QUANTUM WITTEN LOCALIZATION AND ABELIANIZATION FOR QDE SOLUTIONS

EDUARDO GONZÁLEZ AND CHRIS T. WOODWARD

Abstract. We prove a quantum version of the localization formula of Witten [74], see also [70], [61], [78], that relates invariants of a git quotient with the equivariant invariants of the action. Using the formula we prove a quantum version of an abelianization formula of S. Martin [48], relating invariants of geometric invariant theory quotients by a group and its maximal torus, conjectured by Bertram, Ciocan-Fontanine, and Kim [13]. By similar techniques we prove a quantum Lefschetz principle for holomorphic symplectic reductions. As an application, we give a formula for the fundamental solution to the quantum differential equation (qde) for the moduli space of points on the projective line and for the smoothed moduli space of framed sheaves on the projective plane (a Nakajima quiver variety).

Contents

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1. Introduction

1.1. **Quantum Witten localization.** The main result of this paper is a formula relating the equivariant Gromov-Witten graph invariants of a smooth projective variety with group action and the graph invariants of the geometric invariant theory quotient. As a consequence we obtain versions of "quantum abelianization" for graph Gromov-Witten invariants as well as a "quantum Lefschetz" principle for holomorphic symplectic reductions.

To state the main result we introduce the following notation. Let G be a connected complex reductive group acting on a smooth polarized projective variety X. Let $X/\!\!/G$ denote the *git quotient* of X by G, which here means the stack-theoretic quotient of the semistable locus by the group action. We assume that G acts with only finite stabilizers on the semistable locus. In this case the git quotient $X/\!\!/G$ is a smooth proper Deligne-Mumford stack with projective coarse moduli space by Mumford et al [51]. Let $H(X/\!\!/G)$ resp. $H_G(X)$ denote the rational resp. equivariant rational cohomology of $X/\!\!/G$ resp. X. Kirwan's thesis [44] studies the natural map

$$\kappa_{X,G} : H_G(X) \to H(X/\!\!/G)$$

given by restriction to the semistable locus and descent. Integration over $X/\!\!/G$ defines a *trace map*

$$\tau_{X/\!\!/G} : H(X/\!\!/G) \to \mathbb{Q}, \quad h \mapsto \int_{[X/\!\!/G]} h.$$

Naturally one wants to compute the composition of the trace with Kirwan's surjection. For example, one would like to compute the cohomology of $X/\!\!/G$ in terms of the G-equivariant cohomology of X. Witten [74] introduced a strategy, which he termed *non-abelian localization*, to compute the composition $\tau_{X/\!\!/G} \circ \kappa_{X,G}$. His formula involves a trace map

$$\tau_X^G : H_G(X) \to \mathbb{Q}, \quad h \mapsto \int_{X \times \mathfrak{g}_{\mathbb{R}}} h$$

given by integration over X *and* the unitary part $\mathfrak{g}_{\mathbb{R}}$ of the Lie algebra using suitable regularization procedures [60], [61], [78]. In the K-theory version discussed in Paradan [62], the Witten trace is easier to define: it is the invariant part of the index, and no regularization procedure is needed. Witten's localization formula computes the difference between $\tau_{X/\!\!/G} \circ \kappa_{X,G}$ and τ_X^G, that is, the failure of the following diagram to commute:

$$\begin{tikzcd} H_G(X) \arrow[rr, "\kappa_{X,G}"] \arrow[dr, "\tau_X^G"'] & & H(X/\!\!/G) \arrow[dl, "\tau_{X/\!\!/G}"] \\ & \mathbb{Q} & \end{tikzcd} \quad . \tag{1}$$

By Witten's argument in [74], [60] the difference is a non-explicit sum of contributions from the Kirwan-Ness strata of positive dimension:

$$\tau_X^G = \tau_{X/\!\!/G} \circ \kappa_{X,G} + \sum_{[\zeta]\neq 0} \tau_{X,G,\zeta} : H_G(X) \to \mathbb{Q}$$

