown below. The square has side lengths 10 ft. What is the area and perimeter of the pool? Explain your reasoning.
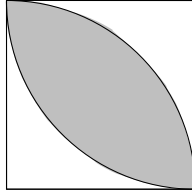
(2) A cafeteria has a recycling container for cans. The recycling container has a lid that is in the shape of a circle with an opening in the center that is also in the shape of a circle. The lid and its dimensions (6 in and 30 in) are shown in the diagram. Find the area and circumference of the lid. Explain your reasoning.

6 inches

30 inches
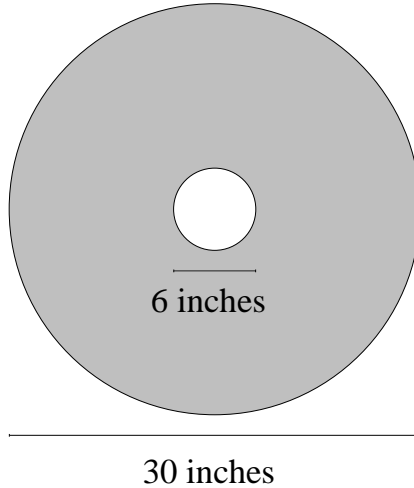
(3) Suppose that a forest on flat terrain has perimeter 20 miles, but there is no information on the shape of the forest. What can you say about the forest's area? Can you find a forest with area 20 square miles and perimeter 20 miles, and what would its shape be?

Workshop 6.5.3.

(a) Find the area of each region below, using the approximation $\pi \approx 3.14$. (Use the symbol $\approx$ in your work, and give two decimal places of accuracy. Make sure to use units correctly, e.g. $\pi r^2 = \pi(2 \text{ units})^2 = \ldots$.)
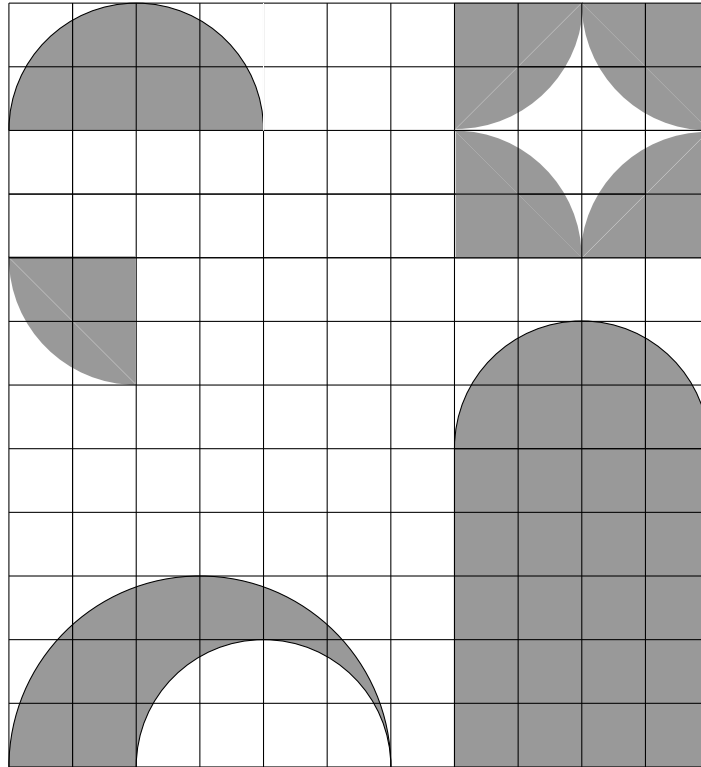
(b) (Adapted from [SM]) The figure is made up of two semicircles and a quarter circle. Find its area and perimeter.
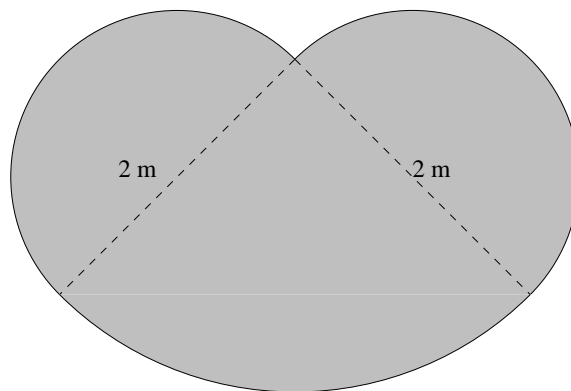
Workshop 6.5.4.

A pizza pie with circumference 30 inches is sliced into eight equal slices. Give answers to (a) and (b) that are *exact*, that is, not approximate. In each case, explain your answer.

(a) What is the perimeter of each slice?

(b) What is the area of each slice, in square inches?

(c) Tom eats one slice. What is the area of the pizza that remains?

(d) Jerry eats one slice, but leaves the outer one inch of crust which he doesn't like. What is the area of the crust of his slice that remains? (Colored paper and scissors may be helpful here.)

(e) What percent of the pizza area has Jerry "wasted", that is, not eaten?

(Here is another version of this problem: use a real pizza, if you can find one. Find the circumference first by measuring the radius and using the formula for circumference. Then gives approximate answers for each part above.)

Workshop 6.5.5.

For this problem use only compass and straightedge. This means that you should carry out the constructions only by drawing circles of arbitrary radius and drawing lines of arbitrary length; measuring distances, angles, or using a right angle are not allowed. If you are using Geogebra, this means that you should only use the line, line segment, or circle tools, and not any of the other tools like the polygon tool.

(1) Draw points A, B and a line segment connecting them. Construct (using circles!) a line that is perpendicular to the line segment $\overline{AB}$ and passes through the point A. Describe your steps and explain why the segment you have drawn is perpendicular to $\overline{AB}$.

(2) Using only circle and lines (not a right angle!) construct a square that has $\overline{AB}$ as one side. Describe your steps and explain why your figure is a square.

(3) Draw a circle. Then, by more circles and lines, construct a square whose vertices all lie on the circle. Describe your steps and explain why your figure is a square.

(4) Using the previous steps in the workshop, using circle and lines to construct a regular octagon. Describe your steps and explain why your figure is a regular octagon.

6.6. **Assessments and sample student work.**

Problem 6.6.1. (Adapted from [MCAS]) A cafeteria has a recycling container for cans. The recycling container has a lid that is in the shape of a circle with an opening in the center that is also in the shape of a circle. The lid and some of its dimensions are shown in the diagram below.

6 inches

30 inches
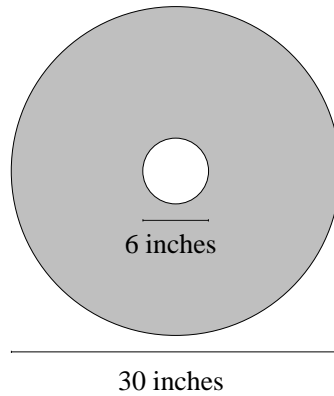
(a) Find the circumference of the lid. (b) Find the area of the lid, including the hole. (c) Find the area of the lid, not including the hole.

Some sample responses are given below. We warn the reader that the sample responses shown below are marked as if the question means the circumference of the outer circle. However, the question should really have specified which circumference is being talked about; annuli have both inner and outer circumferences.

Score 3/4. The student subtracted radii instead of areas to find the area of the difference.

A) C=πd   The circumference of the lid is
    C=3.14(30)   94.2 inches.
    C=94.2 in.

B) A=πr²   The area of the lid including the
    A=3.14(15)²   opening is 706.5 in².
    A=706.5 in²

C) A=πr²   The area of the lid not including
    A=3.14(12)²   the opening is 452.16 in².
    A=452.16 in²

I got the radius of 12 because if the original
diameter is 30 inches, I
subtracted the diameter of
the opening which is 6.
30-6=24. Radius is
half of the diameter, so
I divided 24 by 2 to
get a radius of 12 inches.

Score 2/4. This student got confused about which circumference was being asked for, perhaps because the question was posed vaguely. The student seems not to know how to do the last question.

a. $C = \pi d$
$C = 3.14(6)$
$C = 18.84$
The circumference is 18.84 inches.

b. $A = \pi r^2$
$A = 3.14(225)$
$A = 706.5$
The area is 706.5 inches.^{2}

c. $706.5 - 6^2 = 700.5 =^2$
The area is $700.5 =^2$

Score 1/4.

a.) $C = 2\pi r$
$C = 2(3.14)(15)$
$C = 94.2$ in.
The Circumference of the Recyling Container lid is 94.2 in.

b.) $A = \pi r^2$
$A = 3.14(15)$
$A = 471.^2$in
The area of the lid is 471 in^2.

c) $A = \pi r^2$
$A = 3.14 (15) - 6$
$A = 471.^2$in $- 6$
$A = 41.1$

The area of the lid not including the opening is 41.4 in^2

Score 0/4

A- $30 \times 6 = 180$
B- 30 inches
C- 6 inches

Problem 6.6.2. (Adapted from [MCAS]) The rear window of a car has the shape of a trapezoid with height 21 inches, bottom side with length 48 inches, and top side with length 36 inches. An 18-inch rear window wiper clears a 150° sector of a circle on the rear window, as shown in the diagram below.

150 deg   18 in
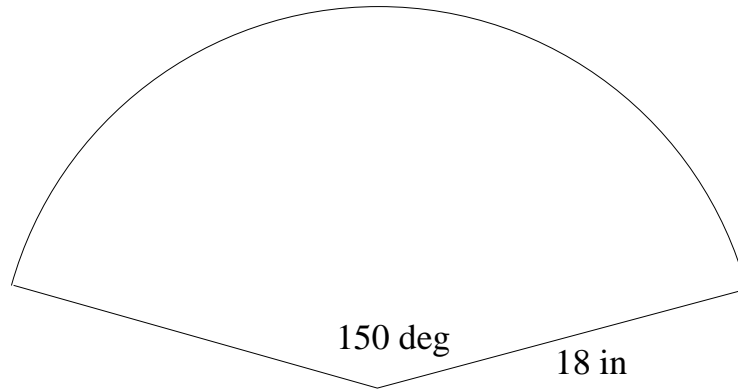
A. Find the area of the rear window.

B. What fractional part of a complete circle is cleared on the rear window by the 18-inch wiper? Show or explain how you got your answer.

C. What is the area, in square inches, of the part of the rear window that is cleared by the wiper? Show or explain how you got your answer.

D. What percent of the area of the entire rear window is cleared by the wiper? Show or explain how you got your answer.

Sample Answer for 3/4. In general incorrect responses showed lack of awareness of how to find the area of a portion of a circle.

7. MOTIONS

A *motion* such as a slide, turn, or flip is a change in position over time. Each figure changes under a motion to a possibly new figure. Some motions are *rigid*, which means that they involve no stretching. Others, like *dilations* (scaling), involve a *scaling factor* as well as a *center* representing the point from the stretching is done. The introduction of motions as a basis for reasoning in geometry is a major (and controversial) feature of the Common Core State Standards.

The objectives of this chapter are: (i) to introduce the notion and precise definitions for translations, reflections, rotations, and dilations; (ii) to describe how to draw these motions when applied to figures; (iii) to discuss the classification of motions of the plane; (iv) to introduce and recognize different kinds of symmetry; and (v) to describe how to draw a shape with given symmetry.

7.1. **Types of motions.**

By a *motion* of a figure we mean a way of changing every point to a (possibly different) other point so that no two points get changed to the same point. For example, rotations (turns), translations (slides), reflections (flips) and dilations (stretches) are motions. Folding in half is not a motion of the plane because two points are "folded" to the same point. See Figure 83 for examples.
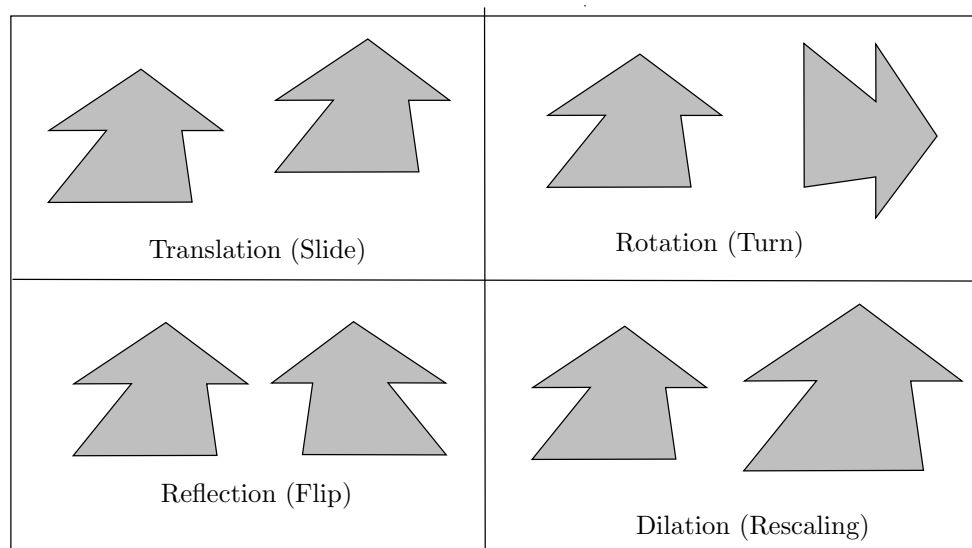
FIGURE 83. Types of motions

One can spend a long time talking about what is meant precisely by a motion (for example, is tearing allowed?), but since we mostly talk about rigid motions and dilations, we will avoid doing so. During board demonstrations of motions, Magna-Tiles may be helpful since they attach to most blackboards and can be easily slid,

turned, and reflected. Cutouts with colored paper with masking tape attached on one side may also be used for translations and rotations, but not reflections since the tape is only on one side.

Two children on a Ferris wheel or merry-go-round stay the same distance apart as the wheel turns. A *rigid motion* is a motion that preserves distances, that is, for *any* two points P, Q that are changed to points P', Q', the distance PQ is equal to the distance $P'Q'$. Motions that are rigid include rotations, translations and reflections. Other motions, such as dilations, do not preserve distances and so are not rigid. For example, a magnifying glass has the effect of dilating a figure: distances seem much larger when looking through the glass.

The following are some sample assessment problems on rigid motions of the plane. Answer them based on what you know already about such motions; the different types of motions will be described precisely in the following sections.

Problem 7.1.1. (Adapted from [NJ])

(a) Which motion shows *only* a slide:
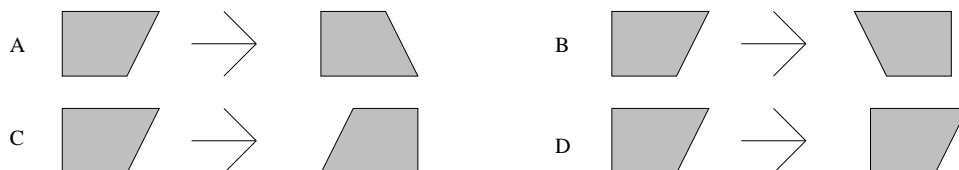
(b) Activity from [NJ]:

Which of the following describes the change from Figure 1 to Figure 2? A. slide B. turn right C. flip D. turn left

Figure 1   Figure 2
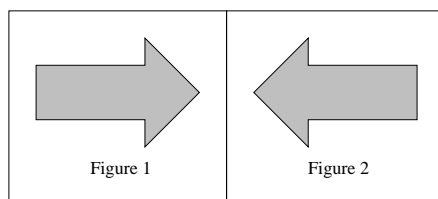
Problem 7.1.2. (Parts 1-3) For each picture below, help Alice to describe precisely how the position of Carlos has changed (comparing "before" and "after" pictures) using your own understanding about motions of a plane such as translations, rotations, and reflections. In each case, give a variety of descriptions, if possible. In the figure, the two ellipses sticking out of Carlos are his feet, seen from above.

Picture 1

Picture 2

Picture 3
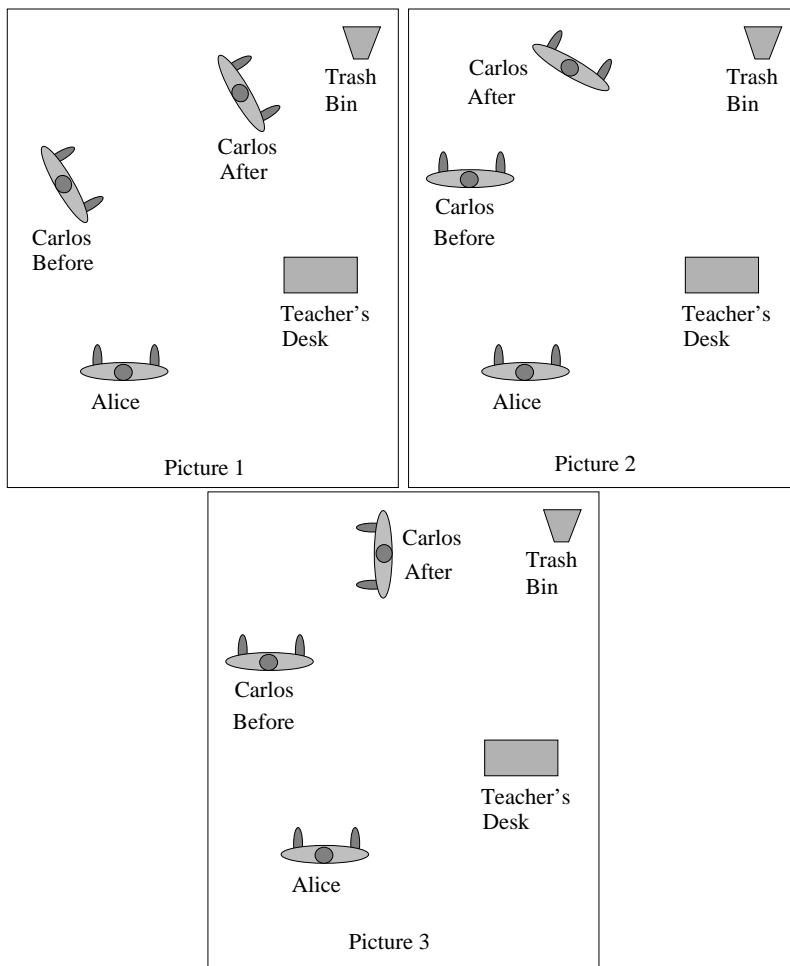
(Part 4) How would your answers change if Carlos were not a person but a (perfectly symmetric) ball?

There are three special kinds of rigid motions of the plane: translations (slides), rotations (turns), and reflections (flips). Each type of motion is specified by different information. A translation of the plane is specified by giving a *distance* and a *direction*. A rotation of the plane is specified by giving a *center*, *direction*, and *amount of rotation*. A reflection of the plane is specified by giving a *line* of reflection. At the end of the chapter, we also discuss dilations which are non-rigid motions. A dilation of the plane is specified by a *scaling factor* and a *center*.

7.2. **Translations.**

A translation is another name for a *slide*. Children will be familiar with the idea of
sliding an object along the floor without turning.

As suggested by the problems above, a translation is specified by a direction and
distance. More precisely, the *translation* of a point P by a distance D along a ray
R is the point obtained by moving from P by distance D along R.

Problem 7.2.1. Find the translation of the point P by the distance D and ray R
given below, using the following steps.

(1) Draw a passing through point P that is parallel to ray R. How do you know
whether the line you have drawn is parallel to the given ray?
(2) Find the point P' at distance D from P along the line L, in the direction
of R.

P •

D

R
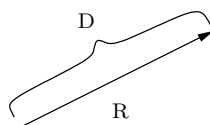
Translations of Figures

The *translation of a figure* means the translation of every point on the figure. To find the translation of a given polygon in the given direction, it's enough to find the translation of each of the vertices. This is because translations (as well as rotations, reflections, and dilations) transform line segments into line segments. An example is shown in the Figure 84. Each vertex has moved the same direction and distance.
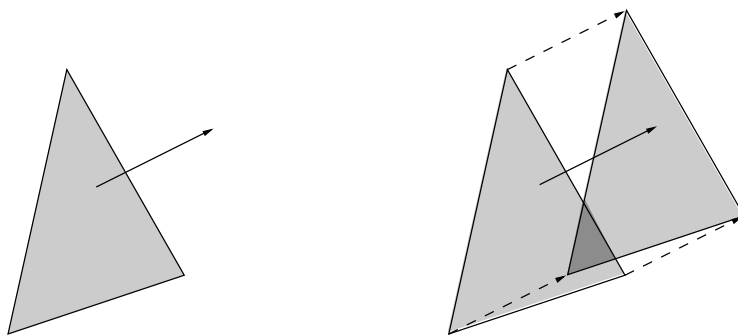
Problem: Draw the translation of the given figure by the given direction and distance (represented by the arrow)

Answer: The translation of the given figure is the new triangle in the figure above.

FIGURE 84. Finding translations of figures

If you have internet access you can click here to bring up a GeoGebra applet which illustrates a translation of a figure:

Translating a figure

Translations of figures can also be done colorfully at the board with the Magna-Tile manipulatives, as in Figure 85. [24] The advantage of using Magna-Tiles in this case is that they allow the instructor to demonstrate the motion in action, by sliding the original object along the drawn rays, and then leaving it on the board in the target position.

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[24]Photo credit I. Radu

FIGURE 85. Translating a Magna-Tile trapezoid

Problem 7.2.2. Find the translation of the given polygon in the given direction and distance.
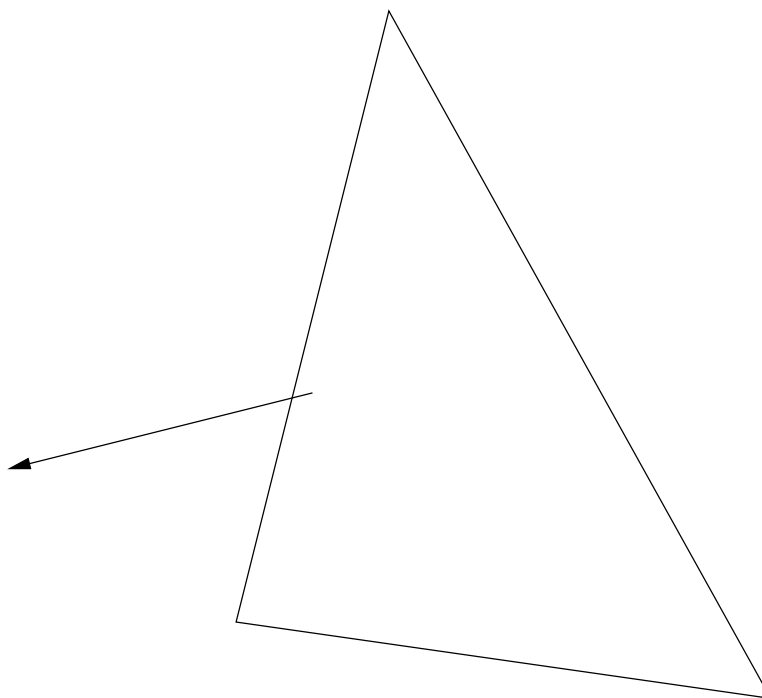
Problem 7.2.3. Start by drawing a simple shape or design that is not symmetrical.

(1) Translate your shape 2 cm east and then translate your resulting shape by 3 cm northeast.
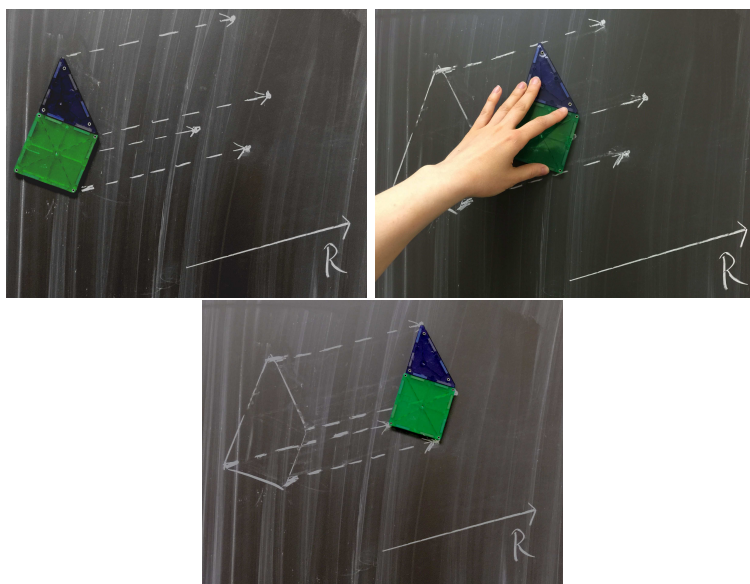(2) What single transformation (reflection, translation, or rotation) will take your initial shape to your final shape?
(3) Can you say anything about the effect of combining two translations in general?
(4) Does the order in which you do the translations matter? Explain your answer.

7.3. **Rotations.**

A rotation is another name for a turn. For the following two exploratory problems, use your own understanding about rotations.

Problem 7.3.1.

(1) Suppose two children sit on opposite ends of a see-saw. How could the movement of each child be described best as a rigid motion?
(2) How does the distance between the children change under the motion?
(3) Suppose a child holds an ice cream cone in her right hand with arm outstretched. What do you think will happen to the ice cream if the child rotates the cone by 90 degrees by rotating her arm by 90 degrees around her shoulder? Try a similar motion yourself with a pencil or pen representing the cone and discuss your answers.
(4) How does the angle formed by the cone and the arm change under the motion?

Problem 7.3.2. Using the following steps, find the rotation of the point P around the center Q by 60 degrees clockwise. You can use a ruler, compass and protractor as you see fit.

(1) Draw the line segment from P to Q.
(2) Find a ray that makes an angle of 60 degrees with the segment $\overline{PQ}$ in a clockwise direction.
(3) Find the point P' on the ray that is the same distance from Q as P.

P

Q

More precisely, the *rotation* of a point Q around a center P by an angle of measure a in a given direction (clockwise or counterclockwise) is the point R whose distance to P is the same as the distance from Q to P, and such that the angle from $\overrightarrow{PQ}$ to $\overrightarrow{PR}$ in the given direction has measure a. See Figure 86.
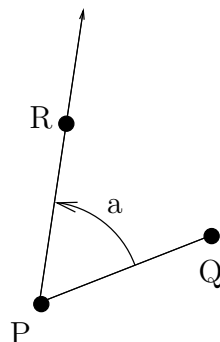
FIGURE 86. Rotation of a point counterclockwise

The *rotation of a figure* or object means the rotation of every point on the figure or object. Rotations of figures or objects usually change the "direction" that the figure or object is pointing. If you have internet access you can click here to bring up a GeoGebra applet which illustrates a rotation of a figure:

Rotating a figure

Problem 7.3.3. Find the rotation of the given shapes around the point P by an angle of 60 degrees counterclockwise.
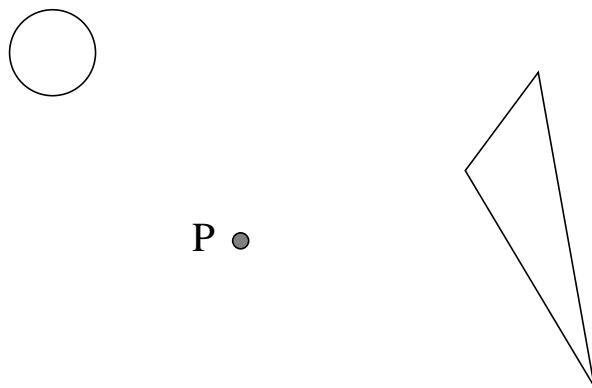
Problem 7.3.4. Start by drawing a simple shape or design that is not symmetrical.

(1) Rotate your shape around a point by 60 degrees clockwise, and then rotate the resulting shape around a different point by 60 degrees counterclockwise.
(2) What single transformation (reflection, translation, or rotation) will take your initial shape to your final shape?
(3) Can you say anything about the effect of combining two rotations in general?
(4) Does the order that you do the rotations matter? Explain your answer.

7.4. **Reflections.**

A reflection is another name for a *flip*. The *reflection* of a shape is often called a *mirror image*, in which the shape has been reflected over a line or plane (in three dimensions, the plane of the mirror). For example, the wings of a butterfly are *reflections* of each other over a line, namely the line corresponding to the body of the butterfly.

Problem 7.4.1. Based on your understanding of reflections in the plane, find the reflection of the point P over the line L given below using a right angle and ruler as appropriate:

(1) Find the line perpendicular to the line L that passes through the through the point P. Denote that line L'.
(2) On L' draw the point P' that is the same distance to L as P, but on the other side of L.
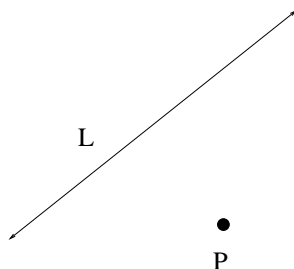
More precisely, the *reflection* of a point P over a line L is the point Q such that $\overleftrightarrow{PQ}$ is perpendicular to line L and the distance from P to L is the same as that from Q to L. The *reflection of a figure* means the reflection of every point on the figure.

To find the reflection of a given polygon over a given line, it's enough to find the reflection of each of the vertices.

Problem 7.4.2. Find the reflection of the shapes over the line shown.
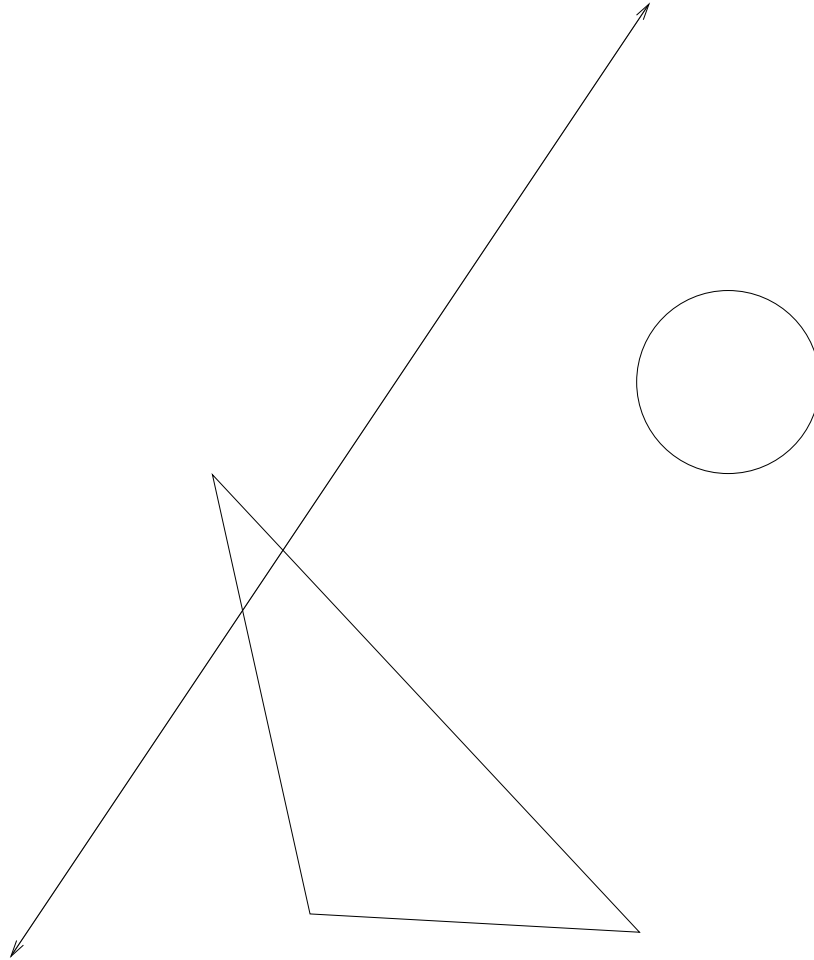
Reflections of figures can also be done colorfully at the board with the Magna-Tile manipulatives, as in Figure 87. [25]

Problem 7.4.3. Start by drawing a simple shape or design that is not symmetrical.

    (1) Reflect your shape over a horizontal line, and then reflect the resulting shape over a line making a 45 degree with the original line.

---

[25]Photo credit I. Radu
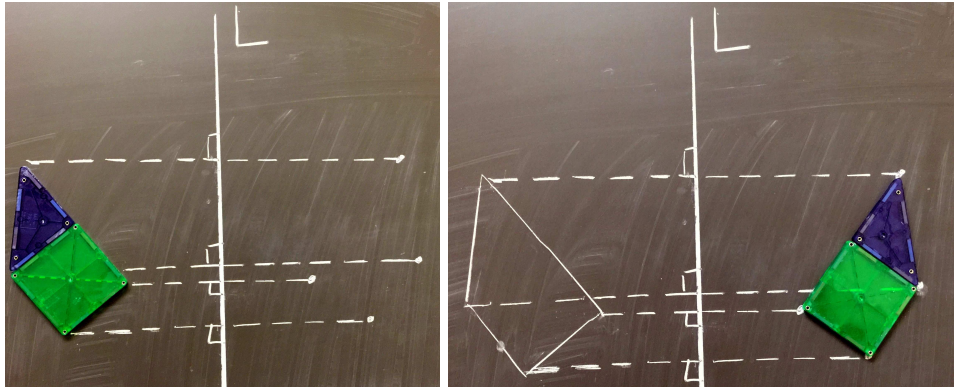
FIGURE 87. Reflecting a Magna-Tile trapezoid

(2) What single transformation (reflection, translation, or rotation) will take your initial shape to your final shape?
(3) Can you say anything about the effect of combining two reflections in general?
(4) Does the order that you do the reflections matter? Explain your answer.

The following problem provides more practice with the three motions covered so far.

Problem 7.4.4.

(1) Consider the translation T that moves point A to point B. Draw the result of moving the shaded region by translation T. Explain the steps you used to move the region.
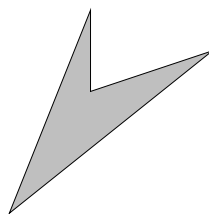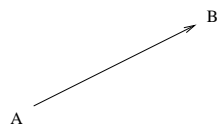
(2) Draw the result of flipping the shaded region below over the line given. Explain your steps.

(3) Draw the result of rotating the shaded region by 30 degrees in the clockwise direction around the given point. Explain your steps.
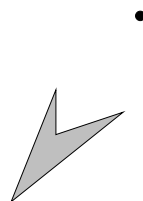
7.5. **Dilations.**

Dilations (rescalings) often occur in real life as *models* of larger objects. For example, toy cars are often made to a $1:24$ *scale*. If the length of the toy car is 6 inches then the length of the real car would be

$$
\begin{aligned}
6 \text{ inches} \ \times \ 24 \ &= \ 144 \text{ inches} \\
&= \ 144 \text{ inches} \times \frac{1 \text{ foot}}{12 \text{ inches}} \\
&= \ \frac{144}{12} \text{ feet} \\
&= \ 12 \text{ feet.}
\end{aligned}
$$

FIGURE 88. A dilation in real life

Problem 7.5.1. Based on your understanding of rescalings (dilations), which of the following are the same for a model car and the real car? Explain briefly.

(1) the ratio of the radius of the tires to the height of the car;
(2) the ratio of the circumference of the tires to the height of the car;
(3) the height of the antenna;
(4) the surface area of the car;
(5) the ratio of the radius of the tires to the volume of the car.

A *rescaling* or *dilation* by a factor s from a point P (called the *center* of the dilation) is a motion of the plane that changes the distance of any point to P by the factor s, and leaves the direction the same. In real life, we usually think of dilations as increasing the size, but in mathematics dilations can also decrease size, as in the case of the model car.

In the definition above, "leaves the direction the same" means that the center P, the point Q, and the rescaled point Q' are colinear. In Figure 89, the point Q' is the dilation of point Q from center P by a scale factor of 2.

Problem 7.5.2. Find the dilation of the point Q from the center P with scale factor 1.5 by the following steps:

(1) Using your ruler, draw the ray from P to Q and measure the distance between P and Q.
(2) Multiply the distance by the scale factor.

---

[26]Red car from Pixabay, released under Creative Commons CC0 into the public domain
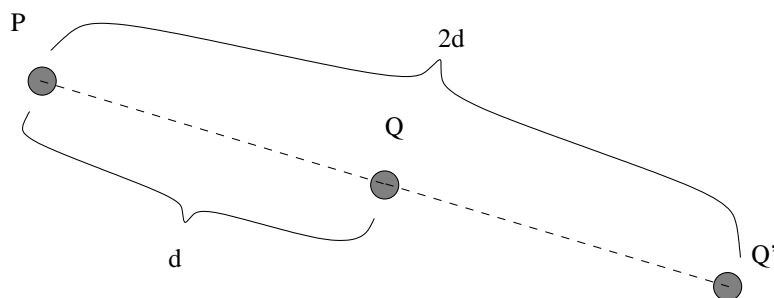
FIGURE 89. A dilation

(3) Using your ruler, draw point Q' on the ray $\overrightarrow{PQ}$, so that distance PQ' is the distance computed in step (2).

The *dilation of a figure* or object means the dilation of every point on the figure or object. A dilation is a motion that is not rigid, that is, it does not preserve distances between points in a figure. Instead, each linear measurement of the figure changes by the scale factor. If you have internet access you can click here to bring up a GeoGebra applet which illustrates a dilation of a figure.

<div align="center">Dilating a figure</div>

Problem 7.5.3. (We suggest doing at most two of the following):

(1) The following picture shows a square with a point at its center. Draw the square whose side lengths are twice as long as the shown square, with the same center.
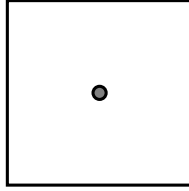
(2) The following picture shows a polygon with a point inside. Draw the figure obtained by scaling by a factor of two from the point.
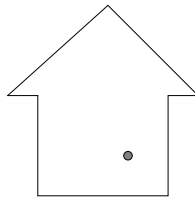
(3) The following picture shows a circle with a point inside. Draw the figure obtained by scaling by a factor of two from the point.
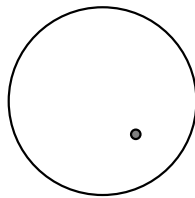
Dilations preserve angles

Suppose a toy sailboat is a model of a real life sail boat that is a dilation by some factor. The measure of the angle at the top of the sail is the same for the toy boat as for the real boat. This property of dilations is called *preservation of angle.* For example, a dilation of a rectangle is again a rectangle, and the dilation of an equilateral triangle is again an equilateral triangle, because the angles before and after the dilation have the same measure. In notation, if P, Q, R are points mapped to P', Q', R' under a dilation then $\angle PQR \cong \angle P'Q'R'$.

A viable argument for preservation of angles under dilation can be given as follows. For points P, Q, R that are mapped by a dilation with scale factor s to points P', Q', R', we will first show that angles $\angle PQR$ and $\angle P'Q'R'$ are congruent in a special case, namely when the center of dilation is Q. Figure 90 shows why the angles are congruent in this case: the rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$ stay the same under the dilation.
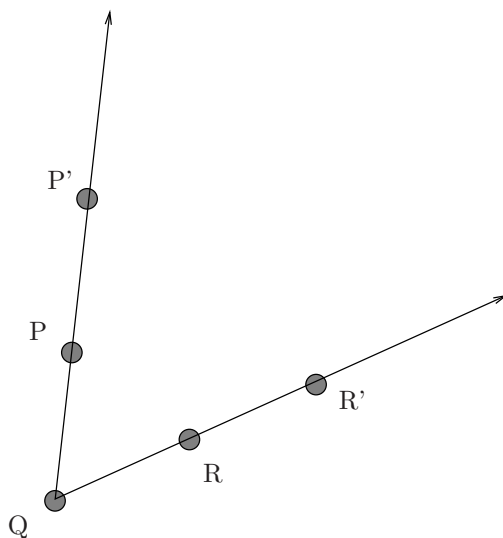
FIGURE 90. Any angle is unchanged by a dilation around its vertex

Next, note that two dilations by the same scale factor differ by a translation. For example, in Figure 91 the triangle $Q_1R_1S_1$ is the dilation of QRS from point P_1, while $Q_2R_2S_2$ is the dilation of the triangle QRS from point P_2 by the same scale factor. Since triangles $Q_1R_1S_1$ and $Q_2R_2S_2$ are congruent and have the same orientation, $Q_2R_2S_2$ is related to $Q_1R_1S_1$ by a translation.

Because of this, *any* dilation of $\angle PQR$ by factor s can be carried out in two stages: first, dilate by factor s from center Q, and then do a translation. Since dilation from Q preserves the angle $\angle PQR$, and translations preserve angles, so does any dilation.
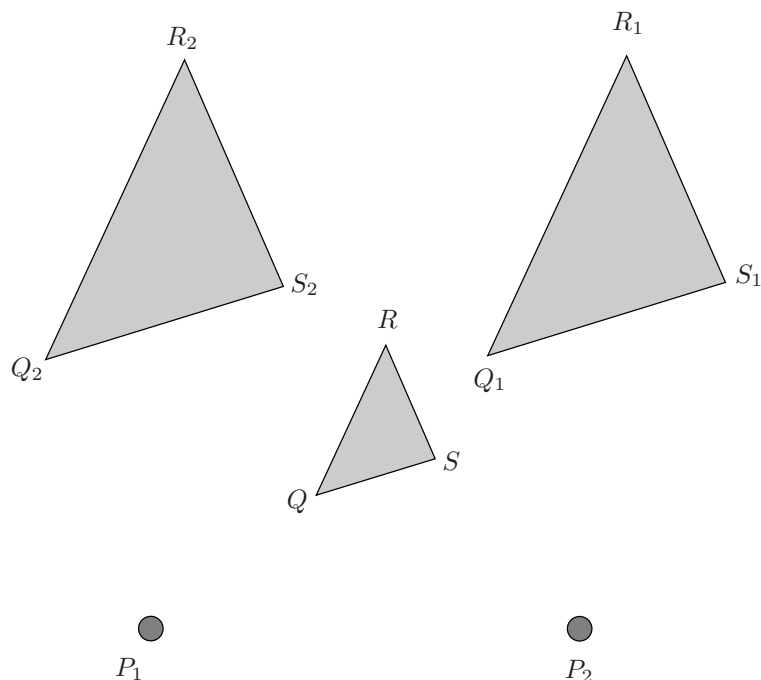
FIGURE 91. Dilations of a figure from two centers are related by a translation

In fact, any motion preserving angles is a combination of a rigid motion and a dilation; we will give a viable argument for this later. An example of a motion that does not preserve angles is the following: the motion of the plane that stretches by a factor of 2 in the x direction only. This is not a combination of a rigid motion and a dilation. For example, if a rectangle is stretched along the x-axis only, the angle formed by the diagonals changes.

Problem 7.5.4. Start by drawing a simple shape or design that is not symmetrical.

(1) Dilate your shape from a point with scale factor 2, and then dilate the resulting shape around a different point by a scale factor of $1/4$.
(2) What single transformation (reflection, translation, rotation, or dilation) will take your initial shape to your final shape?
(3) Can you say anything about the effect of combining two dilations in general? (Hint: it depends whether the two scale factors are reciprocals or not.)
(4) Does the order that you do the dilations matter? Explain your answer.

7.6. **Shears***.

A *shear* is an example of a motion that might be familiar to children who have seen sky-writing. At first, sky-writing is clear but after some time, differences in the wind speed distort the sky-writing. Here is a shear of the letter H, in a horizontal direction shown below the word. The top of the letter moves more to the right than the bottom.
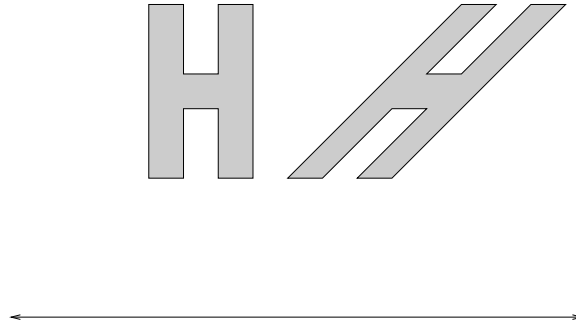
FIGURE 92. A shear

A shear is a motion of the plane that maps lines to lines but is not rigid. A shear is specified by a *shear factor* indicating the amount of shear, a *shear ray*, and a *side* of the shear ray. A *shear* with direction given by a ray $\overrightarrow{PQ}$ and shear factor c is the motion of the plane which moves each point R in the shear direction to a new point R' by a distance equal to c times the distance of the point R to the line $\overleftrightarrow{PQ}$, if the point is on the given side of the ray, and minus c times the distance of the point to the line containing the ray if the point R is on the other side of the line $\overleftrightarrow{PQ}$. See Figure 93 for an example where the shear factor is 1 and the given side of the ray is the shear direction.
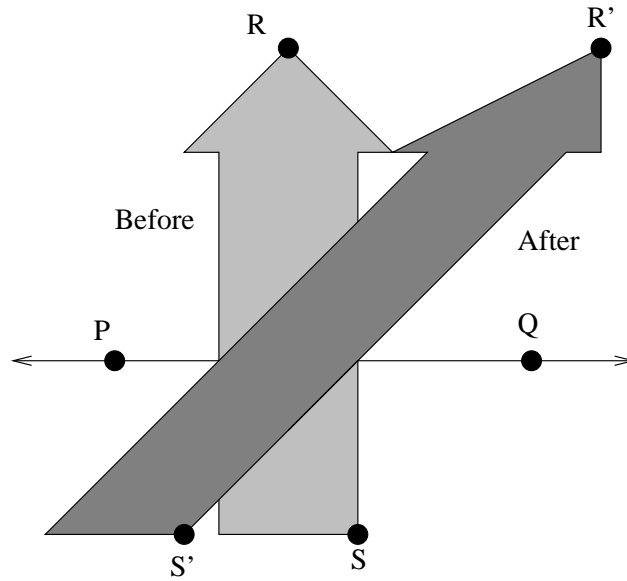
FIGURE 93. A shear

Shears do not preserve distance, but they do preserve area of a region. Break the region up into squares with one side parallel to the direction of the shear. The shear transforms each of these squares into parallelograms with the same base and height, hence the same area. Since the sheared region has area given by the sum of the areas of the parallelograms, it has the same area as the original figure.
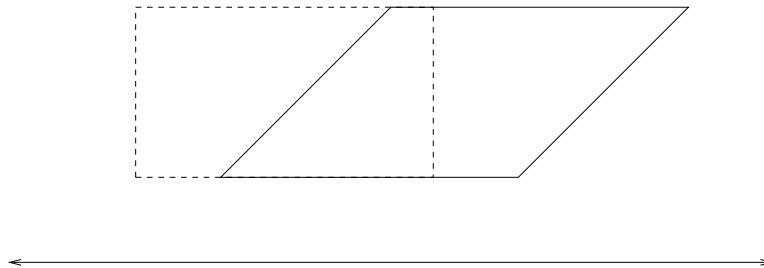
FIGURE 94. Areas of parallelograms are unchanged by shears

Problem 7.6.1. Find the shear of the figure shown below, along the line L shown below, so that

(1) The shear factor is 1;
(2) The shear factor is $-1/2$;
(3) The top vertex in the figure moves four units to the right.

L
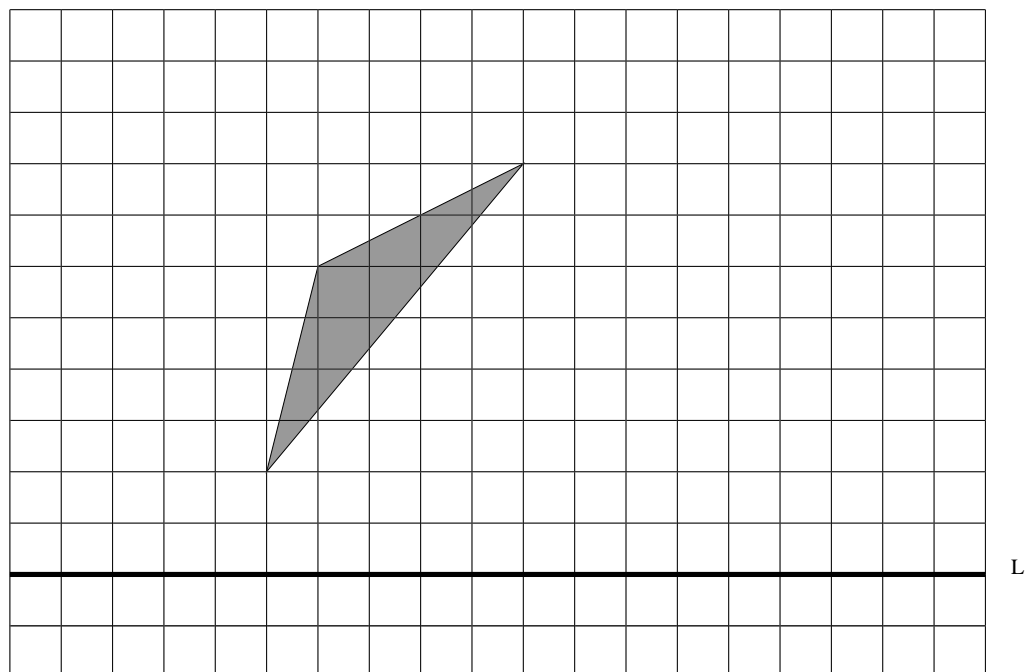
7.7. **Motions of a line.** Instead of starting with motions of a plane we could have first discussed motions that can be performed in a line (that is, without going outside the line.) Motions of a line include translations, reflections, and dilations. There are no "rotations" of a line, unless the angle of rotation is 180 degrees so that the line is turned around. However, a 180 degree rotation of a line is the same as a reflection over the center of the rotation.

Problem 7.7.1. In each case, find a combination of motions of the line (translations, reflections, and dilations) that transforms the points A, B to the corresponding points A', B', using what you understand already about motions. Describe each motion as precisely as possible: as a translation by a certain amount and direction, reflection over a specified point, or dilation from a specified center by a specified scale factor.

(a)

(b)

(c)

(d)

(e)
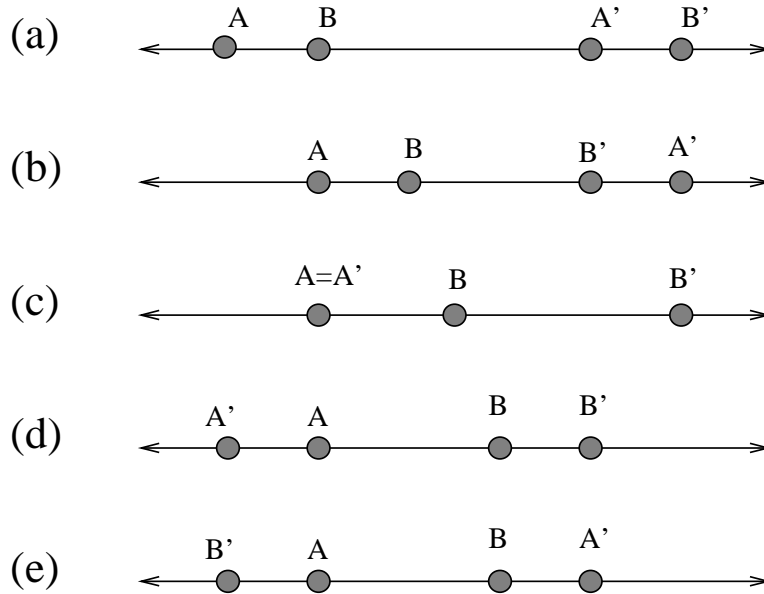
Motion	Type	Precise Description
(a)		
(b)		
(c)		
(d)		
(e)		

Note that each type of motion of a line is specified by different information. A translation of the line is specified by giving a *distance* and a *direction*. A reflection of the line is specified by giving a *point* of reflection. A dilation of the line is specified by a *scaling factor* and a *center*.

7.8. Symmetry.

Many objects in real life (butterflies, wallpaper patterns, wheels) have *symmetry*. A *symmetry* of a figure is a motion that leaves the figure unchanged.

The first type of symmetry we discuss is line symmetry. A *line symmetry* means that the figure is unchanged by reflection over the line. For example:

(1) The butterfly in Figure 95 has (almost exact) line symmetry.

FIGURE 95. Line symmetry in real life

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[27] Image "LucasMorphomenelaus" by Hippolyte Lucas retrieved from http://www.rareprintsgallery.com/. Licensed under Public Domain via http://commons.wikimedia.org/wiki/File:LucasMorphomenelaus.jpg#/media/File:LucasMorphomenelaus.jpg

(2) The letters C, I and O (in some fonts) have horizontal line symmetries, meaning they look the same when turned upside down.

(3) A square has four different kinds of line symmetry, as shown in Figure 96.
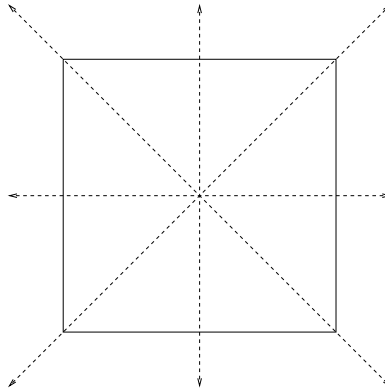
FIGURE 96. Multiple line symmetries

(4) The letters T, I and O have vertical line symmetries, meaning they look the same when flipped from left to right.

Problem 7.8.1. (Adapted from [VA]) What is the total number of lines of symmetry of the letter H?

Problem 7.8.2. Which of the following figures does *not* have a line of symmetry?
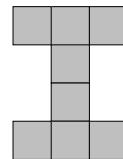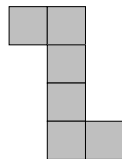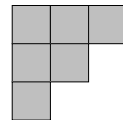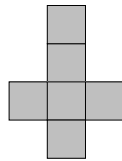
Problem 7.8.3. Activity from [TX]: Which statement is true about the letters A,D,H,M,X?

    (1) They do not have any lines of symmetry.
    (2) They all have at least 1 line of symmetry.
    (3) They all have exactly 1 line of symmetry.
    (4) They all have more than 1 line of symmetry.

Problem 7.8.4. Draw an equilateral triangle. Then, draw all its lines of symmetry.

Next we discuss rotational symmetry. If n is an integer then an *n-fold rotational symmetry* is symmetry around a point consisting of rotations by $360/n$ degrees. For example, squares have 4-fold while equilateral triangles have 3-fold symmetry around their centers.

Problem 7.8.5. Adapted from [CST]) Which figure has a line of symmetry and a rotational symmetry?

A   B   C   D
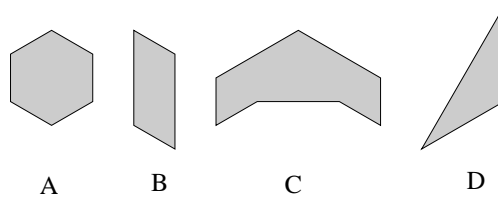
Problem 7.8.6. In the following, words may be written in upper case or lower case.

    (1) Identify an English word with vertical line symmetry.
    (2) Identify an English word with horizontal line symmetry.
    (3) Identify an English word with rotational symmetry.

Problem 7.8.7. If a figure has rotational symmetry with respect to an angle of 100 degrees, what other kinds of rotational symmetry does it have?

Finally we discuss translational symmetry. A *translational symmetry* means that a translational motion leaves the figure unchanged. Only figures that extend infinitely in both directions along some line can have a translational symmetry. For example, graph paper that extends infinitely horizontally and vertically has many translational symmetries (east, north, northeast, ...).

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Drawing figures with given symmetry

To draw a figure with given symmetry, start with a given figure, then apply to it each motion corresponding to the symmetry required by the problem until the figure is symmetric in the required ways. (In principle, this might take infinite time for something like a translational symmetry, but eventually one runs out of page space and can stop the drawing.) The *larger figure* (consisting of all the copies of the original figure) has the required symmetry.

For example, suppose we want to create a figure with a given line symmetry. Draw a figure on one side of the line and then reflect it to create a larger figure (the resulting kite) with reflection symmetry.
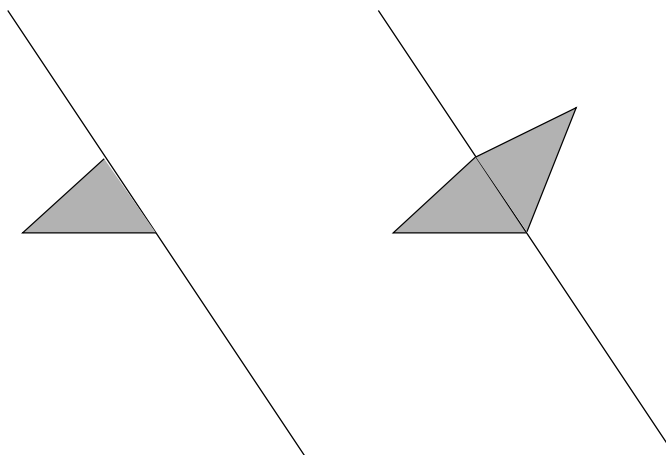
FIGURE 97. Creating figures with line symmetry

Similarly, to draw a figure with a given rotation symmetry we start with a given figure and then rotate it a number of times by the given angle to create a larger figure (starry shape) with the specified rotation symmetry.

To draw a figure with a given translational symmetry we start with a given figure and then (pretend to) translate it infinitely many times in the given direction and by the given distance to create a larger figure (staircase shape) with the specified translation symmetry.

To create figures with several different symmetries, one applies each symmetry repeatedly and hopes that one will obtain a final figure which has the required symmetries.[28]

---

[28]Certain combinations of symmetries cannot be achieved in finite time. For example, if two lines create an irrational angle measure (in degrees) then the only closed figure with both line symmetries is the circle whose center is the intersection of the lines; starting with any other initial figure and applying the symmetries repeatedly will take infinite time.
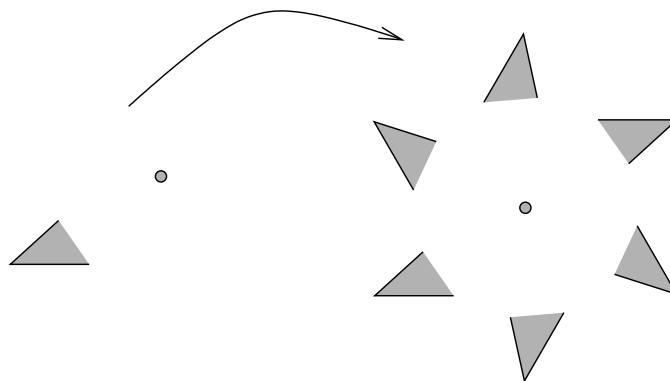
Figure 98. Creating figures with rotational symmetry
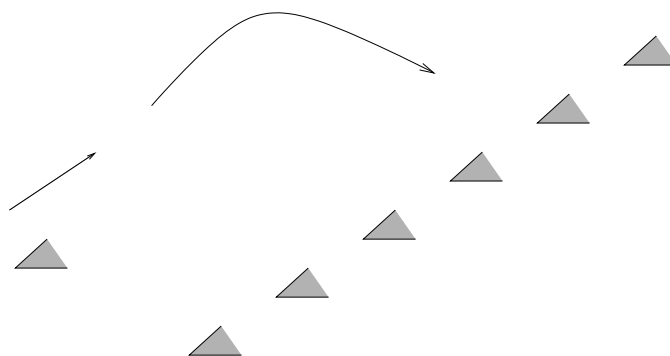
Figure 99. Creating figures with translational symmetry

Drawing figures with several different symmetries can take a long time by hand, even if possible. GeoGebra can help draw the figures more quickly. For example, Figure 100 shows a figure with both reflection and translation symmetry obtained by starting with a triangle and repeatedly translating and reflecting.
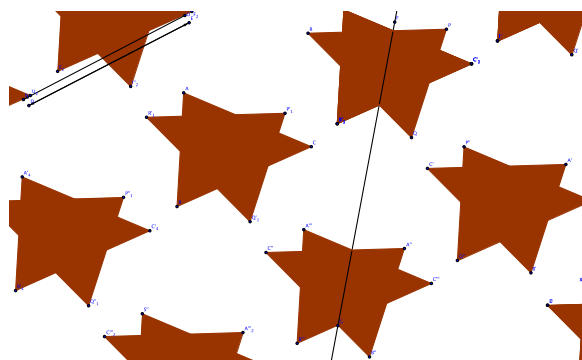
Figure 100. Creating figures with multiple symmetries

Problem 7.8.8. Draw (on paper or with GeoGebra)

(1) a figure with 3-fold symmetry;
(2) a figure with line symmetry;
(3) a figure with both 2-fold and 3-fold symmetry around the same point; (What symmetry does the final figure have?)
(4) a figure with a 3-fold symmetry and a line symmetry;
(5) a figure with a line symmetry and a translation symmetry;
(6) a figure with a 3-fold symmetry and a translation symmetry.

Symmetry is a useful method of *solving problems* in geometry. For example, if two line segments are related by a symmetry then they have the same length, and so if you have already measured one length then you don't need to measure the other. Similarly, the area of a two-dimensional shape with reflection symmetry over a line can be found by finding the area of one-half of the shape (the part on one side of the line) and then multiplying by two.

There are 17 different types of symmetries of the plane such as those above (that is, combinations of translational, rotational, and reflectional symmetry). The Wikipedia entry on wallpaper groups is a good place to learn more.