where $\tau_{X,G,\zeta}$ is the contribution from the stratum with maximally-destabilizing one-parameter subgroup ζ. An explicit formula for the contributions $\tau_{X,G,\zeta}$ was described in papers by Teleman [70] in the case of sheaf cohomology, by Paradan [62] for K-theory of Hamiltonian actions, and in papers by Paradan and Woodward [60], [61], [78] for cohomology of Hamiltonian actions, see also Beasley-Witten [8] which uses the localization formula to compute the Chern-Simons partition function for Seifert manifolds. A different formula computing the composition is given in Jeffrey-Kirwan [42]. A virtual Witten localization formula has recently appeared in Halpern-Leistner [36, (5)]. Because the fixed point contributions in the Witten localization formula are more complicated than those in the usual localization formula, its practical utility is rather limited. Its usefulness lies in proving formal properties of invariants of git quotients. For example, in Teleman [70] the formula is used to study vanishing of higher cohomology; in Halpern-Leistner [36] to study behavior of derived categories under variation of git. Witten localization can also be used to give a purely algebraic proof of Martin's results [48] as we explain below.

The quantum version of Witten's localization formula compares Gromov-Witten invariants of a git quotient with equivariant Gromov-Witten invariants for the action. To state the result let $\omega \in H_2^G(X)$ be the first Chern class of the linearization (that is, the symplectic class) and let

$$\Lambda_X^G = \left\{ \sum_{i=0}^{\infty} c_i q^{d_i}, c_i \in \mathbb{Q}, d_i \in H_2^G(X,\mathbb{Q}),\ \lim_{i\to\infty} \langle d_i, \omega \rangle = \infty \right\}$$

denote the equivariant Novikov field for X. Let

$$QH_G(X) = H_G(X) \otimes \Lambda_X^G$$

denote the equivariant quantum cohomology of X. Virtual integration over the moduli stack of n-marked genus 0 stable maps $\overline{\mathcal{M}}_{0,n}(X)$ for $n \geq 3$ defines a family of formal quantum products

$$\star_h : T_h QH_G(X)^2 \to T_h QH_G(X), \quad h \in QH_G(X).$$

Formal in this setting means that only the Taylor coefficients of the maps are convergent. Define a quantum version of Witten's trace as follows. Let $\mathbb{P} = (\mathbb{C}^2 - \{0\})/\mathbb{C}^\times$ denote the projective line. For $d \in H_2(X,\mathbb{Z})$ let $\overline{\mathcal{M}}_n(\mathbb{P},X,d) := \overline{\mathcal{M}}_{0,n}(\mathbb{P}\times X,(1,d))$ denote the moduli stack of parametrized stable maps from $\mathbb{P}$ to X of class $d \in H_2^G(X,\mathbb{Z})$. The action of G on X induces a natural action on $\overline{\mathcal{M}}_n(\mathbb{P},X,d)$. A natural stability condition for the action is given by requiring that the stable map has generically semistable value [29]. Denote by $\overline{\mathcal{M}}_n(\mathbb{P},X,d)/\!\!/G$ the stack-theoretic quotient of the semistable locus by the group action. By, for example, [31, Lemma 2.6], $\overline{\mathcal{M}}_n(\mathbb{P},X,d)/\!\!/G$ is a proper Deligne-Mumford stack with a perfect relative obstruction theory. Via equivariant formality we may consider $H_2(X,\mathbb{Z})/$ torsion as a subgroup of $H_2^G(X,\mathbb{Q})$. Denote by τ_X^G the formal trace map given by virtual integration over the moduli stacks $\overline{\mathcal{M}}_n(\mathbb{P},X,d)/\!\!/G$:

$$\tau_X^G : QH_G(X) \to \Lambda_X^G, \quad h \mapsto \sum_{n\ge 0, d\in H_2(X,\mathbb{Z})/\,\text{torsion}} (q^d/n!) \int_{[\overline{\mathcal{M}}_n(\mathbb{P},X,d)/\!\!/G]} \mathrm{ev}^*(h\otimes\ldots\otimes h)$$

for $h \in H_G(X)$. The map τ_X^G is a quantum version of Witten's trace in the sense that if one sets $q=0$ and fixes the positions of the markings then one obtains the classical Witten trace for polynomial classes, that is, the integral over $X/\!\!/G$.