Problem 7.8.9. Find all possible symmetries in the following images, from Egypt and Islamic Spain (the Alhambra) respectively, ignoring colors. [29] Each design has line symmetries, translation symmetries, and rotation symmetries. Identify

(1) the lines of the line symmetries,
(2) the directions and amounts of the translation symmetries, and
(3) the centers and angles of the rotation symmetries.
(4) How do the symmetries change if one requires the motions to preserve colors?
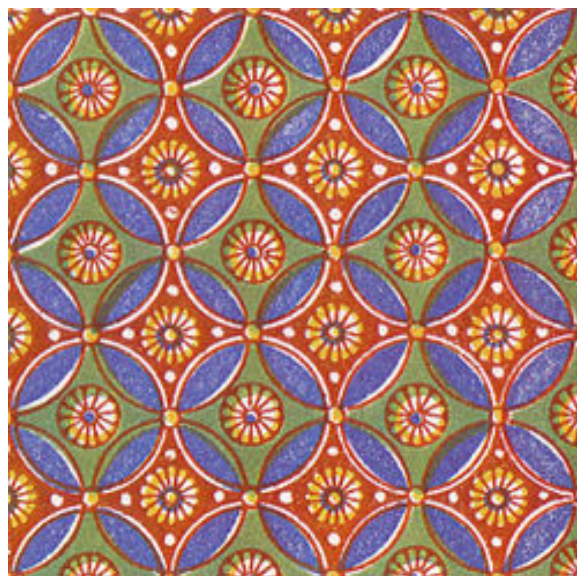
Problem 7.8.10. The following problem requires Geogebra. Using Geogebra's "image" button (on the same menu as ABC) make a wallpaper with an image of an animal or similar by the following steps:

(1) Import an image into Geogebra.
(2) By repeatedly rotating and translating your image (translations should be by the same vector), create a larger pattern with rotational and (approximate) translational symmetry. Here approximate means that as far as you can see on the screen, the image has translational symmetry.
(3) (Challenge) Make a pattern with (approximate) "dilational" symmetry, by repeatedly dilating your original image.

7.9. **Workshops.**

Workshop 7.9.1.

The classroom building has accidentally been built a long way from the ocean. The ocean is represented by the wavy lines in the picture below. Let's pretend that we can move the building using a rigid motion.

(a) Find a translation that moves the building within one inch of the ocean. Draw the translation reasonably precisely, and describe the direction and distance of the translation. Describe what tools you used to draw your answer.
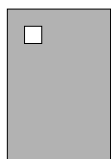
(b) Find a reflection which moves the building within one inch of the ocean. (The building does not have to look "straight up" after the reflection.) Draw the reflection reasonably precisely, and describe the line of reflection. Describe what tools you used to draw your answer.
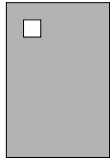
(c) Find a rotation which moves the building within one inch of the ocean. (The building will no longer be upright after the rotation.) Draw the rotation reasonably precisely, and describe the center, direction, and amount of rotation. Describe what tools you used to draw your answer.
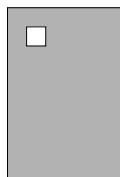
(d) Find a dilation which moves the building within one inch of the ocean. Draw the dilation reasonably precisely, and describe the center and scale factor. (Note that a dilation can *never* be specified only by the scale factor, unless the scale factor is 1.) Describe what tools you used to draw your answer.
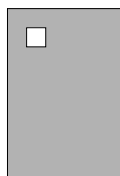
(e) (Challenge) Sometimes a *negative scale factor* is allowed in dilations, which means that each point moves to a point on the *other side* of the center of dilation. Find a dilation with negative scale factor which moves the building within one inch of the ocean. (Hint: the building will be upside down.)
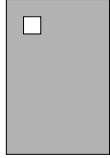
Workshop 7.9.2.

(1) Johnny's ice cream cone is sadly too small. Dilate his ice cream cone by a factor of two, using the vertex of the cone as the center.
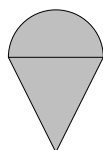
(2) Dilate the cone from (1) by a factor of two, using a center that is outside the cone.

(3) How do the two dilated cones relate? (Hint: it is a rigid motion.)

(4) The figure shows a point and an ice cream cone. Using your ruler, protractor, and/or compass draw the rotation of the ice cream cone around the point by 60 degrees counterclockwise. Explain in a few sentences which tools you used and how. (Hint: the ice cream cone will no longer be right-side-up.)
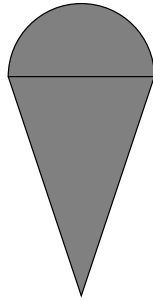
(5) Draw the reflection of the cone over the line shown.
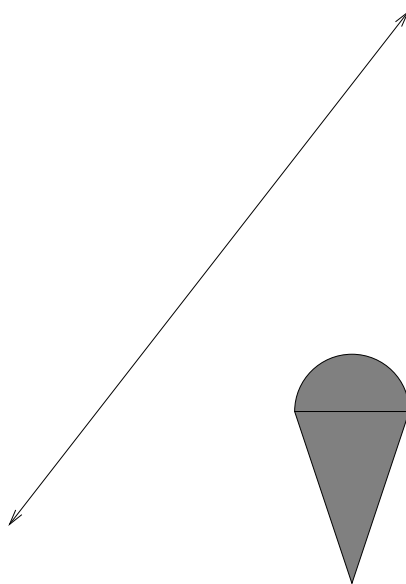
(6) Draw the translation of the cone by one inch up on the figure below.
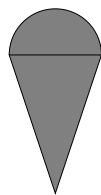
Workshop 7.9.3.

(1) On a new page draw a simple shape or design that is not symmetrical. What can you say about the net effect of rotating your shape about some point and then translating the resulting shape in some direction? Can the net effect be expressed through one single transformation (reflection, translation, or rotation)?

(2) On a new page draw a simple shape or design that is not symmetrical. What can you say about the net effect of dilating the shape from some center and then translating the resulting shape in some direction? Can the net effect be expressed through one single transformation (reflection, translation, rotation, or dilation)?

(3) On a new page draw a simple shape or design that is not symmetrical. What can you say about the net effect of rotating the shape around some center and then reflecting the resulting shape over a line? Can the net effect be expressed through one single transformation (reflection, translation, rotation, or dilation)?

(4) When we apply two transformations to a shape, does it matter which order we apply them? Explain your answer, discussing cases as necessary.

7.10. Assessments and sample student work.

Problem 7.10.1. Adapted from [MCAS]) Hexagon PQRSTU is shown in the diagrams below. In the diagram, Line 1 passes through the midpoints of $\overline{QR}$ and $\overline{UT}$. Line 2 passes through vertices R and U.
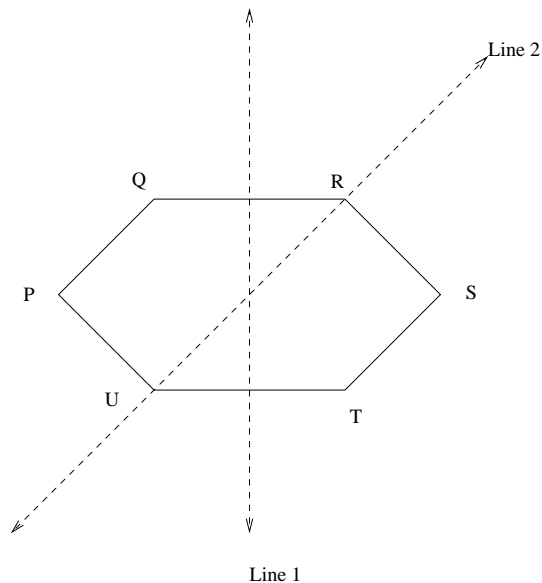
Line 1

1. Is Line 1 a line of symmetry? Explain your reasoning.

2. Is Line 2 a line of symmetry? Explain your reasoning.

3. Is there a line other than Line 1 or Line 2 that is a line of symmetry for hexagon PQRSTU?

(a) If there is another line of symmetry, describe where the line would be on the hexagon.

(b) If there is not another line of symmetry, explain why not.

Sample student work is below.

A. Yes because its split the hexagon in half so it could fold over and be the exactly same size as the other half. (example
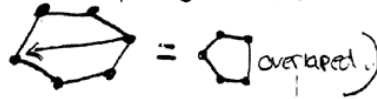
B. No because its split so that. the other has a bigger piece. So it won't match because the unsame size as the other part. (Example isn't the

bigger half

so it doesn't match
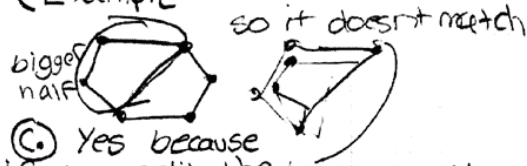
C. Yes because if you split the hexagon like this it would still be half. so it would be the same exact size as the other half. (Example
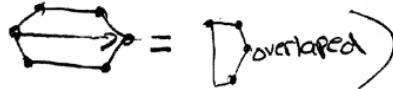
Many students were confused about the definition of line of symmetry. For example, line of symmetry was defined as a line that cuts the shape into two equal areas, or a line that goes from left to right that cuts the shape into two equal shapes.

8. Coordinates and the Pythagorean theorem

The path of a child on the way to school may consist of a number of blocks in one direction, followed by a number of blocks in another direction. So a location near a school can be specified by a pair of numbers, usually called coordinates, giving the number of blocks in each direction that the location is from the school. The distance between two locations given in coordinates can be computed by the *Pythagorean theorem*. Coordinates are also useful in describing motions. For example, the translation of a shape can be given by adding given numbers to the coordinates of each point in the shape.

The objectives of this chapter are: (i) to introduce number lines and coordinate axes; (ii) to discuss and justify the Pythagorean theorem; (iii) to discuss and justify the formula for the distance between two points given in coordinates; and (iv) to describe translations, reflections, rotations, and dilations of the plane in coordinates.

8.1. **Number lines, real numbers, coordinates.**

A *number line* is a line with numbers marked at regular intervals, so that the difference between any two numbers is proportional to their distance on the number line. Each number line has an *origin* represented by the number zero (which may not be shown) as well as a *unit length* representing the distance between any two consecutive integers. In the case of the number line below, negative numbers are represented to the left of zero, and positive numbers are represented to the right.

Figure 101. An example of a number line

For an example in real life, thermometers of the old style (using mercury to measure temperature) are marked with numbers indicating degrees in Fahrenheit and Celsius. See Figure 102. Note that 0 degrees Fahrenheit is not the same as 0 degrees Celsius. [30]

A frequent error that students make when working with number lines is putting the numbers in the spaces between hash marks, instead of at the hash marks.

Problem 8.1.1. For this problem you can use any tools you deem necessary.

(1) Draw 9 on the number line below:

15   18

---

[30]Downloaded from https://www.enasco.com/product/TB15831M June 2016. Used under Fair Use guidelines.

FIGURE 102. Number line on a thermometer

(2) Draw 0 on the number line below:

2

3.333.......

(3) What number does the point P below represent?

32

P

34
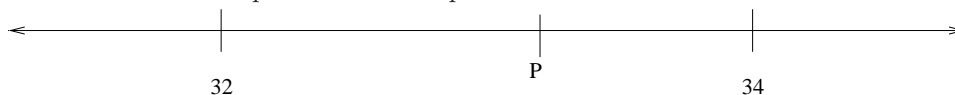
(4) Which of the questions above can be easily answered using only a compass?

Problem 8.1.2. Consider the number line below and a circle of diameter equal to 2 units (with respect to the unit of the given number line).

0   2   4   6   8   10
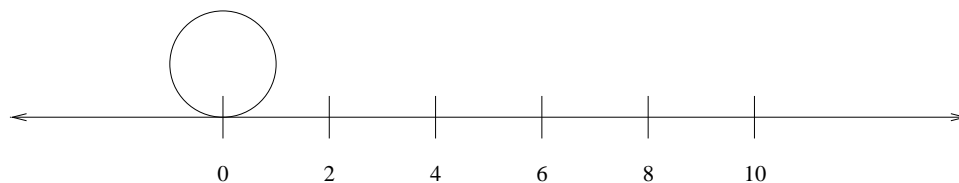
(1) If we straighten the circumference of the circle and place the resulting segment on the number line, starting at the origin, what is the coordinate of the endpoint of the segment?
(2) If we place the same segment on the number line starting at 1, what is the coordinate of the endpoint of the segment?

There is a one-to-one correspondence between real numbers and points on the number line. Recall that a real number is associated with a decimal expansion, either with finitely many or infinitely many digits. For example, any integer is a real number, as is any rational number. The number π is a real number that is not rational, with the decimal expansion $\pi = 3.14159\ldots$. The number π corresponds to a point on the number line in between the numbers 3 and 4, and is about 1/7-th of the way

from 3 to 4. The idea of a real number as cutting the number line into two pieces is commonly used in later math courses.

8.2. **Coordinates in the plane.**

To describe the position of an object on a plane one needs a *pair* of coordinates. *Coordinate axes* are perpendicular number lines intersecting at their origins. Any point in a plane equipped with coordinate axes is described by an ordered pair of real numbers, called the *coordinates* of the point. In notation this means that a point with coordinates (x, y) is obtained by moving x units in the horizontal direction and y units in the vertical direction from the origin. If a coordinate is negative, then the movement along the associated axis is opposite to the positive direction of the number line. As an example, the point $(-1, 2)$ and the square with vertices $(2, 2), (2, -2), (-2, -2), (-2, 2)$ are shown in Figure 103.
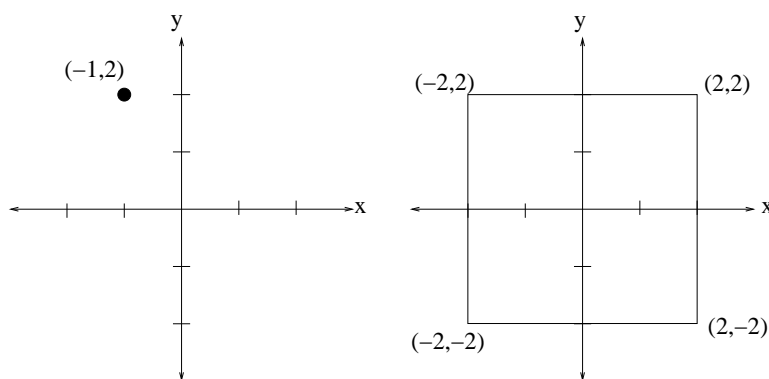
FIGURE 103. A point and a square on the coordinate plane

The terminology *ordered pair* is used on standardized assessments and should be explained to students. The coordinates of a point are the distances required to move from $(0, 0)$ to the point along each of the coordinate axes, in the positive or negative direction depending on the sign of the coordinate. Often the coordinate axes are labelled x, y.

The two number lines do not necessarily need to use the same unit length. For example, if the horizontal axis represents time and the vertical axis represents population of the United States, typical units might be 10 years for the horizontal axis and 1 million people for the vertical axis; these units do not have to be the same length.

Problem 8.2.1. Activity adapted from [MCAS]:

(1) Plot and label point P $(4, 6)$ and point Q $(4, 2)$ on your grid.
(2) Line segment $\overline{PQ}$ is one side of a rectangle. On your grid, draw a rectangle $PQRS$ with a length of 4 units and a width of 2 units.
  (a) Label point R and point S.
  (b) Write the coordinates of point R and point S.
(3) On your grid, draw the two diagonals of rectangle $PQRS$. What are the coordinates of the point where the two diagonals intersect? Explain your answer.
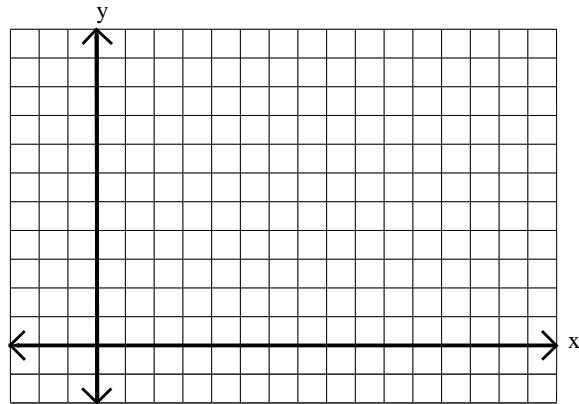
Sample student responses showed that many students know how to draw points with given coordinates, but not how to write down coordinates of given points.

8.3. **The Pythagorean theorem.**

The following is an exploratory problem meant to give some insight into the material developed later in this section.

Problem 8.3.1. Below is a right triangle with squares attached on each side.
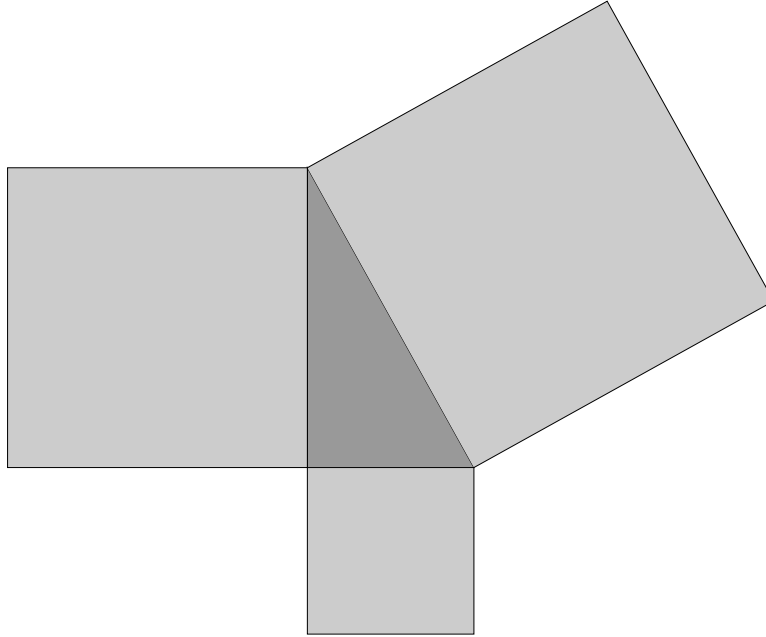
(1) By measuring, compute the area of each square. Make sure to work carefully with units.
(2) How do the areas of the three squares relate?
(3) Do you think that a similar relationship would hold for any triangle? How about for any right triangle?
(4) Can you think of a way of justifying your answer to part (c) using your understanding of lengths of sides of right triangles?

The Pythagorean theorem is a relationship between the lengths of the sides of a right triangle. There are several equivalent formulations.

> (Pythagorean Theorem) For any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

That is, the leg lengths a, b and hypotenuse length c of a right triangle satisfy the equality

$$a^2 + b^2 = c^2.$$

An equivalent form of the equation in the Pythagorean theorem above is

$$c = \sqrt{a^2 + b^2}.$$
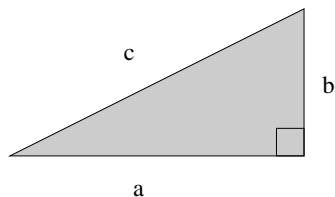
FIGURE 104. A right triangle with hypotenuse c and legs a, b

Working correctly with units in the Pythagorean theorem means putting the units inside the square root sign. For example, if a right triangle has one leg a equal to 2in and second leg b equal to 3in then the triangle has hypotenuse length

$$c = \sqrt{(2\text{in})^2 + (3\text{in})^2} = \sqrt{13\text{in}^2} = \sqrt{13}\text{in}.$$

However, even many mathematicians avoid this and just add the units in at the end to save time.

Problem 8.3.2. Consider the following four right triangles inside a square. Suppose that square has side lengths c, while the right triangles have legs a, b.

(1) What kind of quadrilateral is formed by the sides of the right triangles in the middle of the big square? Explain your answer.
(2) Using principles of area, explain how the areas of the shapes in this figure are related, and discuss their relation to the Pythagorean theorem, if any. [31]
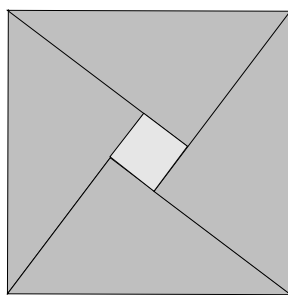
If you have internet access you can click here to bring up a GeoGebra applet which illustrates the Pythagorean theorem:[32]

Justifying the Pythagorean theorem

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[31] Adapted from the 2nd century BCE text Zhou Bi Suan Jing, see http://en.wikipedia.org/wiki/Zhou_Bi_Suan_Jing.

[32] See http://www.dam.brown.edu/people/mumford/blog/2015/Pythagoras.html for more on the history.

Problem 8.3.3. For each of the triangles below, give approximations to the missing lengths, if possible, without measuring. Explain your reasoning, in particular, how you identified any right angles. Make sure to work correctly with units throughout the problem.
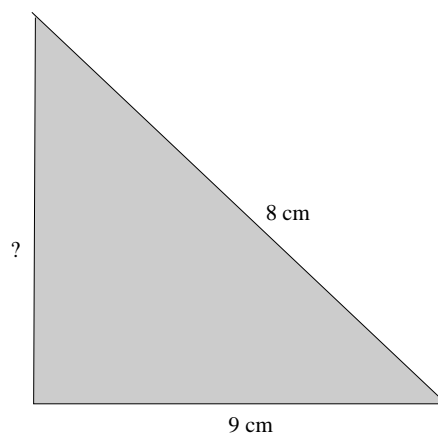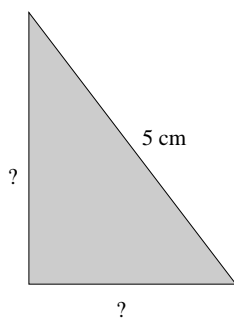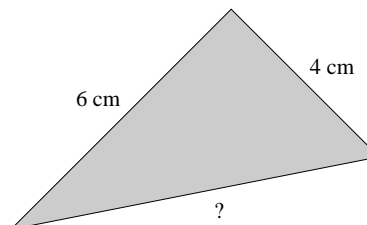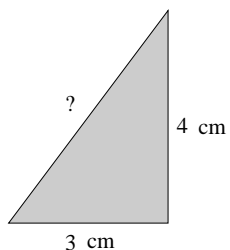
Problem 8.3.4. Here is a sample problem: In a right triangle, a leg is 6in long and the hypotenuse is 7in long. Find the length of the other leg.

Here is a possible student answer: $6^2 + b^2 = 7^2$ $=$ $36 + b^2 = 49$ $=$ $b^2 = 13$ $=$ $b = \sqrt{13} = 3.6$ in.

Discuss the correctness of the answer.

Problem 8.3.5. (Adapted activity from [TX]) Tai sailed east from a Marina for 48 miles, then south for 14 miles as shown in the Figure. What is the shortest distance Tai can sail to return to the Marina?

48 miles

Marina

14 miles
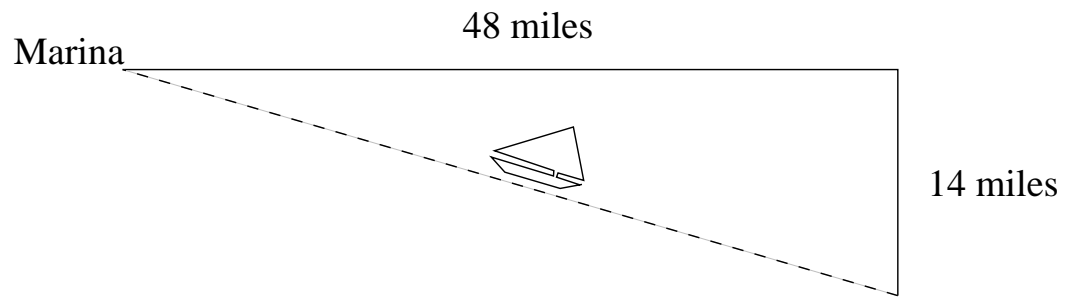
Problem 8.3.6. Find the distance between the points $(-1, 1)$ and $(2, -1)$ without measuring, ignoring units.
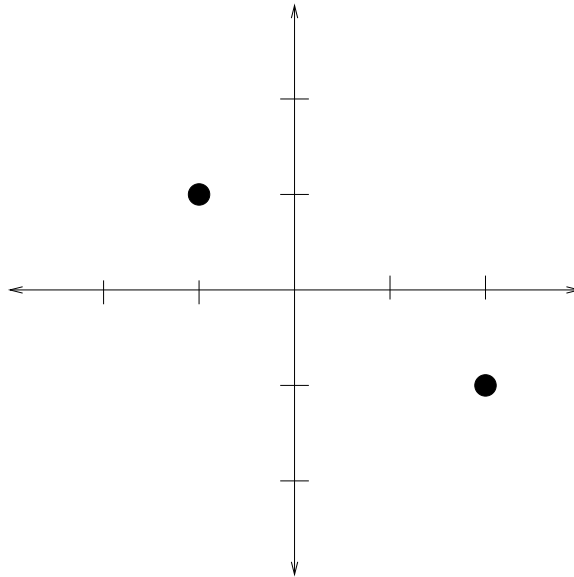
Finding distances using coordinates

The distance between points $(3, 1)$ and $(2, 5)$ is the hypotenuse of the right triangle with these points and also the point $(3, 5)$ as vertices. The leg lengths of this triangle are $3 - 2 = 1$, $5 - 1 = 4$.
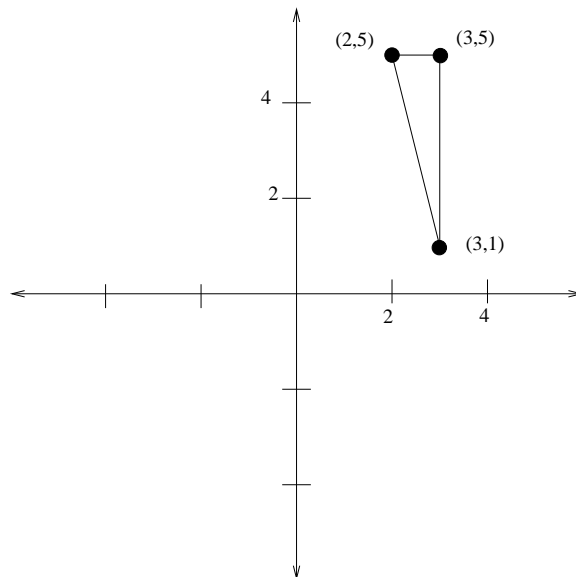
FIGURE 105. A triangle in the coordinate plane

By the Pythagorean theorem the length of the hypotenuse (ignoring units) is

$$\sqrt{(3 - 2)^2 + (5 - 1)^2} = \sqrt{1^2 + 4^2} = \sqrt{17}.$$

Distance Formula

The method used in the previous example works for finding the distance between any pair of points (x_1, y_1) and (x_2, y_2):
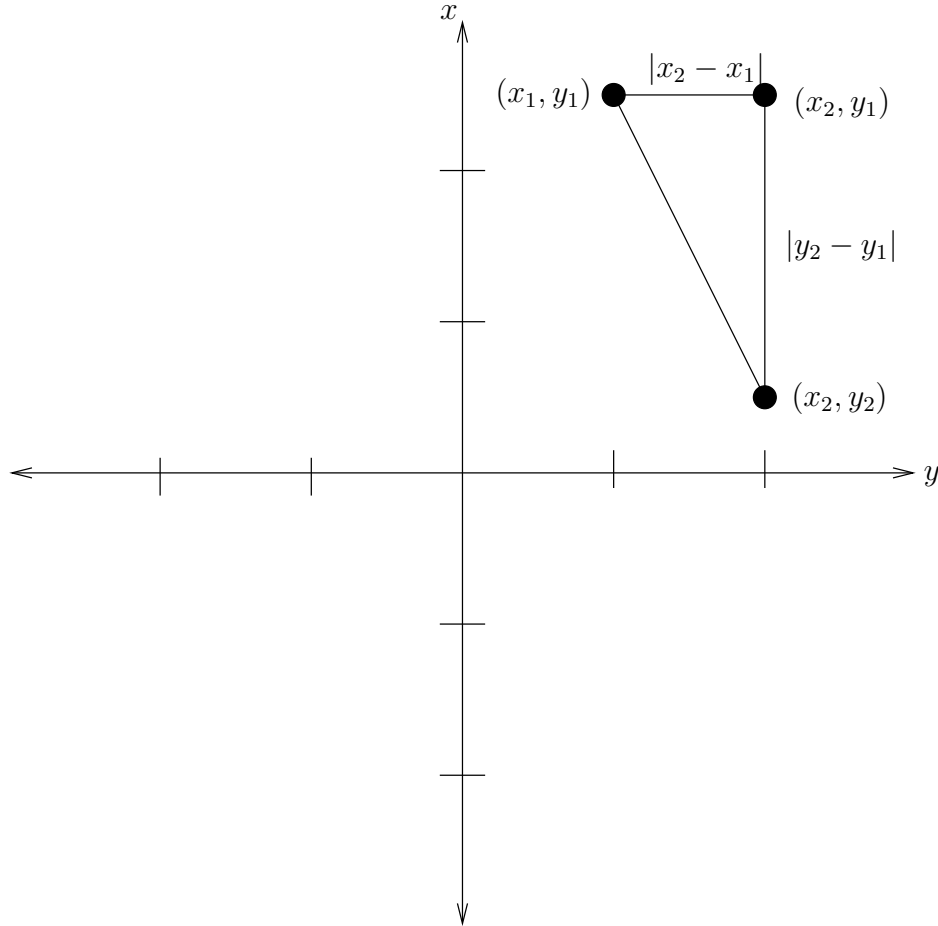
FIGURE 106. The distance formula via the Pythagorean theorem

The right triangle with hypotenuse from (x_1, y_1) to (x_2, y_2) has legs with lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. Applying the Pythagorean theorem to this triangle we obtain

$$\text{distance}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

since the absolute values disappear when taking squares. Taking square roots we obtain a formula for the distance between the two points:

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note that the distance formula only holds if the units on both axes are the same.

Coordinates in space

To describe the position of a point in space, a *triple* of coordinates is needed. A set of *coordinate axes* in space is a triple of perpendicular number lines meeting at the origin, as in Figure 107 (without hash marks, to simplify the picture). [33]
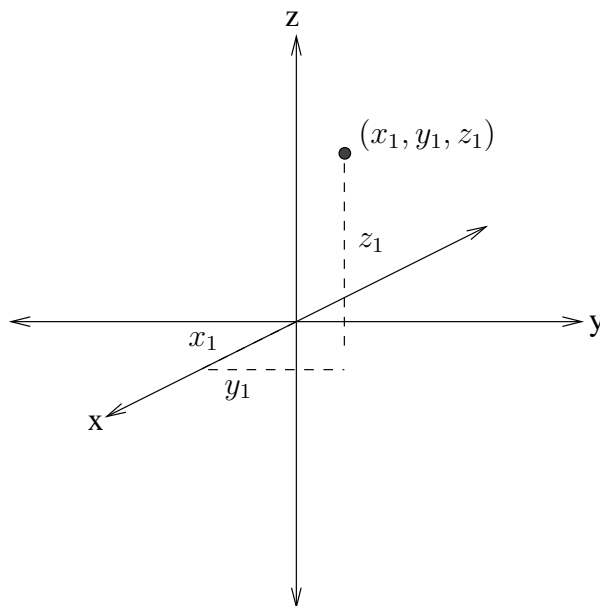
FIGURE 107. Coordinate axes in space

One can think of the three directions at the corner of a box as coordinate axes in the positive directions.

The *coordinates* of a point in space are the three numbers giving the distance necessary to move along each axis from the origin to the given point.

Problem 8.3.7. (1) A pencil box has dimensions 2 in $\times$ 2 in $\times$ 6 in. What is the longest pencil that can fit inside the box?
(2) Suppose two points in space have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Can you give a formula for the distance between the two points?

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[33]Retrieved from http://commons.wikimedia.org/wiki/File:Rectangular_coordinates.svg in October 2014

More activities on the Pythagorean Theorem

Problem 8.3.8. (Activity on special triangles)

(1) A triangle is a $45° − 45° − 90°$ *triangle* if it has interior angles of $45°$, $45°$ and $90°$.
   (a) Which of the following adjectives apply to a $45° − 45° − 90°$ triangle: isosceles, scalene, right, equilateral?
   (b) Find the side lengths of a $45° − 45° − 90°$ *triangle* with hypotenuse 1 unit.
(2) A triangle is a $30° − 60° − 90°$ *triangle* if it has interior angles of $30°$, $60°$ and $90°$.
   (a) Which of the following adjectives apply to a $30° − 60° − 90°$ triangle: isosceles, scalene, right, equilateral?
   (b) Find the side lengths of a $30° − 60° − 90°$ *triangle* with hypotenuse 1 unit. (Hint: show that two such triangles can be put together to make an equilateral triangle.)

Problem 8.3.9. In triangle $\triangle ABC$, D lies on the segment $\overline{AC}$ and $\overline{BD}$ is perpendicular to $\overline{AC}$. If AB is 6 cm, BC is 8 cm, and AC is 10 cm, what is BD?

The converse of the Pythagorean theorem

The Pythagorean theorem works only for right triangles. What happens if the triangle is not a right triangle?

Imagine starting with a right triangle with legs a and b, as in the figure below. Lowering the top vertex decreases a and b without changing c, and makes the top angle larger. The new triangle is an obtuse triangle. On the other hand, raising the top vertex of the right triangle increases a and b without changing c, and makes the top angle smaller. The new triangle is an acute triangle.
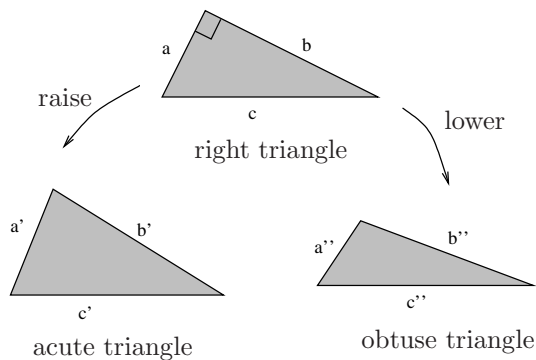
FIGURE 108. Understanding the converse of the Pythagorean theorem

This suggests that if c is the length of the longest side of the triangle then

$$a^2 + b^2 < c^2 \text{ for obtuse triangles}$$
$$a^2 + b^2 > c^2 \text{ for acute triangles}$$

For a more convincing demonstration, suppose that we start with a triangle with an acute angle between the sides labelled a and b as in Figure 109.
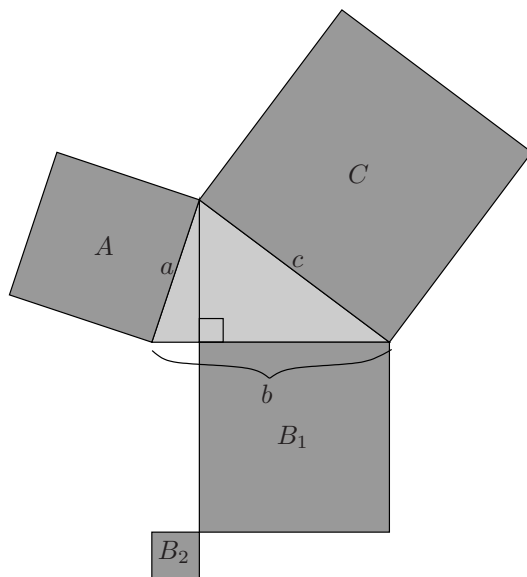
FIGURE 109. Understanding the converse of the Pythagorean theorem, again

Using the Pythagorean theorem twice, convince yourself that

$$C = A + B_1 - B_2.$$

Note the following facts. The areas of the squares at the top are $C = c^2$ and $A = a^2$. Also $B_1 - B_2 < b^2$ since B_1 is the area of a square that is smaller than that with side lengths b. This shows that

$$c^2 < a^2 + b^2$$

for triangles for which the edges labelled a, b form an acute angle. There is a similar discussion if the angle formed by a, b is obtuse.

To summarize, *if two sides of a triangle are not perpendicular, then the sum of the squares of their lengths is not the square of the length of the remaining side.* This is equivalent to the *converse of the Pythagorean theorem* which states:

If the side lengths of a triangle are a, b, c and $a^2 + b^2 = c^2$ then the sides with lengths a, b form a right angle.

8.4. **Motions in coordinates.**

Many motions such as translations have a simple description in terms of coordinates. That is, given the coordinates of a point (called the input) there is a rule for determining the moved point (called the output). The pair of points related by a motion is called an *input-output pair*.

Problem 8.4.1. Guess the rigid motion: The table below contains several input-output pairs.

(1) Deduce the rule and fill in the remaining parts of the table. For the blank rows, fill in both a possible input and possible output pair.

Input	Output	Notes
(3, 5)	(4, 7)	
(-2, 4)	(-1, 6)	
(-3, -2)	(-2, 0)	
(1, -4)	(2, -2)	
	(0, 0)	
	(5, 6)	

(2) What is an algebraic way to describe this rigid motion?

(3) What is a geometric way to describe this rigid motion?

Problem 8.4.2. Here is another rigid motion. The table below contains several input-output pairs.

(1) Deduce the rule and fill in the remaining parts of the table.

Input	Output	Notes / Rule?
(0,0)	(0,0)	
(3,5)	(3,-5)	
(-2,4)	(-2,-4)	
(-3,-2)		
(1,-4)		
	(6,7)	
	(-3,-4)	

(2) What points are not moved at all by this motion?

(3) What algebraic description does this motion have?

(4) What geometric description does this motion have (as opposed to an algebraic description)?

Problem 8.4.3. Here is yet another rigid motion. The table below contains several input-output pairs.

(1) Deduce the rule and fill in the remaining parts of the table. For the blank rows, fill in both a possible input and possible output pair.

Input	output	Notes / Rule?
(0,0)	(0,0)	
(3,5)	(5,3)	
(-2,4)	(4,-2)	
(-3,-6)		
(1,-4)		
	(-3,-4)	
	(5,6)	

(2) If the input is a particular point with coordinates (a, b), what is the output?

(3) What geometric description does this motion have (as opposed to algebraic description)?

(4) What points are not moved at all by this motion? Why?

Problem 8.4.4. The table below contains several input-output pairs.

(1) Deduce the rule and fill in the remaining parts of the table.

Input	Output	Rule?
(0,0)	(0,0)	
(1,0)	(0,-1)	
(2,0)	(0,-2)	
(0,1)	(1,0)	
(0,2)	(2,0)	
(1,1)	(1,-1)	

(2) What point(s) are not moved at all by this motion?

(3) What geometric description does this motion have (as opposed to algebraic description)?

(4) Suppose the input is (a, b). What algebraic expression gives the output? Explain why.

Translations in coordinates

Problem 8.4.5. The circles C and C' drawn below are related by a translation. Find three pairs of corresponding points relating circle C and C'. Then, guess the rule for the translation in coordinates that takes C to C'.
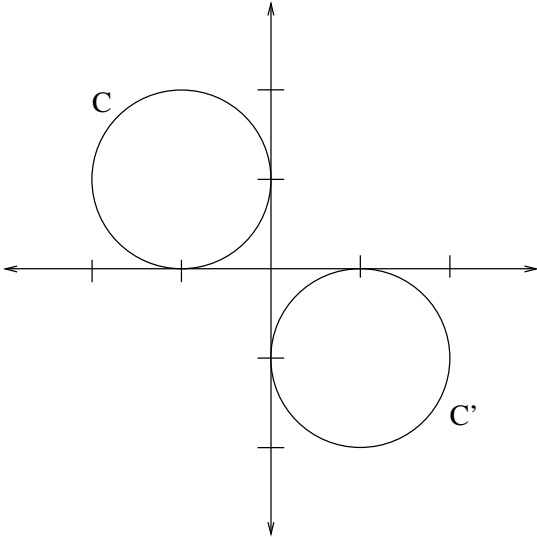
Point	Coordinates of point	Coordinates of corresponding point

Rule: $(x, y) \mapsto$

Reflections and rotations in coordinates

Reflections over certain lines have nice coordinate descriptions. There are also formulas for reflections in coordinates for arbitrary lines. These are outside the scope of this book and we will not describe them here. The following problem is an exploration of line reflections that can be nicely described in coordinates.

Problem 8.4.6. Find the coordinate descriptions of

    (1) Reflection over the y-axis
    (2) Reflection over the line $x = y$
    (3) Reflection over the line $x = -y$.

An easy way to investigate the change in coordinates under a certain rotation is by using a square piece of paper (see Figure 110). By placing one of its corners at the center of rotation and rotating the square in the direction indicated by the problem, one can easily see what happens to the coordinates of key points such as $(0, 1), (1, 1), (1, 0)$ and $(0, -1)$.
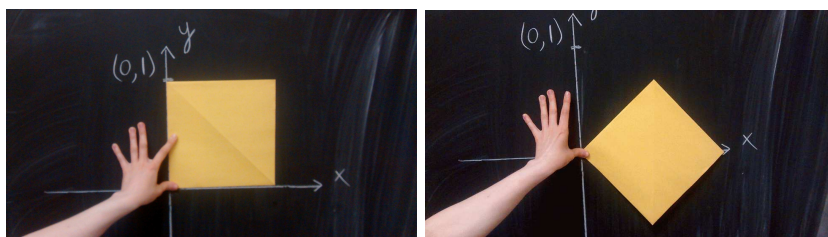
FIGURE 110. Using a square sheet to explore rotation

The coordinate descriptions of most rotations involve trigonometry. However, simple cases often have nice answers which can be discovered by guesswork, often by finding the rotations of a few points such as $(1, 0), (0, 1)$ and $(-1, 0)$, deducing the rule, and then checking the rule on other points.

Here is a sample problem: Find the coordinate description for rotation around $(0, 0)$ by 90 degrees counterclockwise. An answer can be given as follows: The point $(1, 0)$ rotates to $(0, 1)$. The point $(0, 1)$ rotates to $(-1, 0)$. The point $(0, -1)$ rotates to $(1, 0)$. From this one can guess the formula $(x, y) \mapsto (-y, x)$. One can check that this formula works for the point $(-1, -1)$ which rotates to $(1, -1)$.

Problem 8.4.7. Find the coordinate description for rotation around $(0, 0)$ by

    (1) 180 degrees clockwise;
    (2) 270 degrees clockwise.

Here is a sample problem: Given the figure below find (i) the translation by $(2,1)$, (ii) the rotation around $(0, 0)$ by 225 degrees counterclockwise, and (iii) the reflection over the line $x = y$.

A sample answer is below.

slide by (2,1)

225 degrees

x=y
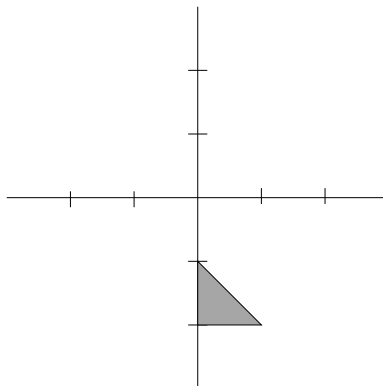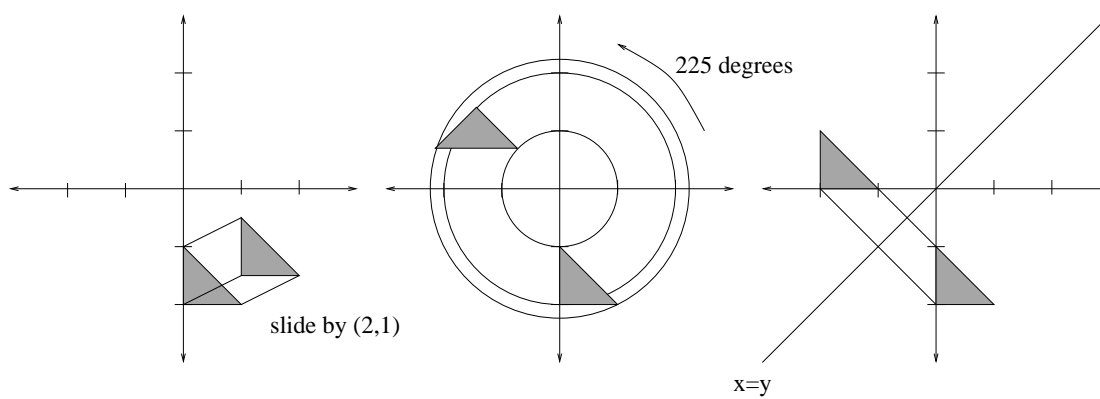
Problem 8.4.8. (Challenge) How many input-output pairs does it take to uniquely identify

    (1) a translation?
    (2) a rotation?
    (3) a reflection?
    (4) an arbitrary rigid motion?

8.5. **Workshops.**

Workshop 8.5.1. (Adapted from [NJ])

A coordinate grid shows Alice's neighborhood. Point A (7,5) represents Alice's house. Her friends Barbara, Carlos, and Dana also live in the neighborhood.
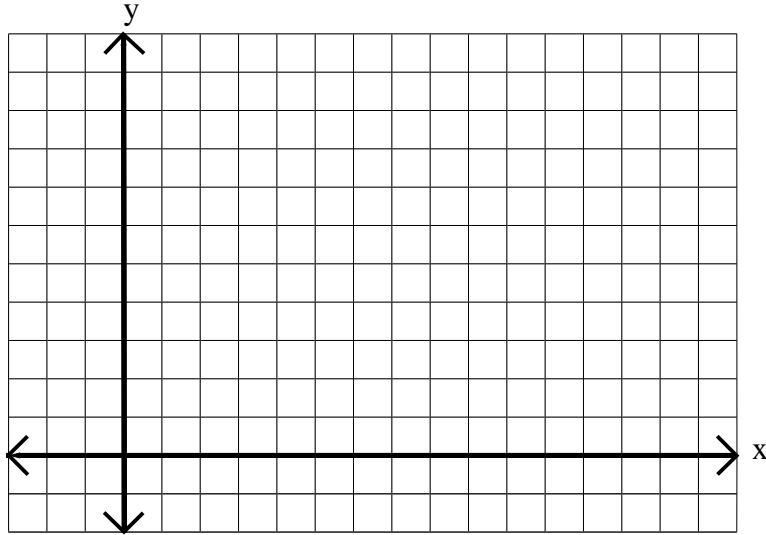
(1) Plot Point A on the grid. Barbara's house is at the point (1, 5). Plot that point and label it Point B. Carlos' house is at the point (1, 2). Plot that point and label it Point C.
(2) If Alice walks to Barbara's house and the two girls then walk to Carlos' house before returning to Alice's, which type of triangle is formed?
(3) Point D represents Dana's house. When the four houses are connected by line segments, they form a trapezoid. Find and label a Point D on the grid. Identify the coordinates of the point you labeled as Point D and explain why quadrilateral ABCD is a trapezoid.
(4) Alice says that the angle ADC measures 45 degrees. What is a possible location (in coordinates) for Dana's house if quadrilateral $ABCD$ is still a trapezoid? Can Dana's house be at more than one location given the above conditions?
(5) (Using the location of Dana's house from part D) What distance does Alice walk if she walks from her house to Dana's, then to Carlos', and then back to her house? Give an exact answer, not an approximation, ignoring units.

Workshop 8.5.2.

(1) The points (0,0) and (3,2) are two vertices of a polygon.
   (a) If the polygon is a square, what are all the possibilities for the other two vertices?
   (b) If the polygon is a nonrectangular parallelogram, what are the other two vertices?
   (c) If the polygon is a right isosceles triangle, what are the possibilities for the other vertex?

(2) The points (3,3) and (2,6) are two vertices of a right isosceles triangle.
   (a) List at least three points that could be the third vertex.
   (b) How many isosceles right triangles with vertices (3,3) and (2,6) can you draw? Explain all the possibilities.

Workshop 8.5.3.

Suppose an Egyptian pyramid (a right pyramid with a square base) has a height of 100 feet and a square base with side lengths of 200 feet.

(1) What are the heights of the triangular faces?
(2) Using thick colored paper and tape, build a model of the pyramid using a scale such as 1 model cm = 10 real feet.
(3) What are the areas of the faces?
(4) What is the distance between two opposite vertices on the bottom of the pyramid?
(5) What is the distance between the top vertex and any of the bottom vertices?

Workshop 8.5.4.

(1) Find the distance between $(-1, -1)$ and $(1, 1)$.
(2) Find the distance between $(-1, -1, -1)$ and $(1, 1, 1)$.
(3) Find the distance between $(1, 0)$ and the line $x = y$.
(4) What is the height of an equilateral triangle with side lengths 1?
(5) What is the height of an isosceles triangle with side lengths $1, 1$ and $1/2$, using the side of length $1/2$ as base?
(6) (Challenge) In an equilateral triangle, the *center* is the point that is the same distance from all the vertices. In an equilateral triangle with side lengths 1 inch, what is the distance from the center to each of the sides?

Workshop 8.5.5.

(1) Draw a right triangle with legs a, b and hypotenuse c. On each edge, draw the square with that side length. Cut out the squares. Show by cutting and re-arranging that the sum of the areas of the smaller squares is the area of the bigger square.

(2) Consider the two squares with edge lengths $a + b$:
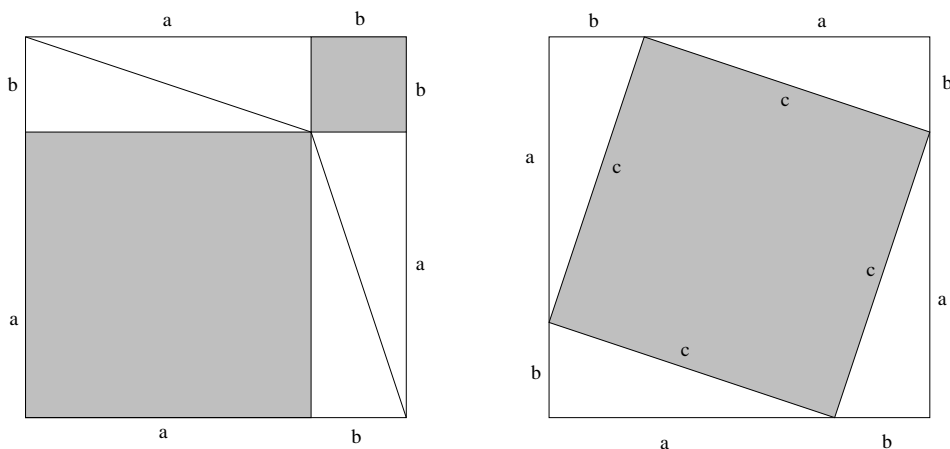
Using additivity of areas, explain why this justifies the Pythagorean theorem.

(3) The *converse* of a logical implication is the reverse implication. The converse of the Pythagorean theorem states that any triangle whose side lengths satisfy $a^2 + b^2 = c^2$ is a right triangle. Give an example of the converse by doing the following:

  (a) Find a set of numbers a, b, c satisfying $a^2 + b^2 = c^2$, not using geometry.
  (b) Draw a triangle with those side lengths. Explain what tools you used in your drawing.
  (c) Find the right angle in the drawing. Explain what tools you used and how.

Workshop 8.5.6.

On graph paper, draw $x-$ and $y-$ axes and plot the following points, labeling them A, B, C, and D: $(4,5), (-4,3), (-3,-5), (2,-1)$.

(1) Plot the locations of these points after they have been reflected across the x-axis, labeling them A', B', C', and D'.

(2) If (a,b) is a point in a coordinate plane, what will its location be after it has been reflected across the x-axis? Explain your answer.

(3) On a different graph paper, plot the locations of A, B, C, D after they have been reflected over the line $x + y = 1$.

(4) (Challenge) If (a,b) is a point in a coordinate plane, what will its location be after it has been reflected across the line $x+y = 1$? Explain your answer.

Workshop 8.5.7.

On graph paper, draw x- and y- axes, and plot the following points, labeling them A, B, C, and D: $(3,4), (-5,2), (-4,-5), (3,-4)$.

(1) Plot points A, B, C, and D and their locations after they have been rotated clockwise by 90 degrees around the origin, labeling them A', B', C', and D'.

(2) If (a,b) is a point in a coordinate plane, what will its location be after it has been rotated 90 degrees around the origin clockwise? Explain your answer.

(3) On a different graph paper, plot the locations of A, B, C, D after they have been rotated clockwise by 45 degrees around the point $(2,2)$, labeling them A'', B'', C'', and D''.

(4) (Challenge) If (a,b) is a point in a coordinate plane, what will its location be after it has been rotated 45 degrees clockwise around the point $(2,2)$? Explain your answer.

8.6. Assessments and sample student work.

Here is an assessment problem adapted from New Jersey [NJ] that illustrates the importance of distinguishing between *properties* of a shape and its *definition*.

Problem 8.6.1. A coordinate grid shows Alice's neighborhood. Point A, at (4,5), represents Alice's house. Her friends Barbara, Carlos, and Dana also live in the neighborhood.

> Part A: Plot Point A on the grid in your answer booklet. Barbara's house is at the point (1, 5). Plot that point and label it Point B. Carlos' house is at the point (1, 2). Plot that point and label it Point C.
>
> Part B: If Alice walks to Barbara's house and the two girls then walk to Carlos' house before returning to Alice's, which type of triangle is formed by their path?
>
> Part C: Point D represents Dana's house. When the four houses are connected by line segments, they form a trapezoid. Find and label a Point D on the grid. Identify the coordinates of the point you labeled as Point D and explain why quadrilateral ABCD is a trapezoid.

A student gave the following answer to Part C: "(7, 2). The quadrilateral is a trapezoid because it has two right angles." The assessment guide states " This is a 2-point response because the student has correctly completed Parts A and B of the problem and identified a Point D that creates a trapezoid, but the explanation is incorrect."

Here is another student answer: "Acute triangle. (5, 5). Two parallel lines." The assessment guide states "This is a 1-point response because the student correctly answers only Part A. The student attempts to answer Parts B and C; however, the responses are incorrect."

Problem 8.6.2. (Adapted from [MCAS]) On the grid in your Student Answer Booklet, copy the x-axis, the y-axis, and triangle LMN shown below.
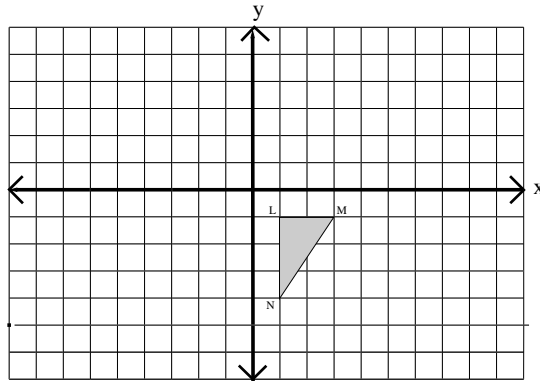
1. On your grid, draw the image of triangle LMN after it is translated 4 units to the left. Label the image PQR. List the coordinates for points P, Q, and R.

2. On your grid, draw the image of triangle LMN after it is translated 6 units up and 3 units to the right. Label the image TUV. List the coordinates for points T, U, and V.

3. On your grid, draw the image of triangle LMN after it is reflected over the x-axis. Label the image XYZ. List the coordinates for points X, Y, and Z.

Here is a student answer which received half-credit 2/4:
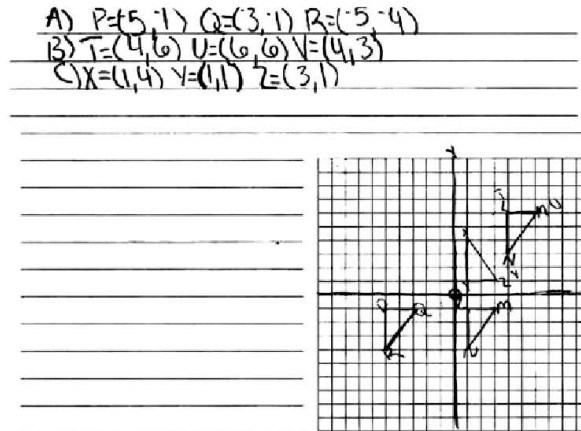
This answer showed a confusion about what it means to move 4 units to the left. Instead of moving each point 4 units to the left, the student created a space of 4 units between the triangles.

Problem 8.6.3. (Adapted from [MCAS])
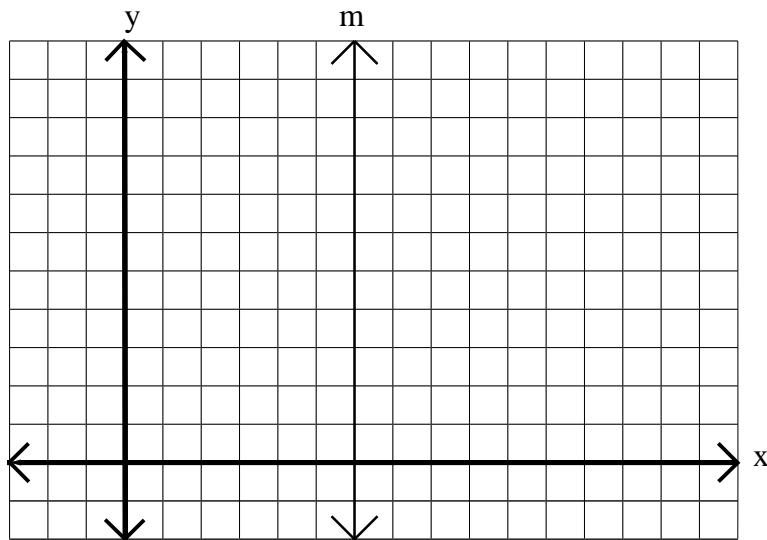
Triangle PQR has vertices located at the following points: P at (1, 6), Q at (4, 2), R at(4, 6)

    (1) Plot points P, Q, and R, and draw triangle PQR on your coordinate grid. Be sure to label the vertices of your triangle with the letters P, Q, and R.
    (2) Draw the reflection of triangle PQR across line m on your coordinate grid. Label this new triangle STU.
    (3) Write the coordinates of points S, T, and U.

Here is a student answer that received 3/4 credit:
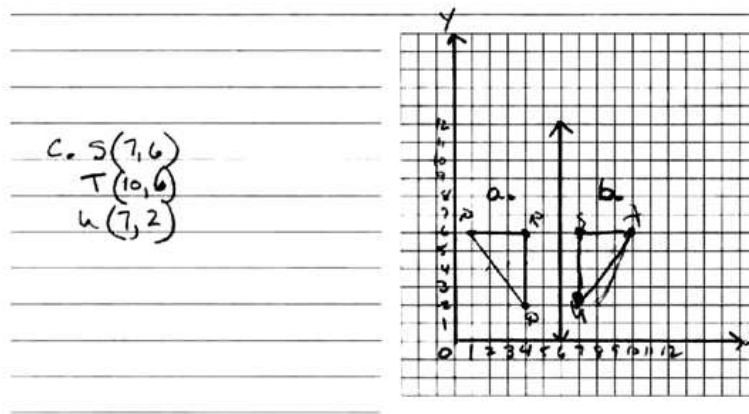
Incorrect student answers often did not recognize that the reflection should be the *same distance* on the other side of the line of reflection, although they often recognized that *distances are preserved* under reflection.

9. Congruence and similarity

Children often play with toy cars that are the same shape as a real car, but a different size. On the other hand, if they have blocks then many of the blocks will be exactly the same size and shape, but just in different locations. In this chapter we will try to describe more precisely how objects such as these are related.

Shapes such as blocks are *congruent* if they have the same size and shape, or more precisely, if there is a combination of flips, turns, and slides that changes one shape to the other.

Shapes such as the toy car and real car are *similar* if one can be changed to the other by flips, turns, slides, and rescalings. The notion of similarity in mathematics is not the same as that used in everyday language. For example, in everyday language, that one house is similar to another house means that the two houses are more or less the same size and style, but in mathematics a house ten times as big as another is similar, as long as *every dimension* is ten times as big. Similar shapes have *congruent angles*, that is, corresponding angles have the same measure. However, the concept of mathematical similarity is often misunderstood: many people think that any two rectangles are similar, but this is not true.

The objectives of this chapter are: (i) to introduce the notions of congruence and similarity both informally and precisely; (ii) to discuss problems with the various definitions; (iii) to describe various methods of reasoning using congruence and similarity; (iv) to apply the concept of similarity to proportional reasoning and maps.

9.1. **Congruence.**

Informally, two figures are *congruent* if they have the same size and shape. More precisely,

> two figures are *congruent* if one can be transformed into the other by a rigid motion.

Problem 9.1.1. Describe three pairs of congruent shapes in the following diagram of a soccer field. Label the figure with symbols as necessary.
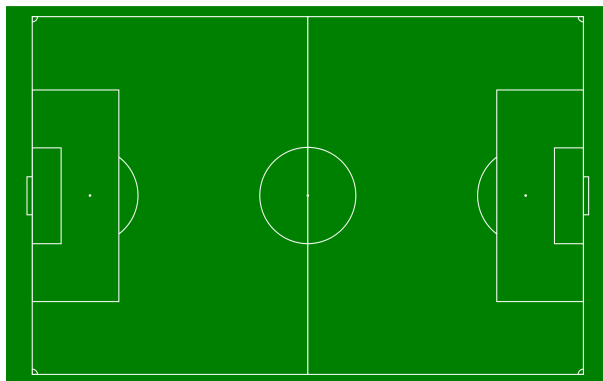
Problem 9.1.2. Agree on an interpretation for each statement and then decide whether it is true. In each case, draw a picture to support your answer. Congruence criteria are discussed in more detail in the next section.

(1) If two triangles have congruent corresponding angles then the triangles are congruent.
(2) If two triangles have corresponding sides of equal lengths then the triangles are congruent.