A quantum version of Kirwan's map counting maps to the quotient stack with semistability enforced at a marked point was introduced in [75], [76], [77]. The quantum Kirwan map is a non-linear map, still denoted $\kappa_{X,G}$,

$$\kappa_{X,G} : QH_G(X) \to QH(X/\!\!/G) \tag{2}$$

with the property that any linearization

$$D_h\kappa_{X,G} : T_hQH_G(X) \to T_{\kappa_{X,G}(h)}QH(X/\!\!/G)$$

is a homomorphism with respect to the quantum products. In particular, if $\kappa_{X,G}(0) = 0$ (which generally happens only in Fano cases) then $D_0\kappa_{X,G}$ is a homomorphism from the small equivariant quantum cohomology $T_0QH_G(X)$ of X to the quantum cohomology $T_0QH(X/\!\!/G)$ of $X/\!\!/G$.

A quantum version of the integration over the geometric invariant theory quotient is defined by a count of stable maps to the graph space. Recall that $\overline{\mathcal{M}}_n(\mathbb{P},X/\!\!/G,d)$ denotes stable maps to $\mathbb{P}\times(X/\!\!/G)$ of class $1,d$. Using the Behrend-Fantechi virtual fundamental classes define

$$\tau_{X/\!\!/G} : QH(X/\!\!/G) \to \Lambda_X^G, \quad h \mapsto \sum_{n\ge 0, d\in H_2(X/\!\!/G,\mathbb{Q})} (q^d/n!) \int_{[\overline{\mathcal{M}}_n(\mathbb{P},X/\!\!/G,d)]} \mathrm{ev}^*(h\otimes\ldots\otimes h)$$

for $h \in H(X/\!\!/G)$. The quantum Witten localization formula gives a precise description of the difference between the traces τ_X^G and $\tau_{X/\!\!/G}\circ\kappa_{X,G}$. That is, it measures the failure of the "quantum integration" to commute with reduction, i.e. the failure of commutativity of the diagram

$$\begin{tikzcd} QH_G(X) \arrow[rr, "\kappa_{X,G}"] \arrow[dr, "\tau_X^G"'] & & QH(X/\!\!/G) \arrow[dl, "\tau_{X/\!\!/G}"] \\ & \Lambda_X^G & \end{tikzcd} \tag{3}$$

As in the classical Witten localization formula [74], the failure to commute is given by a sum of fixed point contributions. Each term is a *gauged Gromov-Witten invariant* $\tau_{X,G,\zeta,\rho}$ associated to the action of centralizers on components of the fixed point variety of some one-parameter subgroup $\exp(\mathbb{C}\zeta) \subset G$, stable with respect to the linearization $\tilde{X}^\rho$ for some $\rho \in (0,\infty)$. The main result is the following:

Theorem 1.1. (Quantum Witten localization) *Let C be a smooth connected projective curve of genus 0, X a smooth projective G-variety, and $\tilde{X}$ a linearization.*

Suppose that for every $\zeta \in \mathfrak{g}$ and $\rho \in (0,\infty)$, stable=semistable for $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}^\rho, \zeta)$, and stable=semistable for the G-action on X. Then

$$\tau_X^G - \tau_{X/\!\!/G} \circ \kappa_{X,G} = \sum_{[\zeta]\neq 0, \rho\in(0,\infty)} \tau_{X,G,\zeta,\rho} : QH_G(X) \to \Lambda_X^G. \tag{4}$$

1.2. **Applications to quantum abelianization.** We give two groups of applications. The first group consist of versions of the *quantum Martin conjecture* of Bertram et al [13] that compares Gromov-Witten invariants of a git quotient $X/\!\!/G$ and the quotient $X/\!\!/T = X^{\mathrm{ss},T}/T$ by a maximal torus $T \subset G$. As an example, we give a formula for the qde solution for the quotient of points on the projective line by its automorphisms, see (7) (8) (9) below.

Before describing the quantum generalization we discuss the classical story of abelianization due to Martin [48]. Let $\nu_{\mathfrak{g}/\mathfrak{t}}$ denote the bundle over $X/\!\!/T$ induced from the trivial bundle with fiber $\mathfrak{g}/\mathfrak{t}$ over X:

$$\nu_{\mathfrak{g}/\mathfrak{t}} = X^{\mathrm{ss},T} \times_T (\mathfrak{g}/\mathfrak{t}).$$

Define the Euler-twisted integration map

$$\tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} : H(X/\!\!/T) \to \mathbb{Q}, \quad h \mapsto \int_{[X/\!\!/T]} h \cup \mathrm{Eul}(\nu_{\mathfrak{g}/\mathfrak{t}}).$$

Let $W = N(T)/T$ denote the Weyl group of $T \subset G$ and

$$\mathrm{r}_T^G : H_G(X) \to H_T(X)$$

the restriction map, inducing an isomorphism $H_G(X) \to H_T(X)^W$.