Problem 9.1.3. Which of the shapes below are congruent? Fill in the table after the figure and give an informal explanation in each case.
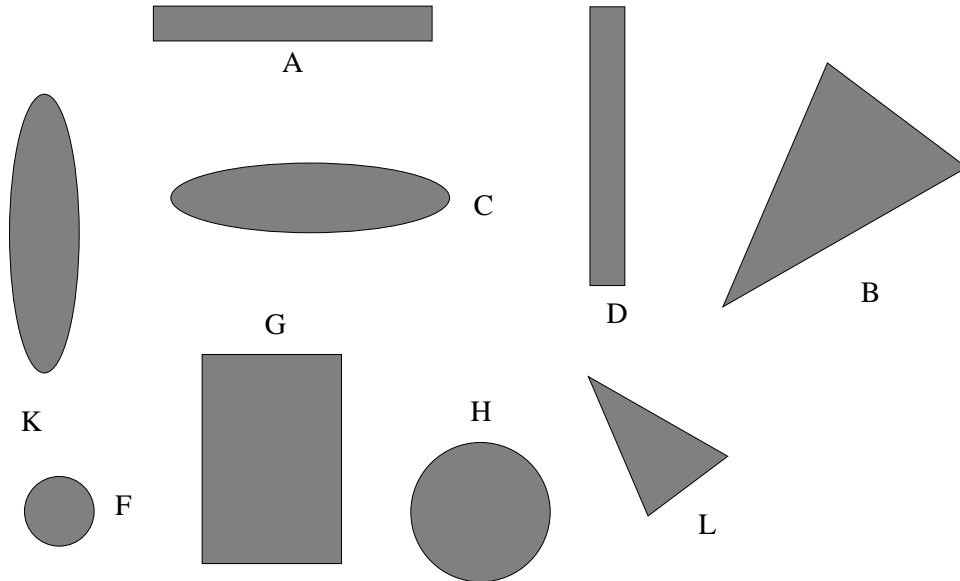
Pair of Shapes	Congruent	Explanation
A,D		
B,L		
F,H		
C,K		
F, H		
B, L		

Problem 9.1.4. (1) Which of the shapes in the figure below are congruent?
  (2) Choose one pair of congruent shapes and give a *precise* description of the rigid motion needed to change one to the other, i.e. translation by ..... in the direction of ...., rotation around the point .... by the angle ....., reflection over the line ..... etc.

A

B

D

C
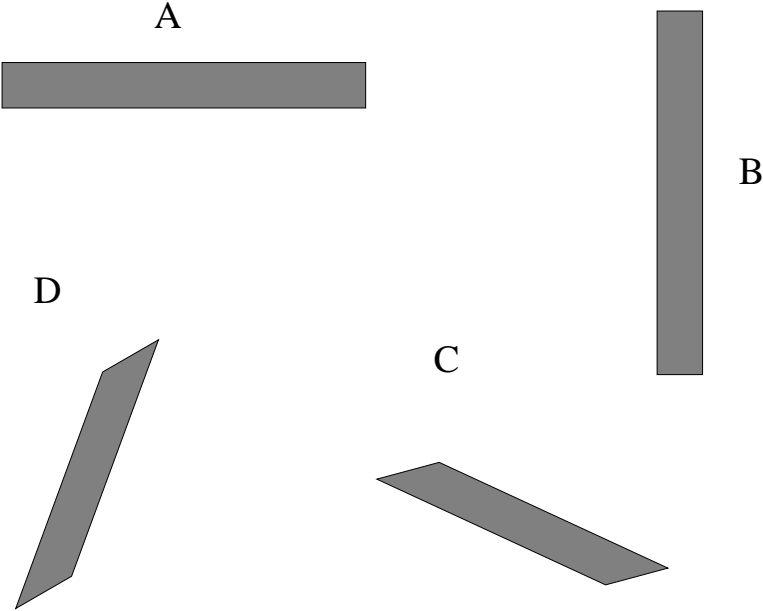
Shape	Shape	List of motions needed	precise description of motions (distances, directions)

9.2. **Congruence of triangles.**

How much information does it take to specify a triangle? The following problem explores this question.

Problem 9.2.1. For this problem you may use a compass, a ruler, and a protractor as necessary.

(1) How many non-congruent triangles can you draw with side lengths 7 cm, 5 cm, and 3 cm?
(2) How many non-congruent triangles can you draw with side lengths 7 cm, 3 cm, and 2 cm?
(3) How many non-congruent quadrilaterals can you draw with side lengths 7 cm, 5 cm, 3 cm, and 2 cm? It may help to experiment with straws for this problem.
(4) You are told that a triangle has side lengths 7 cm and 5 cm, and the angle between them measures 135°. How many non-congruent triangles can you draw with this property?
(5) You are told that a triangle has angles with measures 30° and 135°, and the side between the two angles measures 5 cm. How many non-congruent triangles can you draw with this property?
(6) You are told that a triangle has side lengths 7 cm and 5 cm, and an angle *not* between them measures 45°. How many non-congruent triangles can you draw with this property?
(7) You are told that a triangle has side lengths 7 cm and 5 cm, and an angle *not* between them measures 135°. How many non-congruent triangles can you draw with this property?

Reasoning about geometry involves reasoning about congruence, and especially, reasoning about congruence of triangles. As suggested by the problem above, there are three criteria for congruence of triangles commonly known as SSS, ASA, and SAS. Any two triangles satisfying one of these criteria are congruent.

> Two triangles satisfy the SSS (side-side-side) criterion if they have corresponding sides of the same lengths.

> Two triangles satisfy the ASA (angle-side-angle) criterion if they have two corresponding angles with the same measure, and the corresponding sides between the two angles have the same length.

> Two triangles satisfy the SAS (side-angle-side) criterion if they have two side lengths in common, and the angles formed by the two sides are congruent.

Beware that it is not true that any two triangles with congruent corresponding angles are necessarily congruent. That is, the AAA criterion does not imply congruence. Neither does the SSA criterion (side-side-angle) imply congruence. In the example below, the triangles PQR and PQR' have two side lengths and an angle congruent ($\overline{PQ} \cong \overline{PQ}, \overline{QR} \cong \overline{QR'}, \angle QPR \cong \angle QPR'$) but are not congruent.

However, the side-side-angle criterion *does* hold for acute triangles, as one of the possibilities in Figure 111 (the triangle with the obtuse angle) is ruled out.
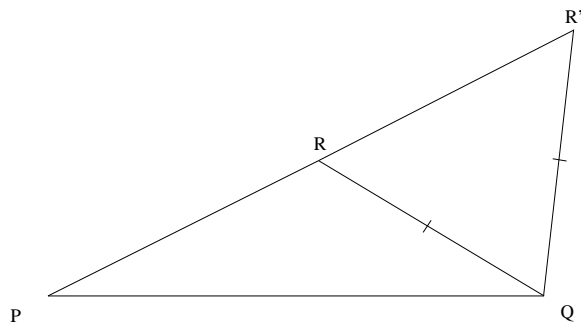
FIGURE 111. The side-side-angle criterion

Justification of the SSS criterion.

Children should be able to give a viable argument for why triangles with the congruent corresponding sides are congruent. Here is one possible argument. Suppose that two triangles have the same side lengths. Slide and turn the first triangle so that the two triangles share a side, say of length a. Suppose the other side lengths of the triangles are b, c, and the sides with these lengths intersect in points P for the first triangle, and Q for the second. Then P lies on the circles of radii b, c whose centers are the endpoints of the side of length a, and the same is true for Q. The two circles intersect in two points, and these points are related by a reflection.
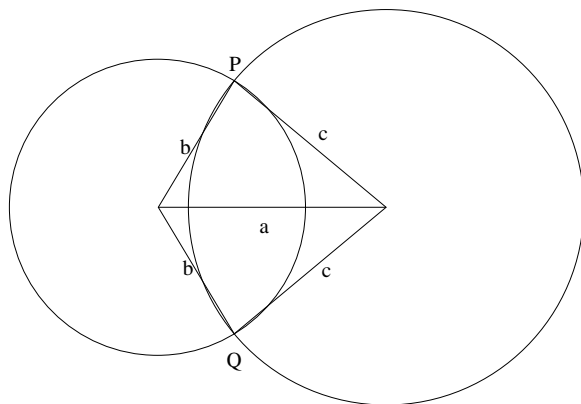
FIGURE 112. Justification of the side-side-side criterion

To see that these are related by a reflection, we can *reason by symmetry*: Each circle is unchanged by reflection over *any line through its center*. So reflecting both circles over the line through their centers changes each point of intersection between the circles to the other point of intersection.

9.3. **Reasoning using congruence.**

Problem 9.3.1. Which figure shows two congruent triangles? Explain your answer carefully using the definition of congruence, that is, what rigid motion shows that the two triangles are congruent?
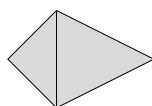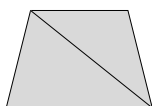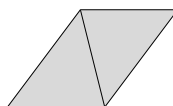
Kite   Trapezoid   Parallelogram   Triangle

Problem 9.3.2. For this problem, you can use hands-on pattern blocks or virtual pattern blocks (e.g., http://www.mathplayground.com/patternblocks.html).

(1) How many different ways are there to build polygons congruent to the yellow hexagon? Explain how you know that they are congruent to the original hexagon.
(2) Using moving principles of area, explore the relationship between the area of the tan rhombus and the area of the orange square. (Hint: use squares, triangles, and tan rhombi to build a "house" two different ways.)

9.4. **Similarity.**

Informally, two objects are *similar* if they have the same shape but different size. For example, different toy models of the same car can have the same shape but different size as in Figure 113.

FIGURE 113. A model car is similar to the original

There are several problems with the informal definition. One of the familiar categories of shapes is the class of rectangles. In that sense all rectangles have the same shape. But do we want to say that every rectangle is similar to every other rectangle? And do we really want to say that two unit squares are not similar because they do have the same size? This issue can be addressed by saying that two figures are similar if they have the same shape but not necessarily the same size. However, the first issue is still not fixed by this version of the definition.

So we will want a better definition. But to make sense of the definition we should "want", let us first look at the notions of "scaling up" and "scaling down" figures in the plane. By *scaling up/down* we mean that every distance in a figure gets multiplied/divided by a constant called the *scale factor*.

Figure 114 shows examples of two figures that are rescalings while Figure 115 shows two figures that are not rescalings of each other.

FIGURE 114. Figures that are rescalings

Similarity via motions

Here is the precise definition of similarity:

FIGURE 115. Figures that are not rescalings

Two figures are called *similar* if one can be changed to the other by a rigid motion and dilation, that is, a combination of transformations that are among the following types: translations, rotations, reflections, or dilations.

A combination of these motions is called a *similarity transformation*. The *scale factor* relating two similar figures is the scale factor in the dilation.

We use the symbol $\sim$ to indicate similarity. For example, if triangles ABC and DEF are similar we write $\Delta ABC \sim \Delta DEF$. (We really prefer the notation $ABC \sim DEF$, but putting Δ in front of the vertices for triangles seems to be standard.)

Problem 9.4.1. Balloons come in different shapes, for example, the usual "round type" and the "skinny" type used by "balloon guys" to make balloon animals. Suppose that a balloon of the round type is blown up by several breaths, and then by several more breaths. Is the new balloon shape similar to the old one? What about for the skinny balloons?

36

Problem 9.4.2. Fill in the table below the figure identifying which of the shapes below are similar, according to the precise definition given above. Use any tools you deem necessary. Explain your answer in terms of the definition.
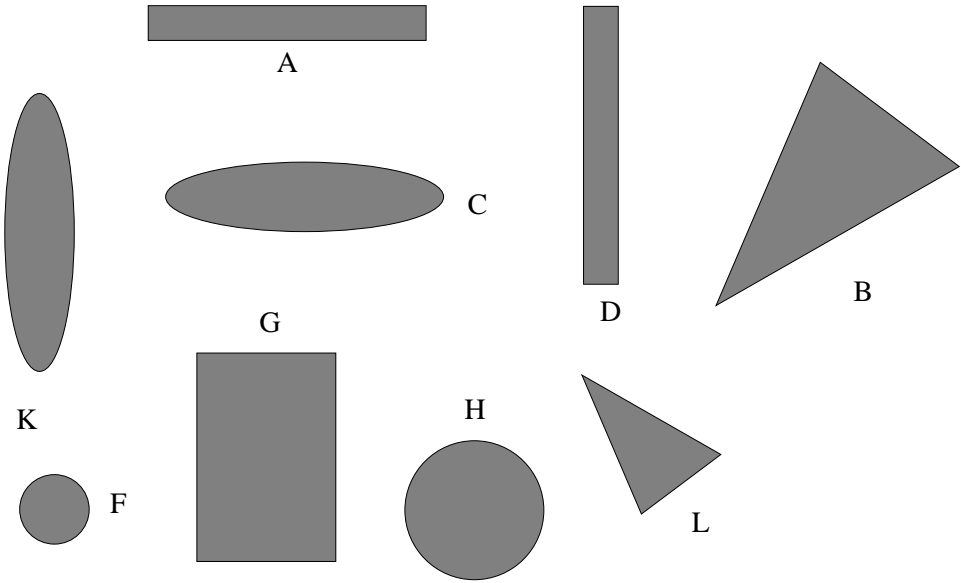
Pair of Shapes	Similar	Explanation
G,D		
C,H		
J,I		
A,K		

Similarity and proportionality

There is a different explanation of similarity given in other commonly used textbooks:

> Two figures are similar if (1) the measures of their corresponding angles are equal and (2) the lengths of their corresponding sides increase by the same factor, called the *scale factor*.

Note that this definition seems not to apply to circles: what is the "side" of a circle? By the definition above, a circle would be similar to an ellipse, which is certainly false!

To avoid this problem, the Common Core State Standards adopt the definition using similarity transformations. Whether the greater precision in the common core definition will be accessible to students is a matter of some debate.

Problem 9.4.3. Toy cars are often made to a 1 : 72 scale. This means that each measurement of the toy car is related to the real car by a factor of 72. Determine whether the following model cars are dilations of each other by measuring several distances and finding the scale factors between the toy cars, if possible. Assuming the left-most car is made on a 1 : 72 scale to the real car, what scale (to the real car) are the other models? Explain your answer.

As this problem suggests, if two figures are both similar to a third figure then the first two figures must be similar. For example, if one toy car is a 1 : 72 model and another toy car is a 1 : 36 model, then the two toy cars are similar with a scale factor of 2.

Similarity and angle preservation

Problem 9.4.4. For this problem, use any tools you deem necessary. Consider the following pair of sailboats.
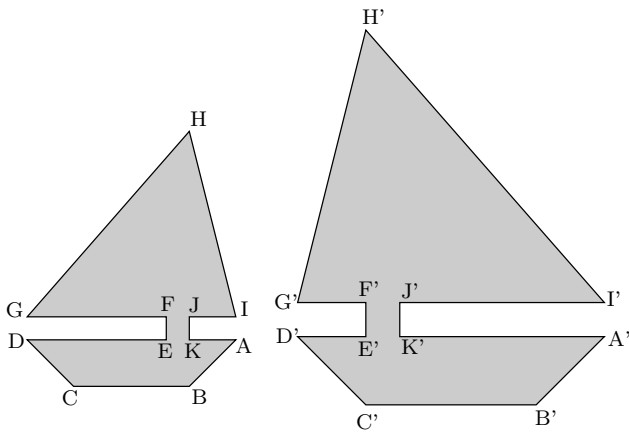
Fill in the following tables.

$AB =$	$A'B' =$	$AB/A'B' =$
$HC =$	$H'C' =$	$HC/H'C' =$
$AD =$	$A'D' =$	$AD/A'D' =$

$m\angle ABC =$	$m\angle A'B'C' =$
$m\angle HAB =$	$m\angle H'A'B' =$

For each table, find one sentence that expresses the relationship expressed in the table. How does this relate to similarity?

Similarity preserves angles and *ratios* of distances: If two figures are similar and P, Q, R are points in the first figure and P', Q', R' the corresponding points in the second, then the angles PQR and $P'Q'R'$ are congruent, while the distances $PQ, P'Q'$, $PR, P'R'$ etc. have the same ratio: $PQ/P'Q' = PR/P'R'$ etc.

Similarity and pattern blocks

Problem 9.4.5. For this problem, you can use hands-on pattern blocks or virtual pattern blocks (e.g., http://www.mathplayground.com/patternblocks.html).

(1) Using only pattern blocks, build a scaled version of the red trapezoid (use scale factor 2). Explain how you know it is similar to the original trapezoid.
  (a) Can you do it in more than one way?
  (b) How many pattern blocks do you need if you want to use as few blocks as possible?
  (c) What if, additionally, you want to maximize the number of red trapezoids used?
  (d) Is there a way to do it using only red trapezoids?
(2) Using only pattern blocks, build a scaled version of the red trapezoid using scale factor 3.
  (a) Try to find multiple ways to do it, while aiming to use up to 11 blocks.
  (b) Is there a way to do it using only red trapezoids?
(3) Using only pattern blocks, build scaled versions of the yellow hexagon. Explain how you know they are similar to the original hexagon.
  (a) For scale factor 2, if you want to use as many yellow hexagons as possible, how many ways are there to build the larger hexagon? Explain your answer briefly.
  (b) For scale factor 3, what is the maximum number of yellow hexagons you can use? Explain your answer briefly.
  (c) For scale factor 4, what is the minimum number of blocks you need? Explain your answer briefly.

The angle-angle-angle criterion

The angle-angle or angle-angle-angle criterion for similarity, abbreviated AA or AAA states:

> Triangles with congruent corresponding angles are similar.

In practice, to verify the criterion it is enough to check that the two triangles have at least two angle measures in common. This is because the sum of the interior angles of a triangle is 180 degrees and so if two angle measures are equal, then the third angle measure is also equal.

Here is a viable argument for the angle-angle-angle criterion. Given two triangles with congruent angles, one can slide, rotate, and flip the first triangle so that the triangles share a vertex and an angle at that vertex and the corresponding angles are in the same direction from that vertex. See for example, the triangles APQ and ABC in the Figure 116. Choose a dilation from the point A that maps the point P to the point B. Since dilations preserve angles and angles $\angle APQ$ and $\angle ABC$ are congruent, this dilation from point A changes the ray $\overrightarrow{PQ}$ to the ray $\overrightarrow{BC}$. So Q transforms to a point on the ray $\overrightarrow{BC}$. Also Q transforms to a point on the ray $\overrightarrow{AQ}$, since this ray is unchanged by this dilation. Combining these facts shows that Q transforms to the unique point on the intersection of $\overrightarrow{BC}$ and $\overrightarrow{AQ}$, which is the point C. Therefore the triangles APQ and ABC are related by a dilation, and are therefore similar.
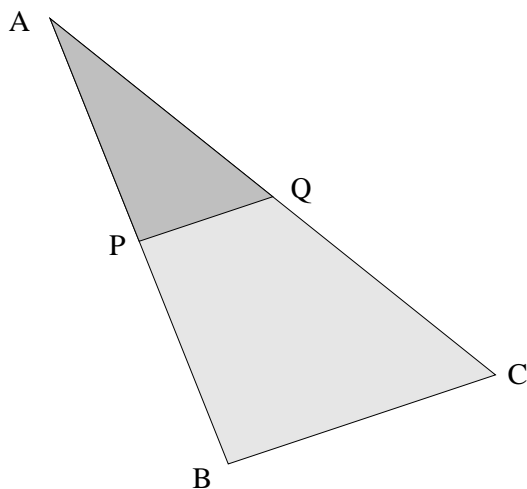
FIGURE 116. Triangles related by a dilation

If you have internet access you can click below to bring up a GeoGebra applet which illustrates the angle-angle-angle criterion:

Demonstrating the angle-angle-angle criterion

Finding a similarity transformation relating similar figures

Given two similar figures, children should be able to find a similarity transformation (combination of translations, reflections, dilations, and rotations) that relates them. Here is a step-by-step process to identify the needed similarity transformation.

(1) Identify triples P, Q, R of points in the first figure and corresponding points P', Q', R' in the second.

(2) Do a translation that changes P to P', if necessary. The points Q, R change into points Q'', R''.

(3) Do a rotation and dilation from P' that changes Q'' to Q', if necessary. The point R'' changes into a point R'''.

(4) If necessary, do a reflection over the line $\overleftrightarrow{P'Q'}$ that changes R''' to R'. This last step is necessary only if the original figure has a different orientation than the final figure.
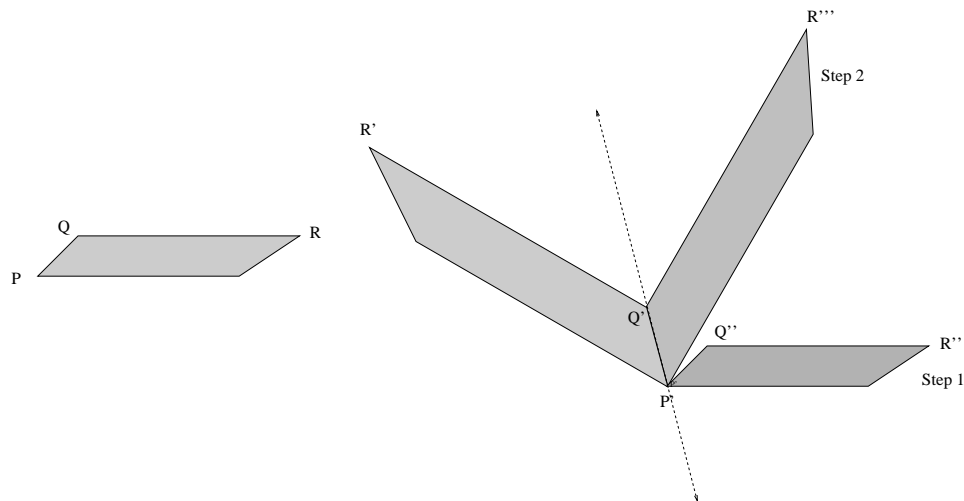
FIGURE 117. Finding a similarity transformation

Problem 9.4.6. Find a similarity transformation that changes the figure on the left to the one on the right. Describe each motion precisely.
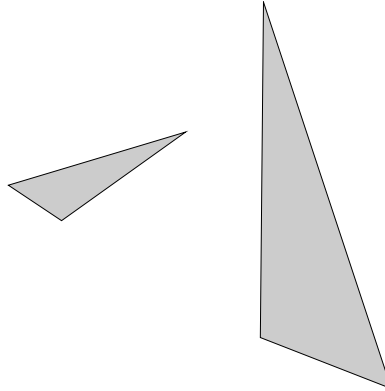
Problem 9.4.7. Determine whether the following figures are similar, and for any two that are similar find the scale factor. For any similar pair, verify your answer by measuring several corresponding distances in each picture.[37]

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[37]Art work copyright M. Woodward 2016, used by permission.

9.5. **Reasoning using similarity.**

Similarity can be used in geometric reasoning in various ways. If two figures are known to be similar, then the *ratios* of distances in one can be determined from the other. *If the scale factor is known*, then the actual lengths or distances in one can be determined from the other.

Problem 9.5.1. If the triangles PQR and STU are similar, and $PQ = 3.4$ cm, $QR = 2.6$ cm, and $SU = 5.2$ cm, find TU.
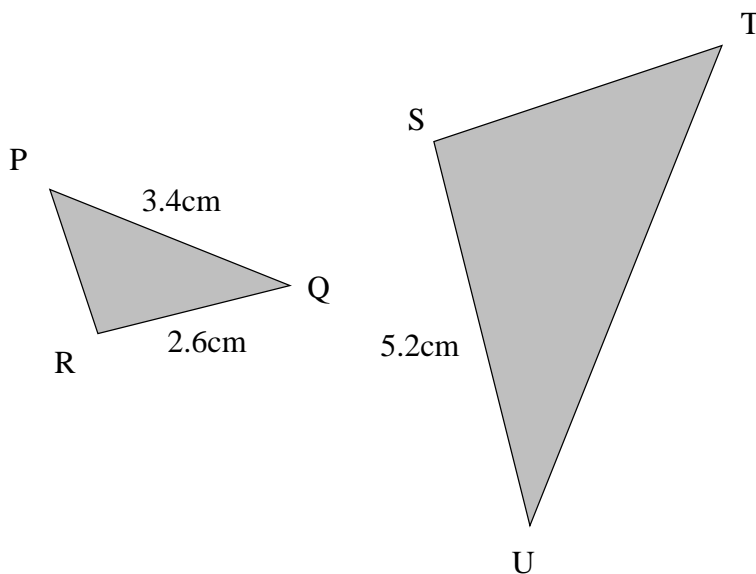
Problem 9.5.2. (Adapted from [CST]) Miranda enlarged a picture proportionally. Her original picture is 4 cm wide and 6 cm long. If the new picture is 20 cm wide, what is its length?

Behavior of area under scaling

The following problem is meant as an exploration of how area changes under scaling.

Problem 9.5.3. adapted from [TX].

(1) Find the area of a square with edge lengths 3 feet.
(2) Find the area of a square with edge lengths 6 feet.
(3) Suppose that one square has twice the side lengths of another square. How do the areas relate?
(4) Suppose that one shape is obtained from another by scaling by a factor of 2. How do the areas of the shapes relate?

As the problem above suggests, the area of a region changes in an easily described way under rescaling. To explore how area changes under scaling, suppose that the first region R_1 is filled with A_1 unit squares, and let R_2 denote the region obtained by rescaling region R_1 by a scale factor s. Let A_2 be the area of the region R_2. Under rescaling, each unit square in the first region R_1 becomes a square of side lengths s in the second region R_2, as shown in Figure 118.
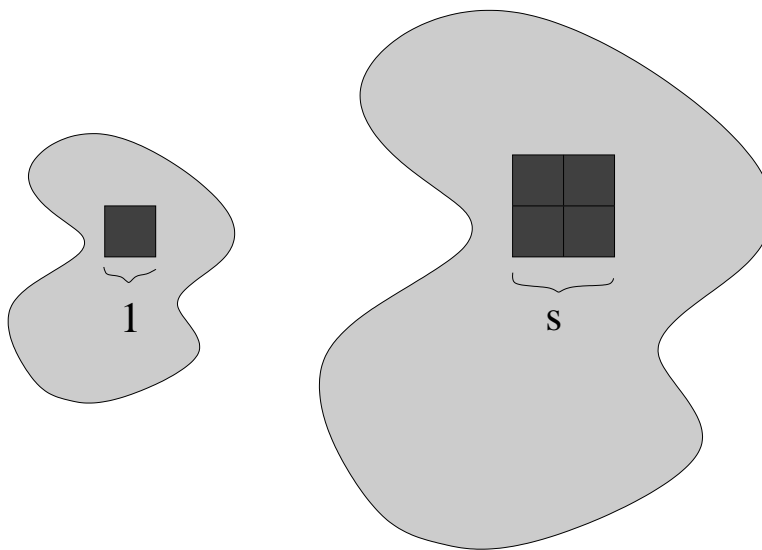
FIGURE 118. Area under rescaling

Each square of side lengths s can be filled with s^2 unit squares (or parts of unit squares). Therefore,

$$
\begin{aligned}
A_2 &= \text{\#unit squares to fill } R_2 \\
&= \text{\# squares of side-length } s \text{ to fill } R_2 \\
&\quad \times \text{\#unit squares per square of side-length } s \\
&= \text{\#unit squares to fill } R_1 \\
&\quad \times \text{\#unit squares per square of side-length-}s \\
&= A_1 \times s^2.
\end{aligned}
$$

Problem 9.5.4. The following two moon shapes are similar. If the area of the big moon is 7 square units, what is the area of the small moon?
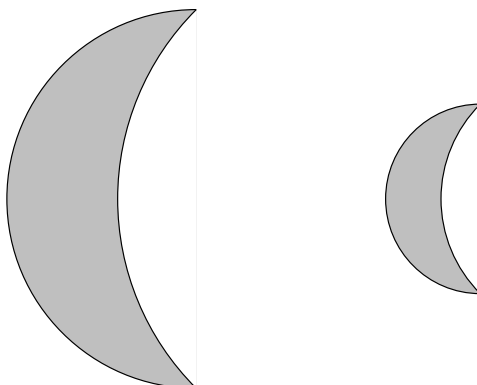
9.6. **Similarity and maps.** A *map* of a reasonably sized part of the earth's surface is a "scaled down" picture of the geography of that part of the world.

Since the earth's surface is curved, a map on a piece of paper is only *approximately* similar to the region of the earth it represents.

Common numerical scales on topographical maps are 1:25,000, 1:50,000, 1:200,000, and 1:500,000. The scale factor 1:25,000 means one inch on the map represents 25,000 inches on the earth's surface. The scale factor 1:500,000 means one inch on the map represents 500,000 inches on the ground. Because the earth's surface is curved and maps are flat, distances on maps are only *approximately* rescaled distances on the earth's surface.

Problem 9.6.1. Find the distance between the capitals of Alabama and Florida using the map of state capitals below.[38]

---

[38]Downloaded from *nationalmap.gov/small_scale/.../states_capitals.pdf* January 2016.

9.7. **Workshops.**

Workshop 9.7.1.

(1) Are the following houses similar? Explain your answer.
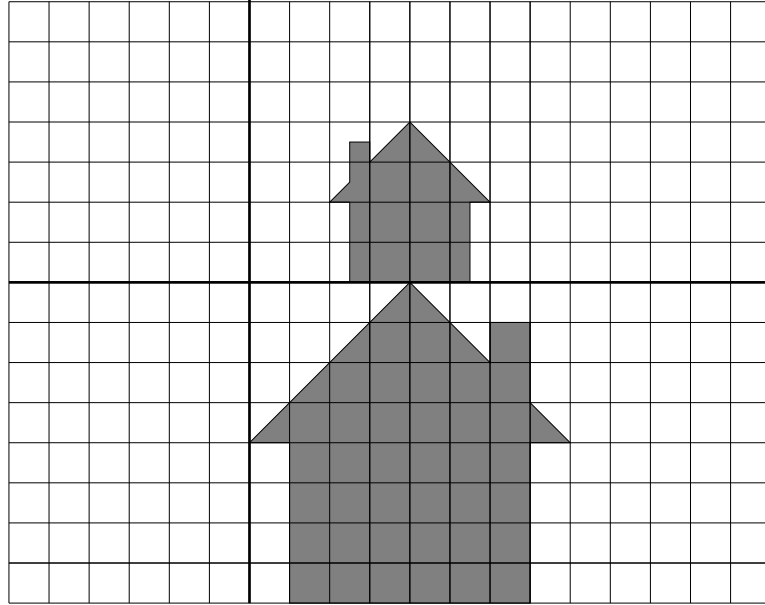
(2) If the houses are similar, what is the scale factor i) from the small house to the big house; and ii) from the big house to the small house?
(3) Find the area of each house (with respect to the square units of the grid). How do the two areas compare to each other? Is there any relationship between the scale factors you found in b) and the areas of the two houses?
(4) Find a sequence of transformations that transforms the house in the 1st quadrant into the one in the 4th quadrant. Each motion should be described precisely.
(5) Is there a unique answer to part (4)? Explain your answer.

Workshop 9.7.2.

(1) (from Beckmann [B]) Johnny is working on the following problem:

A poster that is 4 feet wide and 6 feet long is to be scaled down to a small poster that is 1 foot wide. How long should the poster be?

Johnny solves the problem this way:

One foot is 3 feet less than 4 feet so the length of the small poster should also be 3 feet less than the length of the big poster. This means the small poster should be 6 - 3 = 3 feet long.

Is Johnny's reasoning valid? Why or why not? If not, how might you convince Johnny that his reasoning is not correct? In this case, what would be a correct way to solve the problem?

(2) A museum wants to put a scaled down copy of one of its paintings onto a 3-inch by 5-inch card. The painting is 42 inches by 65 inches. Explain why the copy of the painting cannot fill the whole card without leaving blank spaces (i.e. explain why there will have to be a border). Recommend to the museum a size for the copy of the painting that will fit on the 3-inch by 5-inch card. Show your recommendation here in the 3" × 5" rectangle below.
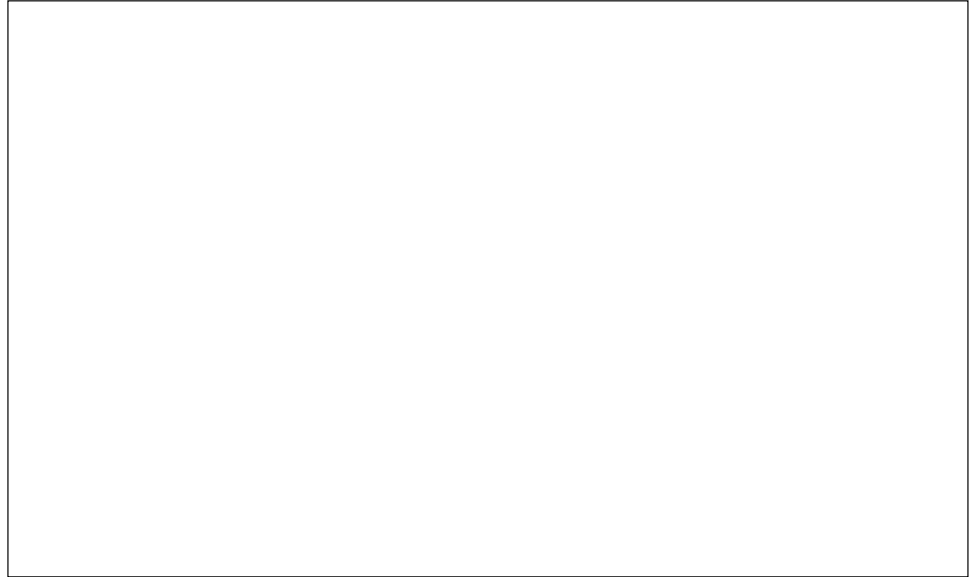
(3) A large American flag can be 5 feet tall by 9 feet 6 inches wide. Suppose Congress decrees that the United States convert to the metric system and the flag should be 1 meter tall. (1 meter is about 3.28 feet). How wide should the new flag be? (Hint: the scale factor, that is, the ratio of proportions, is 1m/3.28ft).

(4) In the games of pool or billiards, a ball bounces off the edge of the playing table with only a small loss of speed. Using similarity, determine what the path of the ball shown below should be, so that it bounces off the wall and goes into the corner pocket.

(5) Using similarity and congruence, explain why the apparent paradox below is not actually a paradox. [39]
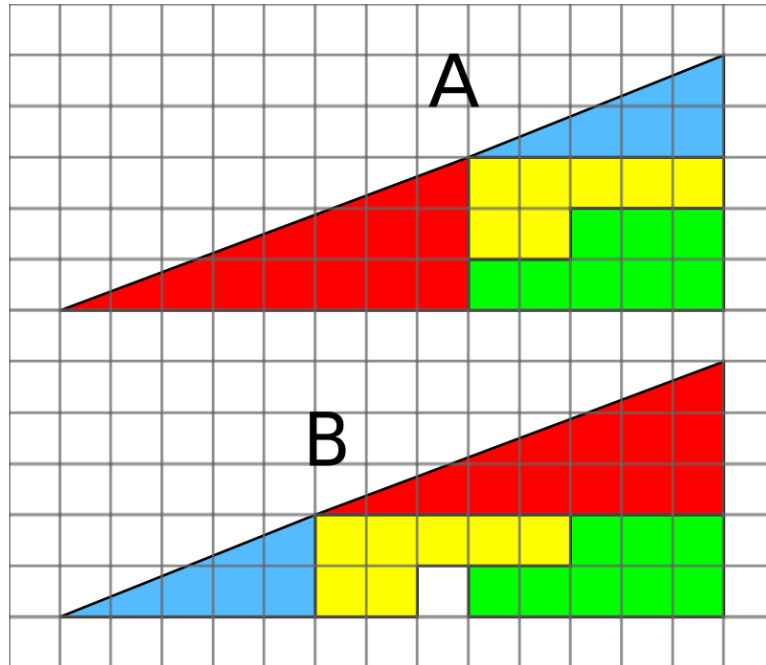
[39]By Krauss - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=37256611

Workshop 9.7.3.

(1) The diagonals of any parallelogram bisect each other. In this workshop you will explain why in two ways.
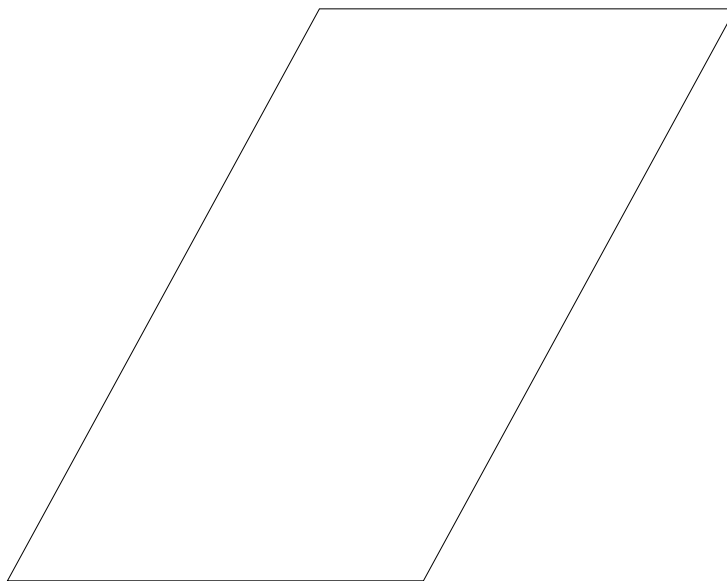
    (a) Draw the diagonals of the parallelograms, and label all points.
    (b) Identify all congruent angles and triangles in the figure.
    (c) Choose a pair of congruent triangles and find a rigid motion that changes one to the other.
    (d) Explain why the lengths of the two parts of each diagonal are the same, using the fact that rigid motions preserve distances.
    (e) Use a criterion for congruence of triangles to show the diagonals of the parallelogram bisect each other.

(2) Investigate the following conjectures.
    (a) Any parallelogram whose diagonals are congruent must be a rectangle. (Hint: show that there are lines of symmetry parallel to the edges.)
    (b) Any quadrilateral with two pairs of opposite congruent sides must be a parallelogram. (Hint: show that there is rotational symmetry around the point where the diagonals intersect.)
    (c) In any kite the diagonals bisect each other.

Workshop 9.7.4.

(1) Suppose that a right triangle with side lengths $3, 4, 5$ is resting on the hypotenuse as base. Find its height exactly, without measuring.
(2) A right triangle with side lengths $3, 4, 5$ is placed on its hypotenuse as base. Its height is the length of a segment perpendicular to the base, that divides the triangle into two other triangles. Explain why these triangles are similar to the original triangle. For each of these triangles find the corresponding scale factor (to the original triangle) and its side lengths.
(3) Suppose that the right triangle as above with side lengths a, b, c has area A. Find the scale factors and the areas of the smaller triangles.
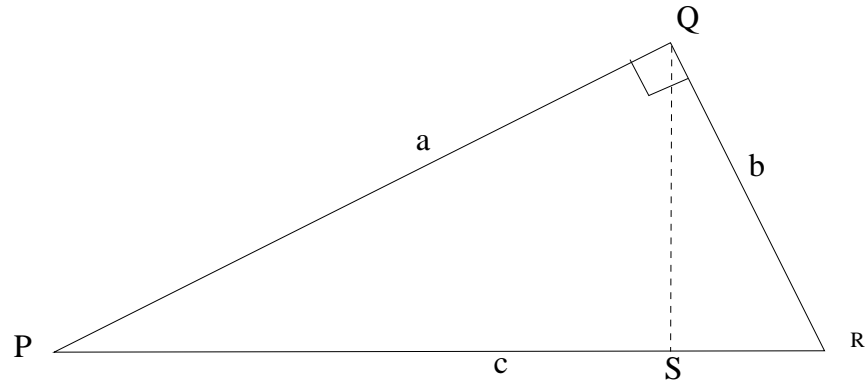
(4) Explain why the areas of the smaller triangles add up to the area A of the larger triangle. Deduce the Pythagorean theorem.

9.8. Assessments and sample student work.

Problem 9.8.1. (from [MCAS], on the use of similarity in maps)

Part of a map is shown below. Each point is labeled to indicate the town that it represents. The map has a scale in which 1 inch represents 20 miles.
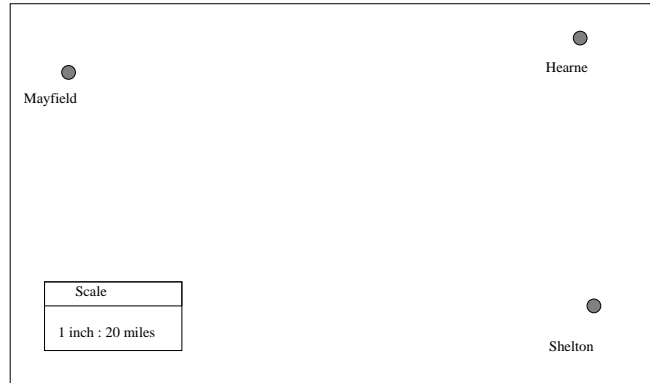
a. Using your ruler, what is the distance, in inches, between Mayfield and Shelton on the map?

b. What is the actual distance, in miles, between the towns of Mayfield and Shelton? Show or explain how you used the scale to get your answer.

c. What is the actual distance, in miles, between the towns of Hearne and Shelton? Show or explain how you used the scale to get your answer.

d. The town of Sawyer is located 50 miles from Mayfield. On the full map, what should be the distance, in inches, between Sawyer and Mayfield? Show or explain how you got your answer.

Here is a sample answer, which received half-credit 2/4: (The figure here is smaller than the exam print-out, so the distances measured do not match).

Here is a sample answer, which received 1/4:

Incorrect answers showed confusion about working with units. The notation "1 inch:20 miles" is inherently confusing as written, because it is hard to figure out whether one is supposed to multiply or divide by 20. It might be helpful to write 1 map inch = 20 real miles. Then the answer might be written 5 map inches = 5 map inches × 20 real miles/ map inch = 100 real miles. Even correct answers fail to recognize a fool-proof procedure for getting the right answer.

10. Solids, volumes, and surface areas

Real life is three-dimensional, and shapes in three dimensions are often called *solids*. Many solids, such as children's blocks, are *polyhedra*, meaning that they have flat faces joined by edges. For example, most people's houses or apartments are polyhedral, meaning they have no curved surfaces. Note that we use the word *solid* to describe a shape such as a house, even though the inside is filled just with air. The *volume* is a measure of how big a solid is, which is different from the height, width, surface area etc. Another example of a solid is a *solid sphere*, such as the earth or the sun, approximately. The ratio of volumes of similar solids can be determined by knowing just the scale factor, without knowing the volumes themselves. For example, the radius of the earth is approximately twice as big as that of the planet Mars, so the volume of the earth is approximately eight times as big as that of Mars.

The objectives of this chapter are (i) to introduce the notions of solids, polyhedra, and faces; (ii) to describe different kinds of solids such as cylinders, prisms, cones and pyramids; (iii) to introduce surface area and volume, and give formulas for the surfaces areas and volumes of some standard shapes; (iv) to explore how volume and surface area change under scaling, and (v) to discuss the different methods for conversion of areas, volumes, and rates between different systems of measurement.

10.1. Solids and polyhedra.

A *solid* is a three-dimensional shape such as a box or ball. A *polyhedron* is, informally, a three-dimensional solid with flat faces and straight edges. We adopt the convention that a polyhedron should have "no holes", that is, is *closed* in the sense that every edge is contained in two faces. There are several versions of a precise definition of polyhedron and polyhedral surface in common use but no universally accepted one.

Polyhedra can be built easily out of Magna-Tiles. Figure 119 shows a few examples of polyhedra built out of Magna-Tiles (the "house" and "tent") as well as non-polyhedra (the ones that are not closed).

A *polyhedral surface* is the surface of a polyhedron. A polyhedral surface is made up of polygonal regions called *faces* of the polyhedron.

Just as we required for polygons that no two edges meet at 180 degrees, for polyhedra we require that no two faces of a polyhedron meet at 180 degrees.

A *solid cube*, or *cube* for short, is a polyhedron with six faces that are all squares and meet at right angles. The surface of a solid cube is a *cubical surface*. A cubical surface can be built out of 6 squares, or 5 squares and two triangles fitting together to a square; in the latter case we still say that the faces of the cube are squares, not triangles. Most people (including most mathematicians) abuse terminology and refer to a cubical surface also as a cube.

A *net* of a polyhedron is a diagram in the plane that can be "folded up" to the surface of the polyhedron.

Figure 119. Polyhedra and non-polyhedra using Magna-Tiles

Figure 120 shows nets for a cube and for a tetrahedron (polyhedron with four faces).
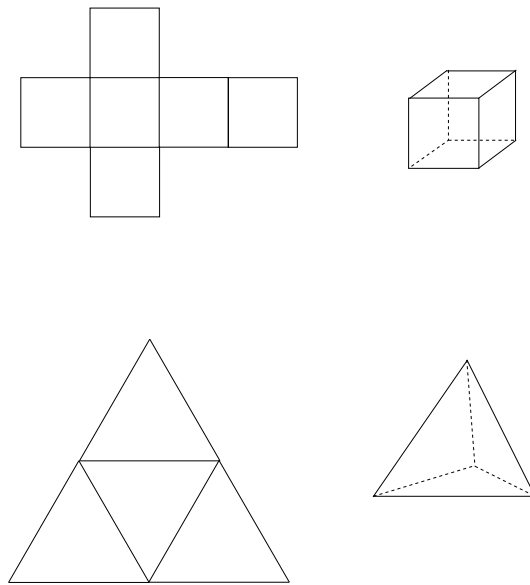
Figure 120. Nets

In both the top and bottom parts of Figure 120, the net is shown on the left and the solid on the right. Note that above we drew the cube so that its edges are parallel on a page. This is actually unrealistic, since the rules of *perspective* (as we learn in art class) dictate that parallel lines in three-dimensional space are actually *converging* to a point on a horizon when drawn on the plane. So a more realistic depiction of a cube is given in Figure 121; see also the photograph of a cube in Figure 133.
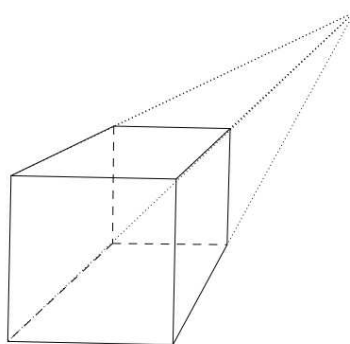
FIGURE 121. A cube in perspective

For more on the mathematics of perspective we recommend the book [VP]. As far as we can tell, most texts and assessments depict cubes without perspective, that is, in a way that is visually inaccurate but perhaps conceptually less confusing than using perspective. If you have internet access, you can bring up an animation of a net using GeoGebra below:

Net for a dodecahedron

A polyhedron is *convex* if the line segment between *any* two points inside the polyhedron stays inside the polyhedron.

Convex polyhedron   Non–convex polyhedron
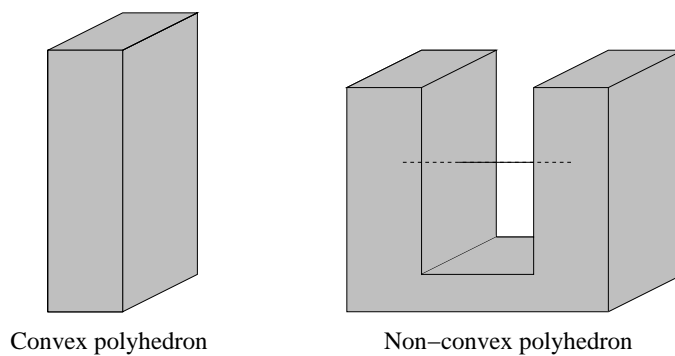
FIGURE 122. Convex and non-convex polyhedra

Problem 10.1.1. (Corrected from [SM]) The figure shows a polyhedron. Which of the following can be a net of the given polyhedron? Note that the polyhedron is shown from the side, so that the actual base and top are "longer" than they appear in the picture.
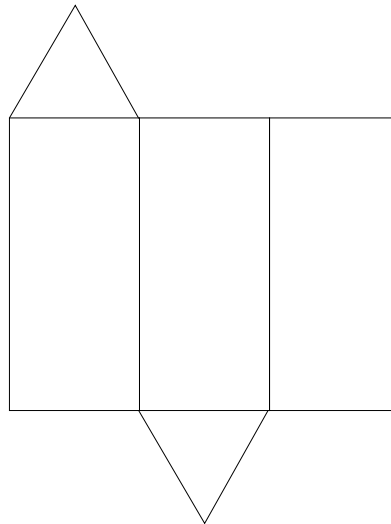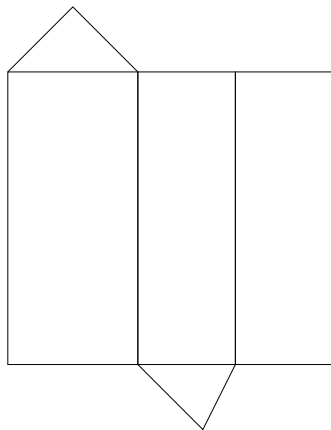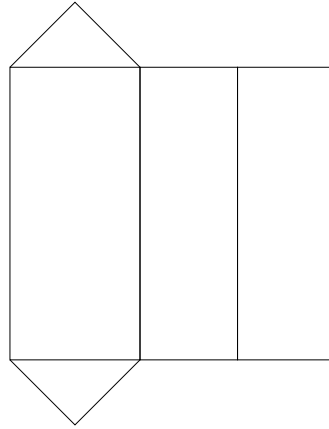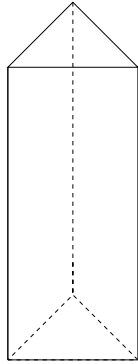
Problem 10.1.2. Which of the following can be the net of a polyhedron?
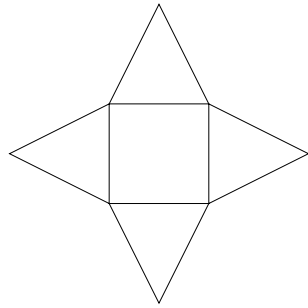
(a)
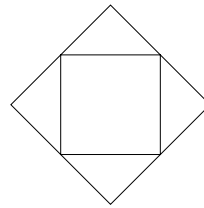
(b)
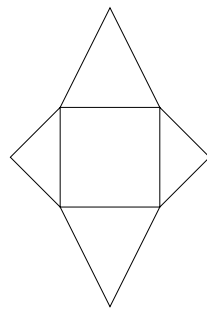
(c)
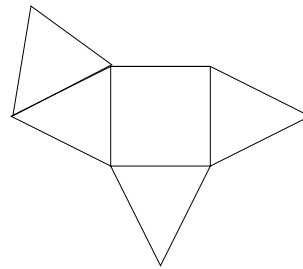
(d)

Problem 10.1.3. Draw a net that can be folded up into

(1) a cube;

(2) an octahedron (regular polyhedron with eight faces; see Figure 128.)

10.2. Cylinders and prisms.

A cylinder is a region of points in space between a *base* and a congruent top, like a soup can. Usually the top face is directly above the bottom face, but more generally a *cylinder* with a two-dimensional (usually circular) base is the region in space obtained by translating the base some distance at an angle from the plane containing the base. If the angle is a right angle, the cylinder is *right*, otherwise the cylinder is *oblique*.

Typically the base of a cylinder is circular, but terminology in more advanced mathematics allows the base of a cylinder to be any shape. The advantage of this approach is that the formulas for the volumes of prisms and cylinders in terms of the area of the base and the height are the same and so do not need to be memorized separately. The top of the cylinder can also be considered a base, by flipping the cylinder over. Thus each cylinder has two bases.

A *prism* is a cylinder on a polygonal base. A prism is called *triangular, rectangular, pentagonal, hexagonal etc.* if its base is a triangle, rectangle, pentagon, hexagon etc. A *right rectangular prism* is what we usually think of as a *box*.

<div align="center">
Right Prism   Oblique Prism

Right Cylinder   Oblique Cylinder
</div>
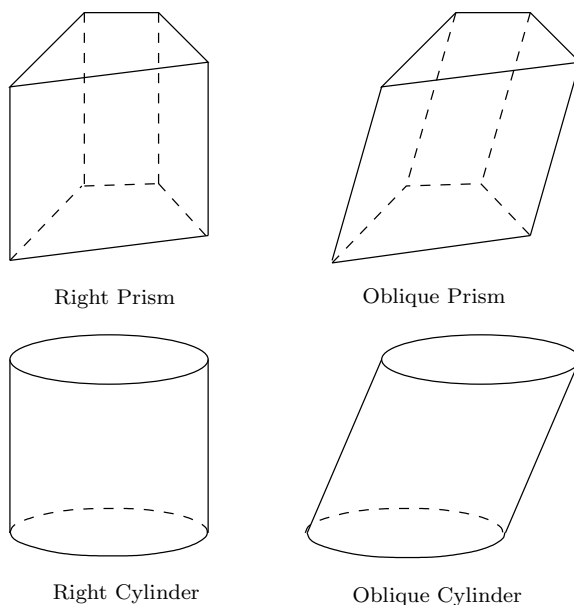
FIGURE 123. Cylinders

A footnote to the Common Core State Standards in first grade states that students do not need to learn formal names such as "right rectangular prism".

Problem 10.2.1.

(1) What kind of polygons are the faces of a cylinder on a polygonal base? Are all the faces congruent?

(2) What can you say about the planes containing the bases of a cylinder on a polygonal base?

Figure 124 shows right and oblique cylinders with bases that are curvy but not circular:
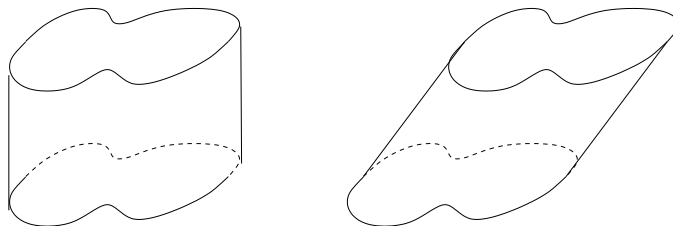
FIGURE 124. Cylinders with curved bases

A *cone* is a region in space of points between a base, which is usually circular, and a point not in the same plane as the base, called the *vertex*.

A *pyramid* is a cone on a polygonal base, so that a *square pyramid* is the kind found in Egypt.

Cone on a square (pyramid)

Standard cone   Cone on a triangle (tetrahedron)
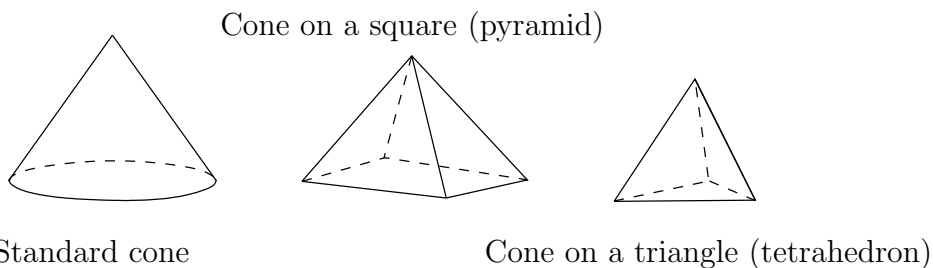
FIGURE 125. Cones and pyramids

For regular polygonal bases, cones can be either right or oblique. A right cone is one in which the center of the polygonal base lies on the perpendicular from the top vertex. Figure 125 shows some right cones. Figure 126 shows some oblique cones. Figure 127 shows a cone with a crazy base.

By convention, if no base or obliqueness is mentioned, then the cone is a right cone with circular base so that the cone looks like an ice cream cone upside down.
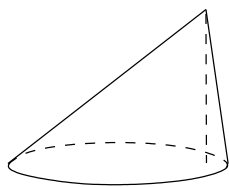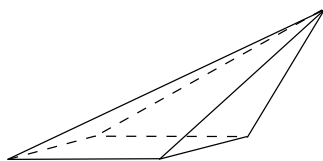
Oblique cone   Oblique cone on a square (pyramid)

Figure 126. Oblique cones
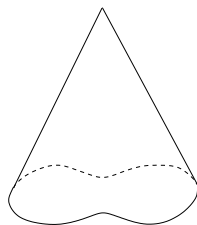
Figure 127. A cone with a curved base

10.3. **Visualizing solids.**

Visualization of solids (being able to picture the solid in your mind based on limited information given) is a skill that can be developed through practice or games and toys such as blocks. In this section we present problems covering various techniques for visualization such as nets, slicing, and depiction of different viewpoints.

Problem 10.3.1. (Adapted from [NJ])
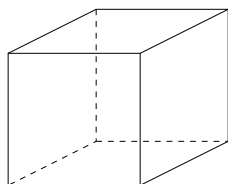
(1) Name each figure.
(2) How many faces does each figure have?
(3) Name one way that the two figures are the same.
(4) Name one way that the two figures are different.

Problem 10.3.2. (From [PARCC])

(1) Which of the following shapes are possible horizontal slices of the cube on the left in Problem 10.3.1? (a) squares (b) rectangles (c) triangles (d) trapezoids.
(2) Which of the following shapes are possible vertical slices of the cube? (a) squares (b) rectangles (c) triangles (d) trapezoids.

(3) Which of the following shapes are possible horizontal slices of the square pyramid shown in Problem 10.3.1? (a) squares (b) rectangles (c) triangles (d) trapezoids.
(4) Which of the following shapes are possible vertical slices of the square pyramid? (a) squares (b) rectangles (c) triangles (d) trapezoids.

Problem 10.3.3. (Adapted from [MI]) Which of the following is a picture of a cone? Of a cylinder? Of a rectangular prism?
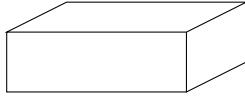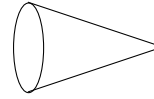
(a)   (b)   (c)

Problem 10.3.4. (Partly from [MCAS]) Using Magna-tiles:

(1) Build a triangular prism.
(2) Build a rectangular prism.
(3) How many faces does a triangular prism have?
(4) How many more faces does a rectangular prism have than a triangular prism has? Show or explain how you got your answer.
(5) How many faces does a hexagonal prism have? Show or explain how you got your answer.

Problem 10.3.5. Which of the following nets can be folded up (possibly after cutting along edges) to obtain the cube at the upper left?
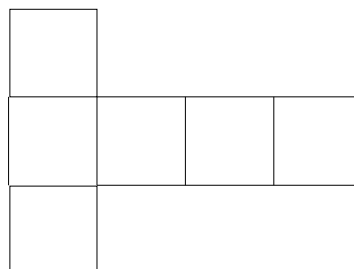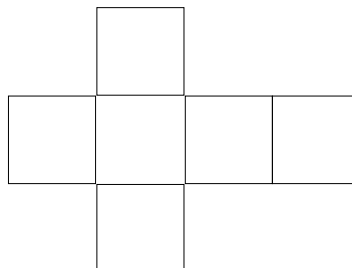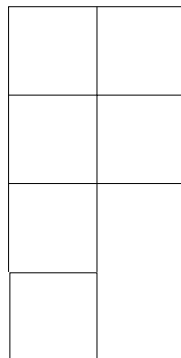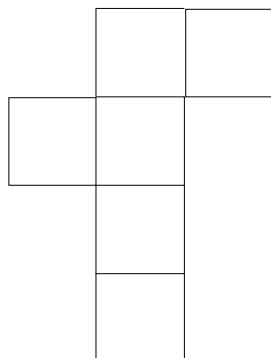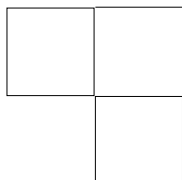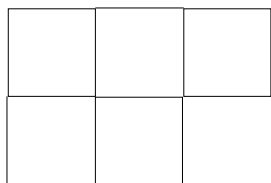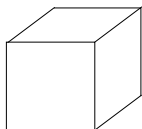
Problem 10.3.6. Which of the following nets can be folded up (possibly after cutting along edges) to obtain a tetrahedron?
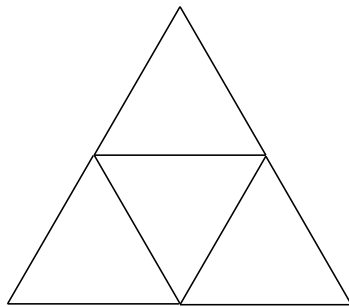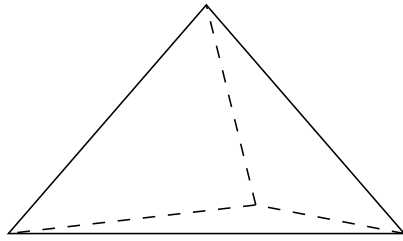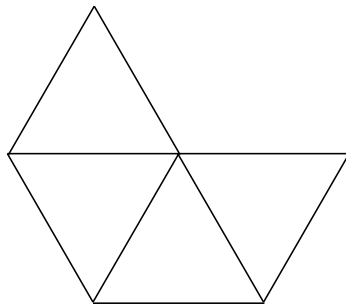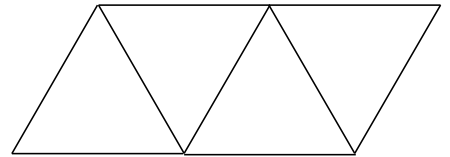
(a) (b) (c)

Problem 10.3.7. (Adapted from [MCAS]): Michael stacked cubes to make the structure shown below. Michael used a total of 14 cubes to make his structure.
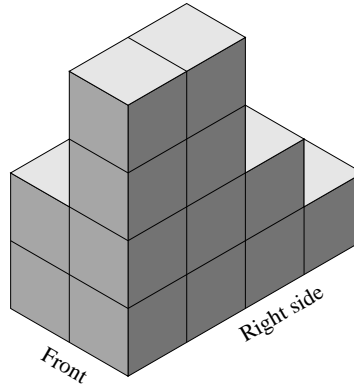
(1) Draw the right side view of Michael's structure. Be sure to label your drawing "right side view."
(2) Draw the front view of Michael's structure. Be sure to label your drawing "front view."
(3) Draw one possible top view for Michael's structure. Be sure to label your drawing "top view." Show or explain how you got your answer.

Spheres

Informally, children are familiar with spheres as shapes of balls. A basketball is the shape of a sphere and is filled with air. A baseball is solid all the way through. In geometry, for a given point P and a positive number r we have the following definitions:

A *sphere* of radius r and center P is the set of points in space that are distance r from P.

A *solid sphere* of radius r and center P is the region in space of points that are distance at most r from P.