Theorem 1.2. (Martin formula [48]) *Let X be a smooth projective G-variety. Suppose that stable=semistable for the actions of T and G on X. Then integration over $X/\!\!/G$ and $X/\!\!/T$ are related by*

$$\tau_{X/\!\!/G} \circ \kappa_{X,G} = |W|^{-1} \tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T} \circ \mathrm{r}_T^G .$$

Furthermore, there exists a surjective map

$$\mu_T^G : H(X/\!\!/T)^W \to H(X/\!\!/G) \tag{5}$$

whose kernel is the annihilator of $\mathrm{Eul}(\nu_{\mathfrak{g}/\mathfrak{t}})$.

In other words, the following diagram commutes:

$$\begin{tikzcd}
 & H_G(X) \cong H_T(X)^W \arrow[dl, "\kappa_{X,G}"'] \arrow[dr, "\kappa_{X,T}"] & \\
H(X/\!\!/G) \arrow[dr, "\tau_{X/\!\!/G}"'] & & H(X/\!\!/T) \arrow[dl, "|W|^{-1}\tau^{\mathfrak{g}/\mathfrak{t}}_{X/\!\!/T}"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Using quantum Witten localization (4) we prove that a formula similar to that in Theorem 1.2 holds in quantum cohomology. Versions of this formula were conjectured and several special cases proved by Hori-Vafa [40, Appendix], Bertram-Ciocan-Fontanine-Kim [13] and Ciocan-Fontanine-Kim-Sabbah [18]. A general result that holds under monotonicity conditions is proved by Schmäshcke [68]. The push-forward in homology $\pi_T^G : H_2^T(X) \to H_2^G(X)$ defines a map of equivariant Novikov rings

$$\pi_T^G : \Lambda_X^T \to \Lambda_X^G, \qquad \sum_{d \in H_2^T(X)} c_d q^d \mapsto \sum_{d \in H_2^G(X)} c_d q^{\pi(d)}. \tag{6}$$

Let $Q\mathbb{H}_G(X) \subset QH_G(X)$ denote the subspace generated by Chern characters of algebraic vector bundles,

$$\mathbb{H}_G(X) := \{ \mathrm{Ch}_G(E) \mid E \to X \ \text{ vector bundle } \}, \quad Q\mathbb{H}_G(X) := \mathbb{H}_G(X) \otimes \Lambda_X^G.$$

The restriction to Chern characters is necessary because our arguments at some point use sheaf cohomology. We denote by

$$\mathrm{r}_T^G : QH_G(X) \to QH_T(X)$$

the map obtained by combining the pull-back $H_G(X) \to H_T(X)$ with the inclusion $\Lambda_X^G \subset \Lambda_X^G$ induced by the inclusion $H_2^G(X,\mathbb{Z}) \cong H_2^T(X,\mathbb{Z})^W \subset H_2^T(X,\mathbb{Z})$.

Theorem 1.3. (Quantum Martin formula) *Let C be a smooth connected projective genus 0 curve and X a smooth linearized projective G-variety. Suppose that stable=semistable for T and G actions on X. The following equality holds on $Q\mathbb{H}_G(X)$:*

$$\begin{aligned} \tau_{X/\!\!/G} \circ \kappa_{X,G} &= |W|^{-1} \pi_T^G \circ \tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}} \circ \kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{r}_T^G \\ &= |W|^{-1} \pi_T^G \circ \tau_{X,T}^{\mathfrak{g}/\mathfrak{t}} \circ \mathrm{r}_T^G : \quad Q\mathbb{H}_G(X) \to \Lambda_X^G. \end{aligned}$$

That is, there is a commutative diagram

$$\begin{tikzcd} Q\mathbb{H}_G(X) \arrow[r] \arrow[d, "\kappa_{X,G}"'] & QH_T(X) \arrow[d, "\kappa_{X,T}^{\mathfrak{g}/\mathfrak{t}}"] \\ QH(X/\!\!/G) \arrow[d, "\tau_{X/\!\!/G}"'] & QH(X/\!\!/T) \arrow[d, "|W|^{-1}\tau_{X/\!\!/T}^{\mathfrak{g}/\mathfrak{t}}"] \\ \Lambda_X^G & \Lambda_X^T \arrow[l] \end{tikzcd}$$