Problem 10.3.8. (1) What geometrical shape is formed by the set of points that lie both on a sphere and in a plane? Describe the various possibilities depending on the position of the plane with respect to the sphere.
(2) What geometrical shape is formed by the set of points that lie both in a solid sphere and in a plane? Describe the various possibilities depending on the position of the plane.
(3) What geometrical shape is formed by the set of points that lie in two different spheres? Describe the various possibilities depending on the position of the spheres.
(4) (Challenge) Describe, as best you can, the geometrical shape formed by the set of points that lie in both a sphere and a solid sphere. Describe the various possibilities depending on the position of the sphere and solid sphere.

10.4. **Regular polyhedra.**

We say that a polyhedron is *regular* if the faces are congruent regular polygons and the angles between the faces at each edge are congruent. In fact, there are only five regular convex polyhedra. To understand why, we introduce a number at each vertex called the *angle defect*, which is the difference between $360°$ and the sum of the angles at the vertex. For example, if the polyhedron is a cube then there are three angles at each vertex summing to $270°$, and so the angle defect is $90°$.

Problem 10.4.1. (1) Find the angle defect at each vertex for the regular tetrahedron.
(2) For an arbitrary regular convex polyhedron, explain why the angle defect is the same for each vertex.
(3) (Challenge) Explain why the angle defect of a convex polyhedron at a vertex is always positive.
(4) Explain why each angle between edges meeting at a vertex must be less than $120°$. (Hint: how many faces meet at each vertex?)
(5) Explain why each face of a regular polyhedron can have at most 5 sides. (Hint: Which regular polygons have angles less than $120°$?)
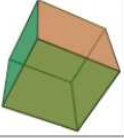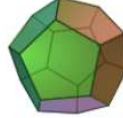
Once one understands that each face can have at most five sides, the classification of regular polyhedra breaks down as follows. For triangular faces, each regular polyhedron with triangular faces can have $3, 4$, or 5 triangles meeting at each vertex. Indeed having 6 triangles meeting at a vertex would yield an angle defect of 0 degrees, which is not possible for polyhedra as in the problem above. These polyhedra are the tetrahedron, octahedron, and icosahedron respectively. For square faces, there can be only three faces meeting each vertex, and one obtains a cube. For pentagonal faces, there is again a single arrangement, resulting in the dodechadron. Here is a nice picture of the convex regular polyhedra, taken from Wikipedia (probably generated with Mathematica), from which the description above was modelled.[40].

Note that in two dimensions, for any natural number greater then two there is a regular polygon with that number of sides. In contrast, in dimension three, there are only a finite number of regular polyhedra.

---

[40]Retrieved 2014 from Wikipedia

There are five convex regular polyhedra, known as the **Platonic solids**:

Main article: Platonic solid

Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}

and four regular star polyhedra, the **Kepler-Poinsot polyhedra**:

Main article: Kepler-Poinsot polyhedra

Small stellated dodecahedron {5/2, 5}	Great stellated dodecahedron {5/2, 3}	Great dodecahedron {5, 5/2}	Great icosahedron {3, 5/2}

FIGURE 128. Convex and non-convex regular polyhedra

10.5. **Regions in space and volumes.**

A child in a classroom can move towards the front or back of the classroom, from side to side, and, when excited, jump up and down. The interior of a classroom is an example of a *region in space*.

> A *region in space* is a part of space in which someone can move in three independent directions (for example, north-south, east-west, and up-down).

The region inside a house is divided into regions between floors: The region under the first floor is often called the *basement* while the region above the top-most ceiling is called the *attic*.

The *surface* of a region in space is its boundary. The surface of the region inside a house is the part of the house usually covered by siding, roofing, and foundation. To give another example, the earth takes up a region in space, whose surface is the surface that we usually stand on. The inside of a balloon is a region in space, whose surface is the balloon itself (pretending that the balloon has no thickness.) An ocean takes up a region in space, whose surface is not only the usual surface of

the ocean but also the surface where the ocean meets the bottom. The interior of a sphere is also a region in space.

The *volume* of a region of space (that is, a solid) is informally the *amount of space* that the solid takes up. More formally, the volume of a region of space is the number of unit cubes (or parts of unit cubes) needed to fill the region without gaps or overlap.

The English word *fill* is a bit confusing, because when we talk about solids such as the solid Earth, the solid is already filled. What we mean is that we first take everything out of the three-dimensional region to make it empty, and then count how many unit cubes are needed to fill it.

Here is a sample problem: Find the volume of the cube on the left of the figure below, using the definition of volume.
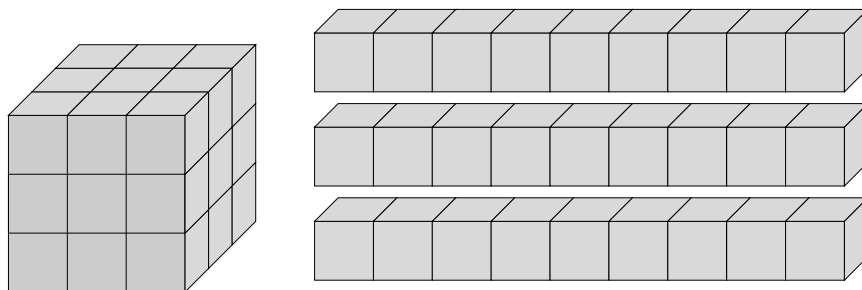
A sample answer is as follows. Since the cube is made up of 3 vertical "layers" and each layer has $3 \times 3 = 9$ unit cubes, we can conclude that the big cube has 3×9 unit cubes and so its volume is 27 cubic units. The left cube is made of 27 unit cubes (cubes of side lengths 1), and so has volume 27. One gets the same volume if all the small cubes are arranged in three rows, as shown in the figure above.

Problem 10.5.1. (From [SM]) The figure below shows a solid consisting of stacked cubes each of edge length 1 in. Find its volume.
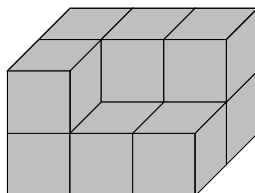
How would your answer change if each cube had edge length 2 in?

Volume of a right rectangular prism

We are interested in investigating the volume of a right rectangular prism, that is, a box. If the length l, height h, and width w are all integers, then we can fill the box by h layers of smaller boxes, each of height 1, and each layer containing w rows of l unit cubes. The total number of unit cubes in one such layer is $l \times w$, so the total number of unit cubes in the prism is $h \times (l \times w)$.
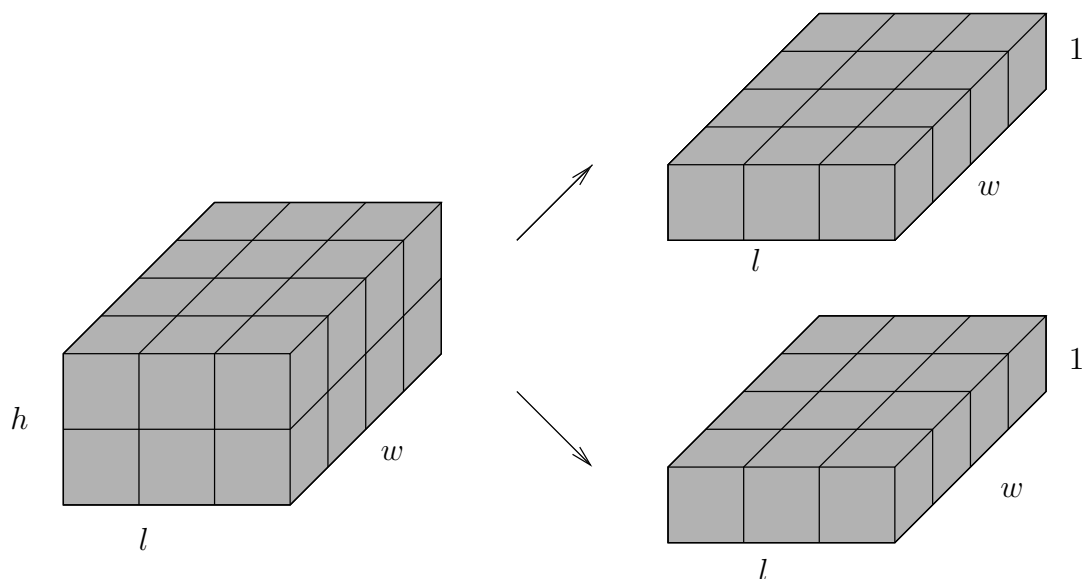
FIGURE 129. Splitting a prism into layers

Figure 129 shows the case $h = 2, l = 3$ and $w = 4$. In this case, the total number of cubes is 24 and each cube has volume one cubic unit, so the volume of the box is 24 cubic units. This procedure suggests that the volume of a box with dimensions l, h, w is

$$
\begin{aligned}
\text{volume} \ &= \ \text{number of cubes} \times \text{volume of each cube} \\
&= \ l \times w \times h.
\end{aligned}
$$

The formula for the volume of a rectangular prism can be extended to the case that the length, height and width are not integers. Suppose that l, h, w are unit fractions of a unit length, say $l = 1/n, h = 1/m, w = 1/p$. Since a cube of unit volume divides into nmp smaller cubes, the cube of dimension l, h, w has volume $1/nmp = lhw$.

Any right rectangular prism with rational side lengths divides into prisms with unit fraction side lengths; combining the arguments above proves the volume formula for these prisms. The following problem explores the formula further in the case of rational side lengths, assuming that the formula is known for integer side lengths and unit fraction side lengths.

Problem 10.5.2. Given a right rectangular prism of side lengths 3/4 units, 4/5 units, and 2/3 of a unit, find its volume in two different ways as follows:

(1) by breaking it up into smaller rectangular prisms;
(2) by forming a larger rectangular prism using a number of copies of the original. How does this support the formula for the volume of a rectangular prism in the case of rational side lengths?

Finally any prism with irrational side lengths may be approximated by those with rational side lengths and one obtains the volume formula as a limit. We will return to the discussion of volumes of prisms with other kinds of bases in Section 10.7.

Problem 10.5.3. Activity on Volumes of Boxes

(1) Consider a rectangular box with height 3 inches on a base that is 4 inches by 5 inches. How many unit cubes can you arrange in a single layer in this box? Note that this language assumes we want to completely cover the base with cubes.
(2) How many such layers can you put into the box to fill it up?
(3) Write the volume of the box both as a sum and as a product.
(4) What shapes make up the boundary of the box? What are their areas? What is the surface area of this box? Draw a net for this box.
(5) Consider a box with dimension 8 inches, 7 inches and 6 inches. What is the maximum number of 2 cm by 2 cm by 2 cm plastic cubes can be stacked neatly inside the box?

Problem 10.5.4. Activity on Volumes of Boxes

(1) Find the volumes of the boxes shown.

4in

4in

10 in

10 inches

area 20 square inches
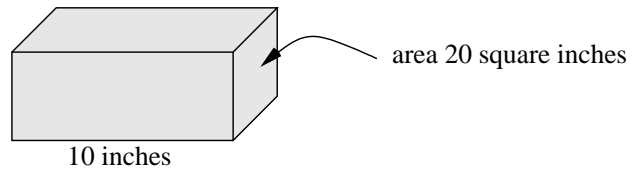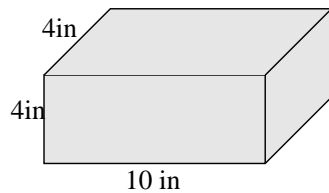
(2) Find the volume of a water tank in the shape of a rectangular prism measuring 5 ft by 10 ft by 10 ft.
(3) A rectangular tank with a base measuring 1ft by 1 ft contains water and a stone. With the stone, the height of the water is 10 in. After the stone is removed, the height of the water is 8 in. Find the volume of the stone in in^3.

Problem 10.5.5. (from [PARCC]) Small cubes with edge lengths of 1/4 inch will be packed into a right rectangular prism with length $4\frac{1}{2}$ in, width 5 in, and height $3\frac{3}{4}$ in. How many small cubes are needed to completely fill the right rectangular prism?

We mention a brief warning on the notation for volume formulas. Some sources use $V = B \times h$ for the formula of the volume of a prism. This has the danger of introducing significant errors. For example, consider the following problem: Find the volume of a prism with square base having lengths 10 feet, and height 20 feet. Many people will write $V = 10 \times 20 = 200$ cubic feet, which is incorrect. This error can be fixed by working correctly with units: If someone writes

$$V = 10\text{ft} \times 20\text{ft} = 200\text{ft}^2,$$

that person can see that they have used the formula incorrectly because the outcome is an answer in square feet, not cubic feet. The correct answer should be

$$V = B \times h = (10\text{ft} \times 10\text{ft}) \times 20\text{ft} = 2000\text{ft}^3.$$

This is why we prefer the better notation $V = A_{\text{base}} \times h$.

Properties of volume

Volume satisfies moving and additivity properties that are similar to those for area. For example, the volume of the region inside the house is the sum of the volumes of the attic, the basement, and the regions between each two consecutive floors. The formal statements are:

> (Congruence/Moving Property) Congruent regions in space have the same volume.
> (Additivity Principle) If a region in space is cut into two parts, then the volumes of the parts sum to the volume of the original region.

Each of these properties can be explained by thinking of a region in space as filled with unit cubes (or parts of unit cubes). Congruent regions can be moved to overlap exactly with each other, and moving does not change the number of cubes needed to fill each region. It follows that the volumes of any two congruent regions (which are the number of unit cubes used to fill them) are equal. For the additivity principle, the number of unit cubes it takes to fill the larger region is the sum of the numbers of unit cubes for each part.

In Chapter 5 we saw that shearing a plane figure does not change its area. Similarly, shearing a three-dimensional object does not change its volume. In the three dimensional context, what we mean by shearing is slicing the region into regions of small vertical height and moving the slices horizontally to form a new shape. For example, a non-right rectangular prism can be thought of as a deck of cards or stack of paper which has not been straightened; straightening the deck so that they form a right rectangular prism is an example of a shear. See Figure 130. In particular, since the base and height of a non-right rectangular prism are the same as those of the right rectangular prism obtained by shearing, we conclude that the volume of a non-right rectangular prism is also given by the formula $V_{\text{prism}} = l \times w \times h$.

FIGURE 130. Shearing a non-right prism into a right prism

10.6. **Surface area.**

The surface of a child is made up of his or her skin, while the surface of the earth is partly covered by the oceans. More generally, the boundary of a solid in space is often called a *surface*. For example, spheres, cubes, and cylinders are surfaces.

The surface can be distinguished from the region inside it by adding the word "solid". For example, the earth is approximately a *solid sphere* also called a *ball*.

The *surface area* of a solid is the area of the surface of the solid, that is, the area of its surface if we flattened it out. Some surfaces cannot be easily flattened but we may view any sufficiently nice surface as made up of approximately flat pieces. We will discuss curved surfaces more in Section 10.8.

The *lateral surface area* of a prism or cylinder is the area of the surface minus the area of any bases, that is, the sides without the top or bottom.

Problem 10.6.1. The figure below shows a solid consisting of stacked cubes each of edge length 1 in. Find its surface area.
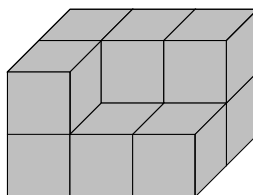
How would your answer change if each cube had edge length 2 in?

Problem 10.6.2. (In this problem, it may be helpful to have at least nine cubes or Lego blocks to help with visualization.) In each diagram below, each square represents a column with the indicated height. For each diagram, draw a picture of the associated building. Use a perspective in which each dimension is visible. (The answer to the first problem is given as an example.) Find its volume and surface area. (Discuss: should the base be included in the surface area?)

Height Diagram

Sample Answer

(a) | 1 | 2 |

(b) | 1 | 2 |
    | | 1 |

(c) | 1 | 2 | 3 |

(d) | 1 | 2 | 3 |
    | | | 2 |
    | | | 1 |
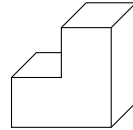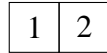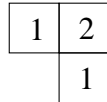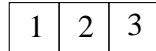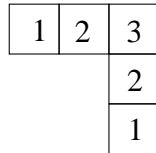
Behavior of surface area and volume under scaling

The following problem is meant as an exploration of how volume and surface area change under scaling.

Problem 10.6.3.

 (1) Find the volume and surface area of a cube with edge lengths 3 feet.
 (2) Find the volume and surface area of a cube with edge lengths 6 feet.
 (3) Suppose that one cube has twice the side lengths of another cube. How do the volumes relate?
 (4) Suppose that one cube has twice the side lengths of another cube. How do the surface areas relate?

As the problem above suggests, the volume and surface area of a region in space changes in an easily described way under rescaling, as shown in Figure 131.

scale factor 2
volume scales by 2x2x2 =8

scale factor 2
surface area scales by 2x2=4
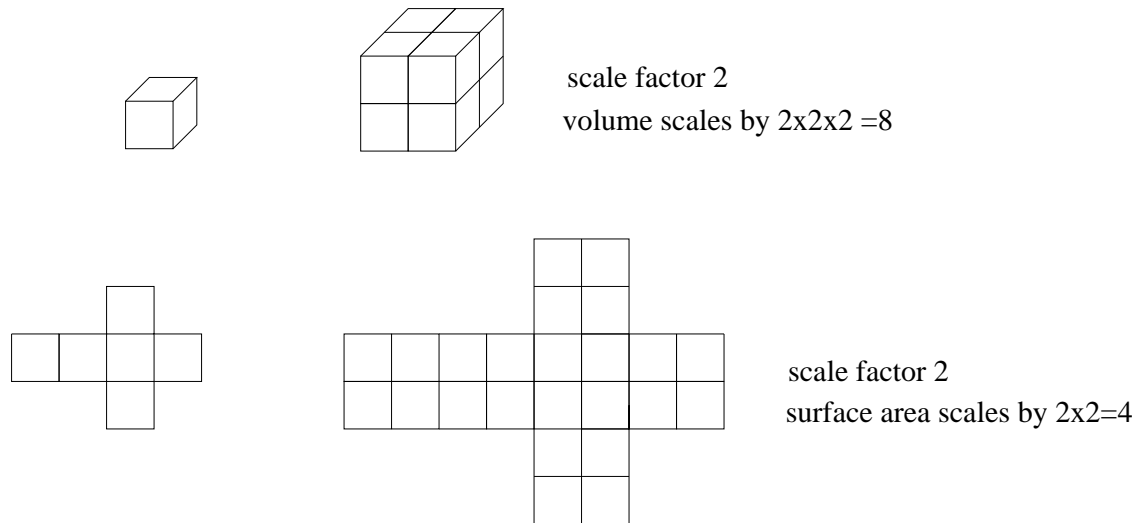
FIGURE 131. Scaling volume and surface area

To explore how volume changes under scaling, suppose that the first region R_1 is filled with V_1 unit cubes, and let R_2 denote the region obtained by rescaling region R_1 by a scale factor s. Let V_2 be the volume of the region R_2. Under rescaling, each unit cube in the first region R_1 becomes a cube of side lengths s in the second region R_2. Each cube of side lengths s can be filled with s^3 unit cubes (or parts of

unit cubes). Therefore,

$$
\begin{aligned}
V_2 &= \text{no. unit cubes to fill } R_2 \\
&= \text{no. cubes of side-length } s \text{ to fill } R_2 \\
&\quad \times \text{ no. unit cubes per cube of side-length } s \\
&= \text{no. unit cubes to fill } R_1 \\
&\quad \times \text{ no. unit cubes per cube of side-length-} s \\
&= V_1 \times s^3.
\end{aligned}
$$

Note that the explanation above does not rely on the region being of any particular shape. For example, suppose you have a sphere of volume 2 cubic feet. If you double the radius, how much is the volume of the resulting sphere? It is not necessary to know the volume formula for the sphere to answer this problem: If R_1 is the region inside the first sphere and R_2 is the region inside the second sphere then the volumes V_1, V_2 are related by

$$
V_2 = V_1 s^3 = 2\text{ft}^3 \times 2^3 = 16\text{ft}^3.
$$

A similar discussion using unit square shows that surface area scales as the *square* of the scale factor:

$$
A_2 = A_1 \times s^2.
$$

In summary, volume scales as the *cube* of the scale factor while surface area scales as the *square*.

Problem 10.6.4. From [MCAS]:

A fish tank at a science museum in the shape of a rectangular prism has a 10 feet by 6 feet base and a height of 5 feet.

(1) What is the volume, in cubic feet, of the fish tank? Show your work.
(2) The tank is made of glass and does not have a cover. What is the surface area, in square feet, of the outside of the fish tank? Show your work.
(3) A new fish tank is being built for the museum. The new tank will have different dimensions than the first tank, but will have the same volume as the first tank. What could the dimensions of the new tank be? Show or explain how you got your answer.

Problem 10.6.5. (Adapted from [VA]) A company makes two similar cylindrical containers. The surface area of the smaller container is 0.8 times that of the larger. If the height of the larger container is 60 cm, what is the height of the smaller container?

10.7. **Volumes and surface areas of cylinders and cones.**

The *height of a prism or cylinder* is the distance between the plane of the top and the plane of the bottom. Notice that this is not the same as the distance from a point on the top to the corresponding point on the base.

The *height of a pyramid or cone* is the distance of the vertex to the plane containing the base.
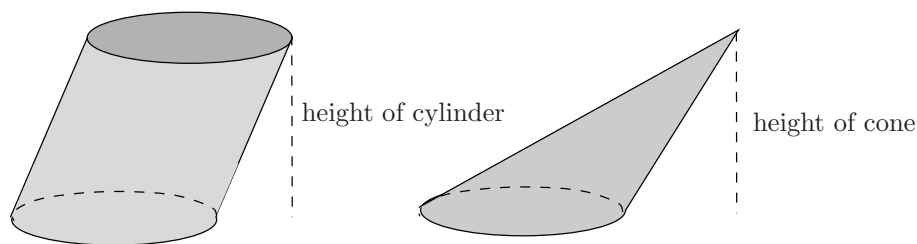
FIGURE 132. Heights of cones and cylinders

Problem 10.7.1. (Requires popped popcorn, paper, and tape)

(1) Take a piece of paper (standard size or similar) and roll it in the short direction to make a cylinder with circular base; use tape to attach the two sides without overlap. Using a ruler find the height of the cylinder and the radius of the base.
(2) Take a different piece of paper of the same size as in the previous part. Bend it in the long direction to make a cylinder of a different shape. Using a ruler find the height of the new cylinder and the radius of the base.
(3) Which cylinder do you think will have the biggest volume, and why?
(4) Test your conjecture by filling one cylinder with popped popcorn, then using that popcorn to try to fill the other cylinder.
(5) In items (1) and (2), how do the radii of your cylinders compare to the dimensions of the paper?

Problem 10.7.2. (From [PARCC]) A right circular cone is shown in the figure. Point A is the vertex and point B lies on the circumference of the base of the cone. The cone has a height of 24 units and a diameter of 20 units. What is the distance from point A to point B?
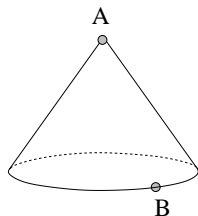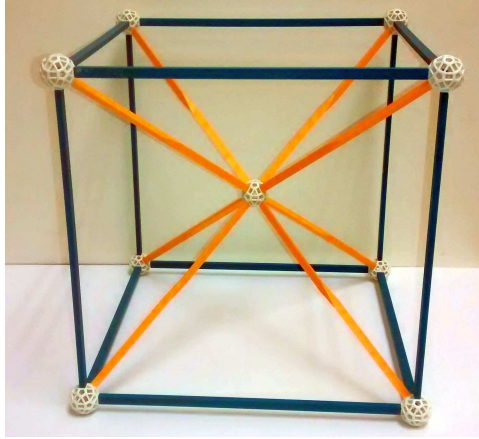
Figure 133. A cube built with Zome tools

Problem 10.7.3. Consider a unit cube with a point at the center. (This can be done with Zome tools if available or with nets. A depiction using Zome tools is shown in Figure 133.)

(1) Draw a line segment between each vertex and the center and visualize the convex pyramidal regions whose edges are these line segments. How many pyramidal regions in space does this divide the interior of the cube into?
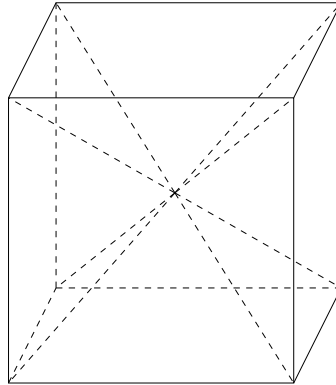
(2) What is the volume of each of these pyramidal regions, based on the fact that they divide the cube and are all congruent?
(3) For each pyramidal region, find the area of the base and the height.
(4) Write an equation expressing the relationship between the volume of each pyramidal region, its base, and its height.

Volume formulas for prisms, pyramids, cylinders and cones

Because of the additivity principle for volume, the volume of a collection of non-overlapping rectangular prisms is the sum of the volume of the individual prisms. For example, if each individual prism in the figure below has base area 1 square unit and height 7 units, then the volume of each is 7 cubic units and the volume of the region in space as a whole is $5 \times 7 \ u^3$, that is, 35 cubic units.
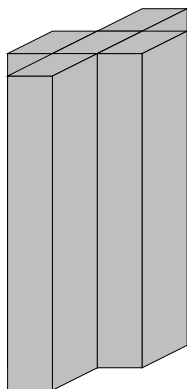
FIGURE 134. A region in space made up from rectangular prisms

Using rectangular prisms one can create many different regions in space. In particular one can cut up a cylinder into pieces, almost all of which are rectangular prisms with the same height. Using the additivity principle for volumes, the volume of the prism cylinder is approximately the sum of the volumes of the rectangular prisms. Using the distributivity property of multiplication over addition, this implies that the *volume of a prism or cylinder* whose base has area A_{base} and whose height is h is

$$V_{\text{cyl}} = A_{\text{base}} \times h.$$

The *volume of a pyramid or cone* must be less than the volume of the prism or cylinder with the same base and height, since it fits inside it.

Another special type of pyramids for which the volume formula is more easily derived are right-angled pyramids: these are pyramids whose top vertex lies above an edge or or vertex of the base. The GeoGebra applet linked below gives a visual explanation of the volume formula for right-angled pyramids with the top vertex lying above a base vertex, as the pyramid shown in Figure 135.

<p style="text-align:center; color:#a00">Three pyramids forming a cube</p>

We already discovered the volume formula for this kind of cone above, in Problem 10.7.3. The volume of a right rectangular pyramid (a pyramid whose top vertex lies above the center of the base) that has base area A_{base} and height h is given by

$$V_{\text{cone}} = (1/3)A_{\text{base}} \times h.$$

Using that volumes are invariant under shears justifies the formula for the volume of a pyramid that is not right. To see that the formula is true for a cone on any base,
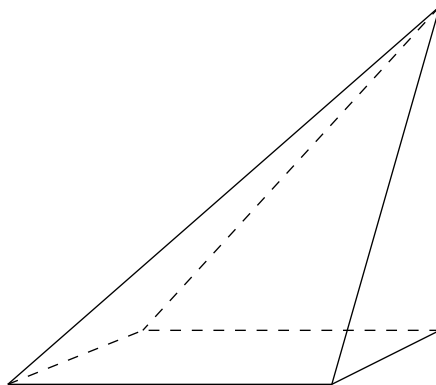
FIGURE 135. A right-angled pyramid whose top vertex lies over a base vertex

note that any cone can be divided into small cones with square bases. The formula for the volume in general follows from the distributivity property, by adding up the volumes of the pieces.

The PARCC assessment formula sheet (see [PARCC]) uses the slightly different notation $(1/3)B \times h$. However, a common error is to write $V = (1/3)$base $\times$ height and then use a *length* of the base instead of the area. This is the kind of mistake that can be avoided by working correctly with units throughout, since only if an *area* is multiplied by a height will the answer become a number of *cubic* units.

Problem 10.7.4. (1) Find the volume of a barrel that has base with area 10 square feet and is 3 feet high.
(2) Find the volume of a barrel that has base with radius 10 feet and is 3 feet high.
(3) Find the volume of a barrel that has base with diameter 10 feet and is 3 feet high.

Problem 10.7.5. (Adapted from [VA]) Which of the following two objects have the same volume?

(1) A cylinder with radius 3 cm and height 4 cm;
(2) A cone with height 2 cm and diameter 6 cm;
(3) A cylinder with diameter 2 cm and height 6 cm.

Problem 10.7.6. From [TX]: A cylindrical glass vase is 6 inches in diameter and 12 inches high. There are 3 inches of sand in the vase. Which of the following is closest to the volume of the sand in the vase?

(1) 85 cubic in.
(2) 254 cubic in.
(3) 54 cubic in.
(4) 339 cubic in.

Problem 10.7.7. A pyramid with square base of sides 200 feet by 200 feet is intended to be built 100 feet high to house the tomb of a king. However, before the pyramid can be finished the king is overthrown by his son and the project is abandoned. The resulting structure is only 50 feet high, and was built from the bottom to top so that the top 50 feet of the would-be-pyramid is missing. What is the volume of the resulting structure?

Surface areas of circular cylinders and cones

Children are familiar with circular cylinders through everyday objects such as soup cans. [41]

<small>FIGURE 136. A cylinder in real life</small>

Problem 10.7.8. A soup can has a radius of 1 in and a height of 4 in. What dimensions should the label (fitting around the can, without tops) have? What is its area?

The surface area of a cylinder can be decomposed into the area of the sides, called the *lateral surface area*, and the areas of the top and the bottom. In the soup can example, the lateral surface area is the *area of the label*. The lateral part of a right cylinder with height h and radius r can be flattened to a rectangle of height h and width $2\pi r$, so the lateral surface area is the same as the area of this rectangle:

$$A_{\text{cyl,lateral}} = 2\pi rh.$$

The *total surface area* of the right cylinder (with the top and bottom) is

$$A_{\text{cyl}} = 2\pi rh + 2\pi r^2.$$

The following problem explores the lateral surface area of a right circular cone.

Problem 10.7.9. The lateral surface of a cone is made from one-fourth of the interior of a circle of radius 8 inches (by joining radii).

    (1) Determine the total surface area of the cone, including the base.
    (2) What is the height of the cone? Explain your answer.

Problem 10.7.10.   (1) Consider a right circular cone with base a circle of radius 3 in and height 4 in. What is the distance between a point on the perimeter of the base and the vertex of the cone?

---

[41] Photo by Jonn Leffmann. Used under the Creative Commons Attribution 3.0 Unported license.

(2) What is the circumference of the circle with radius the distance you found in part (1)? How does it compare to the circumference of the base of the cone?

(3) Express the lateral surface area of the cone in terms of the area of the circle that you found in part (2).

(4) (Challenge) Make a conjecture for the lateral surface area of the right circular cone in terms of the radius of the base and its height.