A similar abelianization formula holds for solutions to quantum differential equations. The $\mathbb{C}^\times$-equivariant extension of the graph potential on a genus zero curve (with the standard $\mathbb{C}^\times$-action by rotations) admits a factorization into *localized graph Gromov-Witten potentials*

$$\tau_{X/\!\!/G,\pm} : QH(X/\!\!/G) \to QH_{\mathbb{C}^\times}(X/\!\!/G).$$

Here, as in the remainder of the paper, $QH_{\mathbb{C}^\times}$ denotes the completion of the $\mathbb{C}^\times$-equivariant cohomology with the equivariant generator inverted and completed: If the equivariant generator is denoted $\zeta \in H(B\mathbb{C}^\times)$ then

$$QH_{\mathbb{C}^\times}(X/\!\!/G) = QH(X/\!\!/G)[\zeta, \zeta^{-1}]].$$

In the literature these potentials are often call J-functions or one-point descendant potentials [28]. Denote by

$$\mu_T^G : QH(X/\!\!/G) \to QH(X/\!\!/T)$$

the map combining Martin's map of (5) with the canonical map of Novikov rings π_T^G of (6).

Theorem 1.4. (Abelianization for qde solutions) *Suppose that X is a smooth linearized projective G-variety, and stable=semistable for the T and G-actions on X. Then*

$$\begin{aligned} \tau_{X/\!\!/G,\pm} \circ \kappa_{X,G} &= \mu_T^G \circ \tau^{\mathfrak{g}/\mathfrak{t}}_{X/\!\!/T,\pm} \circ \kappa^{\mathfrak{g}/\mathfrak{t}}_{X,T} \circ \mathrm{r}_T^G \\ &= \mu_T^G \circ \tau^{\mathfrak{g}/\mathfrak{t}}_{X,T,\pm} \circ \mathrm{r}_T^G : Q\mathbb{H}_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G). \end{aligned}$$

The argument extends to quasiprojective targets under suitable properness conditions. In particular, it holds for targets that are G-vector spaces X satisfying a certain convexity condition, see Theorem 3.9 below. In Examples 4.1, 1.5 we apply the formula to give formulas for the solution to the quantum differential equation for the Grassmannians and moduli of points on the projective line.

Example 1.5. (Moduli of points on the projective line) We consider the git quotient for the diagonal action of $SL(2,\mathbb{C})$ on $(\mathbb{P}^1)^{2k+1}$ with linearization on each factor the same. The git quotient is

$$Y = \left\{ (x_1, \ldots, x_{2k+1}) \in (\mathbb{P}^1)^{2k+1} \mid \sup_{x \in \mathbb{P}} \#\{x_i = x\} \le k \right\} / SL(2,\mathbb{C}).$$

In order to apply our results we realize the moduli space of points as a git quotient of a vector space. The product $(\mathbb{P}^1)^{2k+1}$ is the git quotient of $X = \mathbb{C}^{4k+2}$ by the diagonal action of $(\mathbb{C}^\times)^{2k+1}$. Thus,

$$Y = X/\!\!/G, \quad X = \mathbb{C}^{4k+2}, \quad G = (\mathbb{C}^\times)^{2k+1} \times SL(2,\mathbb{C}).$$

The diagonal subgroup $\mathbb{C}^\times$ in the first factor acts on X with all positive weights, so X is convex. Since X is G-equivariantly Fano the quantum Kirwan map is the identity on $QH_G^{\le 2}(X)$ for reasons of dimension by (27). By abelianization the localized graph potential is given by

$$\tau_{X/\!\!/G,\pm} = \mu_T^G \circ \tau_{X,T,\pm} \circ \mathrm{r}_T^G \in QH_{\mathbb{C}^\times}(X/\!\!/G). \tag{7}$$