10.8. **Volumes and surface areas of spheres.**

The eighth-grade Common Core State Standards ask students to know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems. The inclusion of these formulas in the standards is somewhat debatable. In the age of technology, knowing how to use a formula is a skill that doesn't qualify someone for any job. This is because computers can do "plug and chug" mathematics much better than humans can. Almost all middle school students and teachers do not know the origin of the formula, and even many professional mathematicians mistakenly believe that one needs calculus. On the other hand, one might say that it is helpful to know that there is a formula.

Given that these formulas are on the Common Core State Standards, we will forge ahead and explain how one can understand them, without using any mathematics beyond similarity of triangles. We start with the formula for the surface area of a sphere, which is easier to understand than the volume in the sense that one can explore surface area experimentally using fruit peels. (One can also go the other way, trying to understand the volumes of spheres first and then surface areas.)

Problem 10.8.1. Cut up an orange peel horizontally, still on the orange, into four pieces of equal height. (This requires a knife and ruler but may be done in advance for safety.)
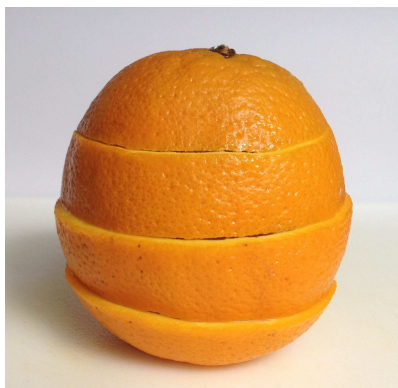
(1) Make a conjecture about how the areas of the pieces compare to each other. Check your conjecture by cutting the non-circular pieces into smaller parts and re-arranging so that the parts of a non-circular piece fit on top of a circular piece.

(2) How does the surface area of the orange with radius r compare to the lateral surface area of the cylinder with radius r and height $2r$? Explain your answer without using the formulas for the surface areas of either shape. You may wish to rearrange the pieces of the peel on top of a rectangle representing the lateral surface of the cylinder.

(3) What is the lateral surface area of the cylinder with radius r and height $2r$?

(4) Find the formula for the surface area of the sphere in terms of its radius, using your comparison with the cylinder.

Surface area of spheres

The justification for the formula for the surface area of the sphere depends on the additivity principle for surface area. We cut the surface into pieces of equal height, and add them up to get the total surface area. Each piece, as shown in Figure 137, is approximately an annulus. In order to compute the area of each piece, we need to know the circumference of the annulus as well as the width. It turns out that, amazingly, each piece has exactly the same area. As one moves higher in the sphere, the pieces have smaller circumference but larger width. These two effects cancel.
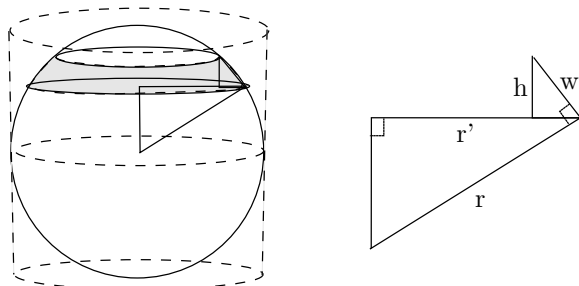
FIGURE 137. Cutting the sphere horizontally into pieces

To understand this cancellation, we consider the two triangles in Figure 137. In fact the two triangles in the figure are approximately similar. This is an application of the angle-angle-angle criterion. The radius of the sphere meets the surface of the sphere at a right angle, so the angles of the small and large right triangles meeting at that point are approximately complementary. Since the acute angles in a right triangle are always complementary, each angle in the large right triangle has the same measure as some angle in the small right triangle, and the triangles are similar by the angle-angle criterion.

This gives a formula for the area of each piece, as follows. Let r be the radius of the sphere, r' the radius of the bottom circle of the shaded region, h the height of the shaded region, and w the width of the peel layer on the outside of the orange. Because the triangles are approximately similar we have that

$$r'/r \approx h/w \quad \text{or equivalently} \quad r'w \approx rh.$$

Each piece, after flattening, is approximately a rectangle of length $2\pi r'$ and width w and so has area

$$(2) \qquad A_{\text{piece}} \approx \text{length} \times \text{width} = 2\pi r'w \approx 2\pi rh.$$

Having found the area of each piece, we can now find the surface area of the sphere by adding up the areas of all the pieces. If the number of pieces is n then each piece has height

$$h = \text{height of sphere/number of pieces} = 2r/n.$$

So the total area of the sphere is approximately

$$
\begin{aligned}
A_{\text{sphere}} &= \text{number of pieces} \times \text{area of each piece} \\
&\approx n \times 2\pi rh \\
&= n \times 2\pi r(2r/n) \\
&= 4\pi r^2
\end{aligned}
$$

which is Archimedes' formula. By dividing the sphere into pieces, of smaller and smaller height one sees Archimedes' formula is exact.

Surface area of cylinders and spheres, compared

Another way of viewing this argument is to say that the sphere has the same surface area as the lateral surface area of the cylinder in which it is inscribed, that is,

$$\text{perimeter of base} \times \text{height} = 2\pi r \times 2r = 4\pi r^2.$$

This is because each piece of the sphere (layer of peel) has the same area as a piece of the sphere near the equator. The piece of the sphere near the equator is approximately equal in shape to a piece of the cylinder with the same height and radius as the sphere. After replacing each piece of the sphere with the corresponding part of the cylinder, one gets a cylinder with the same height and radius as the original sphere. Archimedes was so proud of this result that he ordered it to be etched onto his tombstone.

Problem 10.8.2. Suppose a can of tennis balls contains three tennis balls. Which is greater, the lateral surface area of the can or the total surface area of the three balls?[42]
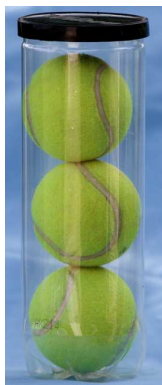
Problem 10.8.3. Find the surface area of the sphere with radius

(a) 1 cm;

(b) 4 miles;

---

[42]Photo copyrightTom Hutton 2016. Used with permission.

(c) 5 feet.

First express the answers using π, and then give an approximation for each to the nearest unit.

Volumes of spheres

The volume of a sphere can be found by using the additivity principle for volume. We divide the sphere into pieces each of which is approximately a pyramid with a trapezoidal or triangular base as follows. Cut the sphere of radius r horizontally into pieces of equal height. Then cut the sphere into congruent pieces of equal angle from the vertical axis as shown in Figure 138. Connect each piece to the center of the sphere to form an approximate pyramid. The volume of the sphere is the sum of the volumes of these pieces.
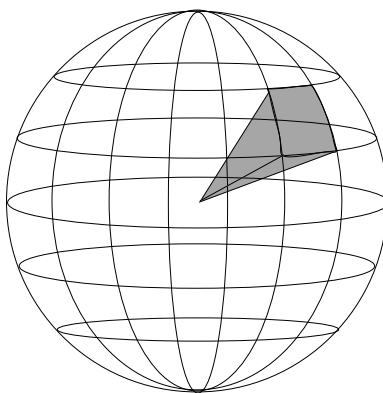
FIGURE 138. Dividing the region inside the sphere into pieces of equal volume

We will now explain why any two of the smaller pyramidal-type regions of space inside the sphere have the same volume. Each of the pieces of the surface of the sphere cut up in this way has the same area by (2). If there are p pieces, then the area of the base of each pyramidal shape is

$$A_{\text{base}} = A_{\text{sphere}}/p = 4\pi r^2/p.$$

Each piece is approximately a pyramid with height r. Using the formula for the volume of a pyramid we get

$$V_{\text{piece}} \approx (1/3)\text{area of base} \times \text{height} \quad = \quad (1/3)(4\pi r^2/p) \times r$$
$$= \quad (4/3)\pi r^3/p.$$

Finally we add up the volumes of the pieces to get the volume of the sphere. Since the number of pieces is p we get

$$V_{\text{sphere}} \quad = \quad \text{number of pieces} \times \text{volume of each piece}$$
$$\approx \quad p \times (4/3)\pi r^3/p$$
$$= \quad (4/3)\pi r^3.$$

The approximations in the above calculation turn into equalities as the sphere is divided into smaller and smaller pieces. That is, the *volume of a sphere* of radius r is

$$V_{\text{sphere}} = (4/3)\pi r^3.$$

Problem 10.8.4. (a) (A similar problem can be found in the Everyday Mathematics [EM] 6th grade curriculum.) Estimate the volume of the body of a stuffed animal, by approximating each piece of the body as one of the types of solids we have discussed above. [43]

Body Part	Radius or Length/Width (cm)	Height (cm)	Volume formula	Volume	Numb. Parts	Total Volume
					1	

(b) Suppose you have a (right, rectangular) pool in your back yard of dimensions 20 ft by 20 ft and depth 5 ft. How could you measure the volume of your body? You have the following tools, not all of which may be necessary: a ruler, a hose (to fill the pool), and a cup.

(c) Measure the dimensions of a can of root beer and a root beer glass, similar to those shown in the Figure below. Find the volume of both. Will the contents of the can fit in the glass? (Hint: One cannot just take the average of the areas of the top and bottom of the glass and multiply by the height. [44] [45]

---

[43]The original problem asked to use the students body, but teachers in practice avoided the problem due to sensitivities of the children. For this reason, a stuffed animal was substituted.

[44]Used under GNU Free Documentation license. https://commons.wikimedia.org/wiki/File:Rootbeerfoam2.JPG

[45]From https://s3.amazonaws.com/everipedia-storage/ProfilePics/69693548642.PNG

10.9. **Conversion of area, volumes, and rates.**

Unit conversion at its core involves proportional reasoning. Each quantity of some unit is equivalent to a quantity of another unit determined by the conversion factor. Two methods for unit conversion are *substitution* and *ratios*. As we mentioned earlier, in the substitution method, one unit is replaced with an equivalent number of units of the desired type (for example, 1 ft is replaced with 12 inches); in the ratio method, the given quantity is multiplied by a ratio of units that is equal to 1 (for example, the ratio (1 ft)/(12 inches).

Substitution

Replace the unit that needs to be converted with the equivalent unit, and then simplify. For example, to convert 50 inches^2 to square feet we write

$$50 \text{ inches}^2 = 50((1/12)\text{ft})^2 = 50/144 \text{ ft}^2.$$

As seen in the examples below, square units get multiplied by the conversion factor twice and cubic units get multiplied three times.

> Problem: Convert 80 square miles to square kilometers. Sample answer: 80 miles$^2 \approx 80(1.6\text{km})^2 = 80(1.6)^2\text{km}^2 = 80(2.56)\text{km}^2 = 204.8\text{km}^2$.
> Problem: Convert 10 cubic feet to cubic meters. Sample answer: 10 ft$^3 \approx$ 10 $(1/3.2\text{m})^3 = 10/(3.2)^3\text{m}^3$ which is about .305m^3.

Ratios To convert using ratios, multiply by the ratio of the two equivalent units (which is equal to 1) to an appropriate power, and then simplify. Since multiplying by one does not change quantities, this method produces an equivalent expression in different units.

Here is a sample problem: Convert 50 square inches to square feet.

A sample answer is as follows. $50 \text{ inches}^2 = 50 \text{ inches}^2(\frac{1 \text{ ft}}{12 \text{ inches}})^2 = \frac{50}{144} \text{ ft}^2$.

Problem 10.9.1. Convert the following using either method.

    (1) 10 square miles, to square feet.
    (2) 10 cubic feet, to cubic inches.
    (3) 100 square kilometers, to square centimeters.

Problem 10.9.2. Convert each of the following measurements, by *both* methods (substitution and ratios)

    (1) 175 inches to yards.
    (2) 100 meters to feet.
    (3) 70 miles per hour to kilometers per minute.
    (4) 1 square mile to square feet. You may approximate.
    (5) 10 ft × 44 yards × 2 inches to cubic feet.
    (6) 1 cubic meter to cubic inches.

The following is a non-traditional conversion problem in which you are required to determine the conversion factors.

Problem 10.9.3. The inhabitants of a remote island do not use money; instead, they barter goods. Four ducks are worth two blankets. Three blankets are worth two coats. Tom has many ducks and needs a coat. Considering the given barter rates, how many ducks must he exchange for a coat?

    (1) Solve the problem using substitution.
    (2) Solve the problem using the ratio method.
    (3) Can you solve this problem using any other method?

Problem 10.9.4. Water drips out of a tub faucet at 2 deciliters per minute. The capacity of the tub is 0.24 cubic meters. How long will it take for 2/3 of the tub to be filled with water if no water drains? Express your answer in hours. (Note: 1 liter = 1 cubic decimeter).

10.10. **Workshops.**

Workshop 10.10.1.

The figure below shows a one-story house formed by gluing together a right rectangular prism and a right pyramid with rectangular base. Make sure to explain your work briefly for each part.
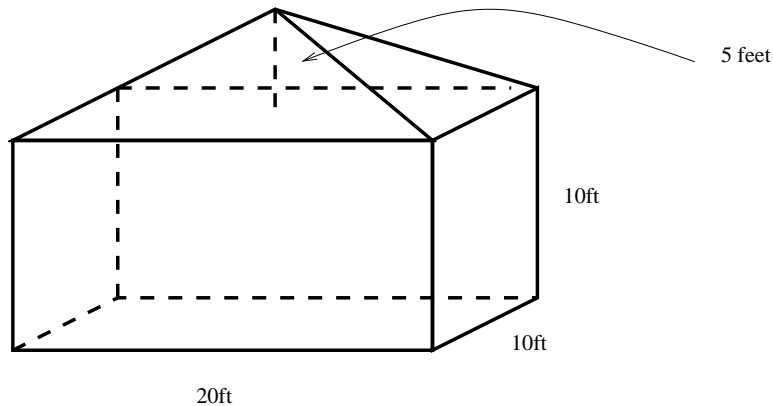
(1) Find the volume of the house.
(2) Find the surface area of the house, including the floor area.
(3) Convert the volume of the house to cubic yards.
(4) By what factor should the house be rescaled so that the floor area is doubled?
(5) By what factor should the house be rescaled so that the volume of the house is doubled?

Workshop 10.10.2. from [MCAS]:

A water dunking tank at a carnival is in the shape of a right circular cylinder. Its height is 5 feet, and the radius of the base is 3 feet.

(1) What is the lateral surface area, in square feet, of the tank? Show your work.
(2) On the first day of the carnival, the dunking tank was filled with water to a height of 4 feet. What was the volume, in cubic feet, of the water in the tank on the first day of the carnival? Show your work.
(3) A teacher volunteering for charity at the carnival, gets dunked. Suppose that the height of the water rises 2 inches. Deduce the teacher's volume.

At the end of the second day of the carnival, some water was drained from the tank. The volume of water drained was 35.3 cubic feet.

(4) Using your answer from part (2), determine the height, in feet, of the water remaining in the tank after the water was drained at the end of the second day. Show your work.

  The water that was drained from the tank was poured into containers, each in the shape of a right rectangular prism. Each container was 2 feet in length, 1.5 feet in width, and 3 feet in height.
(5) What was the least number of containers needed to hold all the water that was drained at the end of the second day? Show your work.

Workshop 10.10.3.

A right pyramid with triangular base has base side lengths 200 feet and height 200 feet. Find

(1) the volume of the pyramid;
(2) the area of the base;
(3) the height of each of the triangular sides (Recall: this is the distance from the top vertex to the base of the triangular side, not to the base of the pyramid.)
(4) the area of each triangular side; and
(5) the distance from any of the corners of the base to the vertex at the top of the pyramid. (You will need to approximate a square root for this. Any reasonable answer is fine.)

Workshop 10.10.4.

(1) Suppose that one swimming pool is three times as long, deep, and wide as another swimming pool. Suppose it takes 20 hours to fill the small pool with a hose. How many hours will it take to fill the large pool?

(2) A recipe for apple pie calls for 6 regular size apples. You have a bag of apples whose length, height, and width are half those of regular apples. How many apples do you need for the apple pie?

(3) Suppose that a balloon (not necessarily spherical) is of the type that as it inflates, each of the resulting shapes is similar to each other. Suppose that four breaths blow up a balloon to a diameter of 1 ft, where by diameter we mean the distance between points on the balloon that are furthest away from each other. What is the diameter after an additional breath? (Assume that each breath takes up equal volume. Hint: By what factor does the volume increase on the fifth breath? Deduce from this factor the scale factor for the fifth breath.)

(4) A sphere has volume $2\,\text{ft}^3$. A second sphere has radius twice that of the first sphere. Find the volume of the second sphere.

(5) A sphere has radius $2\,\text{ft}$. A second sphere has volume twice that of the first sphere. Find the radius of the second sphere.

Workshop 10.10.5. (Sea level and global warming)

(1) Estimate how long it takes to fly half way around the world, and the approximate speed. (Hint: it takes about 6 hours to fly from New York to Los Angeles and the Pacific ocean is VERY large. Planes travel about ten times as fast as cars.) There is no exactly right answer, just reasonable guesses.

(2) Estimate the circumference of the earth, using this data. Using this estimate the radius of the earth.

(3) Using (2) estimate the surface area of the earth.

(4) The Greenland ice sheet is almost 2,400 kilometers long in a north-south direction, and its greatest width is 1,100 kilometers at a latitude of 77° N, near its northern margin. The thickness is generally more than 2 km (see picture) and over 3 km at its thickest point. Approximately how much ice (in cubic miles) is contained in the Greenland ice sheet?

(5) How much would sea level rise if half the ice in Greenland melted (in meters and in feet)? (Hints: You need the approximate surface area of the Earth from (3) for this as well as the fact that about 2/3 of the Earth's surface is ocean and would remain so after the melting. Also, the sea level rise is approximately the volume of the ice divided by the ocean surface area.

[46]Retrieved from https://commons.wikimedia.org/wiki/File:Greenland_ice_sheet_AMSL_thickness_map-en.png March 2016. Public domain illustration obtained from NOAA.

10.11. **Assessments and sample student work.**

Problem 10.11.1. (Adapted from [MCAS])

Erika has a cylindrical container with a diameter of 6 inches and a height of 1.5 feet.

    (1) What is the height, in inches, of the container? Show or explain how you got your answer.
    (2) What is the volume, in cubic inches, of the container? Show or explain how you got your answer. (Use 3.14 for π.)
    (3) Erika filled the container with 250 cubic inches of sand. What is the approximate height, in inches, of the sand that Erika put in the container? Show or explain how you got your answer.

A sample answer for half-credit is given below.

Incorrect answers showed incorrect use of units.

Problem 10.11.2. (from [MCAS]) Ray had a block of wood in the shape of a rectangular prism:

4 inches
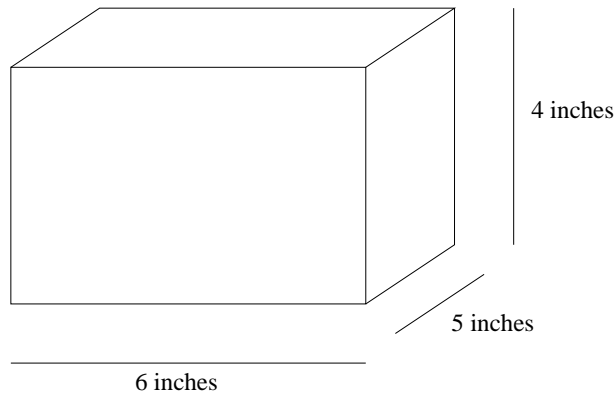
5 inches

6 inches

    (1) Ray painted the front face of the block red. What is the area, in square inches, of the face he painted red? Show or explain how you got your answer.
    (2) Ray painted the top and bottom faces of the block black. What is the area, in square inches, of the faces he painted black? Show or explain how you got your answer.
    (3) Ray painted the other faces of the block white. What is the area, in square inches, of the faces he painted white? Show or explain how you got your answer.

A sample answer for half-credit is shown below:

Incorrect answers showed awareness of the formula area = base x height, but confusion about what to do with it. Also, incorrect use of units was common, even in answers marked as correct, and many answers did not give sufficient detail of work.

11. Instructor's guide to activities

In this Chapter we comment on some of the subtleties the Instructor may wish to mention regarding the material. In each chapter, we have selected several problems and workshops from the text and then discussed some of the struggles and questions of the teacher-learners who have worked through the problems previously. Since the text was originally used with in-service teachers, much of the advice primarily concerns that audience, but we have found that many of the same issues arise with undergraduates as well. Our hope is that this guide will allow the Instructor to better communicate with the students for a more effective course.

Chapter 2

- In the section "Working correctly with units" on page 14, the instructor might want to emphasize that units satisfy the same associativity and distributivity properties as numbers, as in the equality $2\text{in} + 3\text{in} + 2\text{in} + 3\text{in} = (2 + 3 + 2 + 3)\text{in}$.
- In the Problem 2.4.1, the instructor might want to emphasize that there is some ambiguity in the figures, and bring up the question of whether, in shape 5, point B is on the line?
- In Problem 2.6.1 the instructor might want to mention or have teacher-learners discover the formula for conversion of change of temperatures: A Celsius change ΔT_C produces a shift of $\Delta T_F = (9°F/5°C)\Delta T_C$ in Fahrenheit.
- In Problem 2.7.1 two possible errors are lining up the end of the ruler instead of zero with the beginning of the object, and using the wrong side of the ruler for a given unit system.

Chapter 3

- In the section "Measuring angles with a protractor" on page 34 the instructor might want to emphasize that it might be the case that the rays in the angle are not drawn with enough length to reach the edge of the protractor. In this case, the rays have to be extended (using the straight part of the protractor) before the angle can be measured. Also, it might be that the angle to be measured has measure more than 180 degrees. In that case, one can draw the ray opposite to the first ray, measure the angle of the second ray from that, and add 180 degrees to the answer.
- In Chapter 3.1 the instructor might want to mention that the length of the rays shown in an angle, are not related to "how big" the angle is.
- In Chapter 3.1 the instructor might want to note that the use of the same number of arcs may be used to indicate congruence of angles. However, a single arc used in two different angles may or may not indicate congruence.
- In Chapter 3.1 the letter a is often used for both the angle and the measure of it. So it may be correct to write both $a = 90°$ and $ma = 180°$, where

in the first case a indicates the measure and in the second case the angle itself.

- In Problem 3.2.2, the grid may be used to find the slope of the perpendicular line.
- In the basic facts about angles on page 46, the instructor should explain what "corresponding angles" mean.
- The additivity principle for angles on page 51 is justified in the workshops, and not in the text.
- In Problem 3.3.5 students should be encouraged to justify carefully the linearity of the angle that they construct.
- In the Problem 3.3.7 the instructor may wish to discuss how the answer to the problem might change depending on whether square is considered a trapezoid or not. Based on the definition in this text, a square is considered a trapezoid.
- In Problem 3.5.2, it might help to ask whether the person at the bottom of the pool has to look higher or lower than if the water was not there in order to see their friend.
- The problem 3.7.3 (b) about billiards is challenging; a hint is to think about the reflection of the pocket over the horizontal line representing the edge of the table.

Chapter 4

- In Problem 4.2.2 it would be good to agree on some ground rules on what are acceptable combinations of pattern blocks (whether vertices are required to match) through classroom discussion before doing the problem. There is also an interesting general question of whether given n, one can build a convex n-gon using pattern blocks.
- When covering the material on page 80 the instructor may wish to discuss the following: What guides how much you put in definitions? Do you put everything or just the minimal amount in definitions? Could we remove something from the definition of a square and still have the same object? (That is, should definitions be "minimal" to specify the object?)
- Problem 4.3.1 is one of the first *classification* problems in the text. One of the things that is most difficult for teacher-learners is the idea of parsing different mathematical objects into classes and subclasses. The subclass of a class idea is particularly challenging; for instance, many teacher-learners will not realize that equilateral triangles are *also* isosceles. Number (5) is especially tricky for them because the overlap between the acute, right, and obtuse angle classifications and the scalene, isosceles and equilateral classifications is subtle. This is also a good place to encourage them to draw a variety of different types of examples for each classification: for example sometimes teacher-learners only draw acute isosceles triangles, leading to mistakes further on later problems.
- Regarding Problem 4.4.1, many times people want to classify shapes by 'look', meaning they have a standard picture of what, say, a trapezoid looks

like in their head, and anything that looks different isn't a trapezoid. This leads to mistakes because rectangles, for example, don't 'look like' trapezoids, but of course *still are* trapezoids. The purpose of this exercise is to get them out of this habit and get them thinking about the mathematical definitions. As the hint implies, part i) is tricky because they will rarely attempt to draw a quadrilateral where the right angles are not adjacent, because it doesn't seem natural. Also, this problem assumes that the audience had basic familiarity with all of these shapes. In-service teachers and most pre-service teachers will recall this information. The instructor may want to adjust the problem depending on the audience.

- In Problem 4.4.2, the instructor may want to briefly discuss the phrasing of the question.
- Regarding Problem 4.5.2, we have noticed that there is a wide variety in familiarity with Venn Diagrams. (It is why this first example is worked out explicitly.) It may be worth working out more simple examples before using them to classify longer lists of shapes.
- Problem 4.7.2 is another example where the people might resort to physically measuring segments. The instructor should encourage teacher-learners to come up with more abstract arguments about why the segments are or are not bisected.
- The last workshop concerns a formula called *Pick's theorem.* More explanation of the formula can be found by clicking here, or cut-and-paste http://www.jamestanton.com/wp-content/uploads/2009/04/ picks_theorem_focus_web-version.pdf into your browser. Another part of this workshop which might be added is the following: Express the sum $1+2+\ldots+30$ as the number of lattice points in a lattice polygon. Find the area of the polygon. Find the number lattice points (hence the sum above) using Pick's theorem.
- In Problem 4.7.4, students should be encouraged to make extensions to the line segments shown in order to help solve the problem. In general, the instructor may wish to emphasize that an important part of problem solving in geometry is to make additions to a given figure, showing and choosing notation for additional angles, points, etc. It is possible to solve the problem only using the facts discussed in this section. It may be helpful for students to have string to measure perimeters.

Chapter 5

- In Problem 5.3.4, a sample answer is as follows: The playground will fit 48 children since

$$\frac{30 \text{ feet} \times 40 \text{ feet}}{(25 \text{ feet}^2/ \text{ child}} = \frac{1200}{25} \frac{\text{feet}^2 \times \text{child}}{\text{feet}^2}$$
$$= 48 \text{ children}.$$

- On page 105, the instructor may wish to discuss possible justifications for the area of a rectangle before launching into the various cases. In particular it may not be clear at all that the formula needs justification in the case that the height and base are not integers.

- On page 116, the area formula for trapezoids is justified in the workshops, and so not in the text.
- In the justification for the area formula for a triangle in Workshop 5.7.3, the teacher-learners should be encouraged to discuss how one knows that two congruent triangles form a parallelogram.
- Workshop 5.3.7: For part (a), the instructor may wish to encourage teacher-learners to avoid using the Pythagorean theorem, instead computing the area of the white square by using the fact that a triangle is half a rectangle. Parts (b) and (c) reinforce this idea: in particular they should realize that exact answers (i.e. not counting fractional parts of squares) are possible.
- For Problem 5.5.4, the argument may be related to the arguments necessary for Workshop 4.7.4.
- In Workshop 5.7.4, note that the problem can be solved in two ways. Each way ignores one piece of the given information, so the problem given actually contains extraneous information.

Chapter 6

- In Problem 6.1.2, it is difficult to give a full explanation for why the best coverage while minimizing the number of towers is achieved through a triangular lattice. Our intention with this part of the problem is to give the teacher-learners an opportunity to informally consider this question by comparing the coverage from at least two different patterns.
- In Problem 6.1.5 (7) the instructor might wish to discuss the connection with the triangle inequalities.
- In Problem 6.1.6, the instructor may wish to refer back to Workshop 4.7.2 which explained the bisecting natures of diagonals in rhombi.
- In Problem 6.3.1, the instructor may wish to have the slices cut up in advance.
- In Problem 6.5.2 (3), possible shapes of the forest include shapes with holes; alternatively one can start with a square and remove square corners until the area of the forest is the desired one.

Chapter 7

- In Figure 83, the instructor may wish to engage the students in discussing where the centers, lines, etc. of dilation, reflection, etc. are in each case.
- In Problem 7.1.1 part (b), the correct answer is claimed to be (C). However, there seems to be nothing wrong with answers (B) and (D); a turn doesn't have to be 90 degrees.
- Before describing procedures for producing motions, the instructor might wish to have a brief discussion about whether the teacher-learners know how to produce them.
- In Problem 7.1.2, the class discussion should cover what needs to be specific for each motion to be described fully.
- In Problem 7.3.1 part (c), the phrasing is left deliberately vague to allow for rotations in multiple planes and multiple beginning positions of the arm.

The context of this problem is also a good opportunity to discuss precision of language.

- In Workshop 7.9.3 (1), the instructor may wish to encourage teacher-learners to consider the more general case where the angle of rotation is not specified.

Chapter 8

- In the discussion of the number line, the instructor might want to discuss whether if there is just one number, say 13, can one figure out where to put 15.

- Before discussing the Pythagorean theorem, the instructor may wish to discuss possible statements, for example, whether $a^2 + c^2 = b^2$ is a correct statement of the Pythagorean theorem in the absence of additional details on the meaning of these variables.

- In Problem 8.3.3, in one case there is no solution, while in another there are multiple solutions. These problems are meant to encourage students to question the validity of assumptions.

- Issues to discuss in Problem 8.3.4 include the following: The student misuses the equal sign to indicate equivalence of equations, omits units, and uses equals when he/she means approximates.

- The justification of the SSS criterion is also given in a workshop.

- In Problem 8.3.5, some students may answer that the shortest distance is the least of the options given.

Chapter 9

- Acceptable answers in Problem 9.1.3 involve specifying rigid motions in the case of congruent figures or referring to corresponding distances or angles that are not congruent, in the case of non-congruent figures. The instructor may have to provide some guidance about the specificity required for the answers.

- In Problem 9.2.1, the instructor may with to encourage teacher-learners to consider each issue in a more general sense. Note that in the third part, there are infinitely many possibilities.

- In Problem 9.4.2, the instructor may again have to provide guidance about the level of detail provided in the answers.

Chapter 10

- In Problem 10.3.1, possibly the authors of the exam want the answer "cube" or "pyramid". But it could be other shapes, for example, a prism on a non-rectangular base. Possibly the only correct answer is "prism on a parallelogram" for the first picture and "cone on a parallelogram" for the second. It would be interesting to know whether any teacher-learners pointed out that one cannot deduce that a solid is a cube from such a picture.

- In Problem 10.3.2 (1), note that trapezoids are defined in 4.4 as quadrilaterals with at least one pair of parallel sides, whereas other texts define trapezoids as having exactly one pair of parallel sides.

- In Problem 10.5.1, the figure can be viewed in multiple ways. So in theory, multiple answers are possible, although most will view the figure as a box with two additional cubes attached.

- In Problem 10.6.2, the buildings may be drawn from a number of perspectives. Some perspectives may not reveal all of the components of the building.

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