The maximal torus of G is $T \cong (\mathbb{C}^\times)^{2k+2}$ embedded as the subgroup of products of diagonal matrices. The canonical identifications as symmetric polynomials gives

$$H_2^G(X) \cong H_2^{(\mathbb{C}^\times)^{2k+1}}(X) \cong \mathbb{Z}^{2k+1}, \quad H_2^T(X) \cong H_2^{(\mathbb{C}^\times)^{2k+2}}(X) \cong \mathfrak{t}_{\mathbb{Z}} \cong \mathbb{Z}^{2k+2}.$$

The weights for the T-action on X are written in terms of the standard basis $\epsilon_1, \ldots, \epsilon_{2k+2}$

$$\epsilon_1 + \epsilon_{2k+2}, \epsilon_1 - \epsilon_{2k+2}, \epsilon_2 + \epsilon_{2k+2}, \epsilon_2 - \epsilon_{2k+2}, \ldots \epsilon_{2k+1} - \epsilon_{2k+2} \in \mathfrak{t}_{\mathbb{Z}}^\vee.$$

Let $\theta_1,\ldots,\theta_{2k+2} \in H^2_T(X)$ denote the generators corresponding to the splitting $T = (\mathbb{C}^\times)^{2k+2}$. For any $2k+2$-tuple of non-negative integers $\underline{d} = (d_1,\ldots,d_{2k+2})$, $\theta = \sum c_i\theta_i$ with $c_i \in \mathbb{Z}$, define a factorial-like product

$$\Delta_{\underline{d}}(\theta) := \frac{\prod_{l=-\infty}^{\theta\cdot\underline{d}}(\theta + l\zeta)}{\prod_{l=-\infty}^{0}(\theta + l\zeta)}.$$

The localized potential $\tau_{X,T,,\pm}$ has restriction

$$\tau_{X,T,\pm}|QH^{\leq 2}(X/\!\!/G) \subset QH^{\leq 2}_G(X) \subset QH^{\leq 2}_T(X)$$

given by

$$\begin{aligned}(8)\quad & (\tau_{X,T,\pm}|QH^{\leq 2}(X/\!\!/G)) : QH^{\leq 2}(X/\!\!/G) \to QH^{\leq 2}(X/\!\!/G)[[\zeta^{-1}]] \\ & (t_0+t_1\theta_1+\ldots+t_{2k+2}\theta_{2k+2}) \mapsto \sum_{\underline{d}} q^{\underline{d}} e^{t_0+(t_1(\theta_1+d_1\zeta)+\ldots+t_{2k+2}(\theta_{2k+2}+d_{2k+2}\zeta))/\zeta}\tau_{X,T,\pm}(\underline{d})\end{aligned}$$

where

$$(9)\quad \tau_{X,T,\pm}(\underline{d}) := e^{\sum_{j=1}^{2k+2} d_jt_j} \frac{\Delta_{\underline{d}}(2\theta_{2k+2})\Delta_{\underline{d}}(-2\theta_{2k+2})}{\Delta_{\underline{d}}(\theta_1+\theta_{2k+2})\Delta_{\underline{d}}(\theta_1-\theta_{2k+2})\ldots\Delta_{\underline{d}}(\theta_{2k+1}-\theta_{2k+2})}.$$

Combining (7), (8), (9) gives a formula for a qde solution.

Using the result on abelianization of qde solutions we obtain the following relationship between quantum cohomology rings of abelian and non-abelian quotients which was proved in monotone cases by Schmäschke [68]. Let $T \subset G$ a maximal torus as above. Denote by

$$\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_- \subset \mathfrak{t}^\vee_{\mathbb{Z}}$$

the set $\mathcal{R}$ of roots, partitioned into positive $\mathcal{R}_+$ and negative $\mathcal{R}_-$ roots. Consider the decomposition of the Lie algebra into root spaces,

$$(10)\qquad \mathfrak{g} \cong \mathfrak{t} \oplus \bigoplus_{\alpha\in\mathcal{R}_-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\mathcal{R}_+} \mathfrak{g}_\alpha.$$

Denote the Euler class of $\mathfrak{g}/\mathfrak{t}$

$$e = e_-e_+, \quad e_\pm = \prod_{\alpha\in\mathcal{R}_\pm} \alpha \in S(\mathfrak{t}^\vee) \subset QH_T(X).$$

The image of $e_\pm$ under the linearized quantum Kirwan map is denoted

$$D_h\kappa^{\mathfrak{g}/\mathfrak{t}}_{X,T}(e_\pm) \in QH(X/\!\!/T), \quad D_h\kappa^{\mathfrak{g}/\mathfrak{t}}_{X,T}(e_+) = (-1)^r D_h\kappa^{\mathfrak{g}/\mathfrak{t}}_{X,T}(e_-).$$

Either e_- or e_+ works equally well in the formulas below.

Theorem 1.6. (Comparison of quantum cohomology rings) *For any $h \in QH_G(X)$ there exists a canonical surjection*

$$T_{\kappa_{X,T}(h)}QH(X/\!\!/T)^W \to T_{\kappa_{X,G}(h)}QH(X/\!\!/G)$$

whose kernel is the annihilator of $D_h\kappa_{X,T}(e_\pm)$:

$$T_{\kappa_{X,G}(h)}QH(X/\!\!/G) = T_{\kappa_{X,T}(h)}QH(X/\!\!/T)^W / \operatorname{ann}(D_h\kappa_{X,T}(e_\pm)).$$

Example 1.7. (Quantum cohomology of the Grassmannian) The Grassmannian $\mathrm{Gr}(k,n)$ is the git quotient $X/\!\!/G$ of $X = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by the action of $G = GL(k)$. The maximal torus is $T = GL(1)^k$ and the abelian quotient

$$X/\!\!/T \cong (\mathbb{P}^{n-1})^k.$$

The standard presentation of $QH((\mathbb{P}^{n-1})^k)$ is

$$QH((\mathbb{P}^{n-1})^k) = \Lambda_X^G[H_1, \ldots, H_k], \quad H_i^n = q, i = 1, \ldots, k$$

where H_i is the hyperplane class on the i-th factor. The Weyl group $W = S_k$ is the k-symmetric group. The W-invariant part of the cohomology ring $QH((\mathbb{P}^{n-1})^k)$ is generated by the Schur polynomials

$$\chi_{\lambda^\vee}(H_1, \ldots, H_n) = \prod_{w \in W} \frac{(-1)^{l(w)} H^{w(\lambda+\rho)-\rho}}{\prod_{i<j}(H_j - H_i)}$$

where

$$H^\lambda = H_1^{\lambda_1} \ldots H_k^{\lambda_k}, \quad \rho = (1, \ldots, k).$$

Now since $\kappa_{X,T}$ has no quantum corrections we have

$$D_0 \kappa_{X,T}(e_\pm) = \pm \prod_{i<j} (H_j - H_i).$$

Hence for any $\mu \in \mathbb{Z}^k$, the elements

$$\chi_{\lambda+n\mu}(H_1, \ldots, H_n) - \chi_{\lambda^\vee}(H_1, \ldots, H_n)$$

are in the annihilator of $D_0\kappa_{X,T}(e_\pm)$. So we have relations

$$\chi_{\lambda+n\mu} = q^{\mu_1+\ldots+\mu_k} \chi_{\lambda^\vee} \in QH(\mathrm{Gr}(k,n)), \quad \lambda, \mu \in \mathbb{Z}^k.$$

These are the usual relations in the cohomology of the Grassmannian describing the cohomology as a truncation of the polynomial representation ring of $GL(k)$ in for example Bertram-Ciocan-Fontanine-Fulton [11].

1.3. **Applications to holomorphic symplectic quotients.** A second application of the quantum Witten localization formula is to the quantum Lefschetz principle, by which Gromov-Witten invariants of complete intersections are expressed in terms of Euler-twisted Gromov-Witten invariants of the ambient space. This extends the quantum Lefschetz principle beyond cases where the bundle is *concavex*, that is, a direct sum of convex and concave line bundles [20], [23]. As an example, we give a formula for the qde solution of the ADHM quiver variety, Theorem 1.10 below.

We introduce the following notation. Suppose that $X/\!\!/G$ is a git quotient as above, and V is a G-representation. Then

$$V/\!\!/G = (V|X^{\mathrm{ss}})/G \to X/\!\!/G$$

is the associated bundle on the git quotient. Suppose that the bundle $V/\!\!/G$ admits a section

$$\Phi : X/\!\!/G \to V/\!\!/G$$

induced from an equivariant map $\Phi : X \to V$. Denote the level sets

$$Z /\!\!/ G = (\Phi /\!\!/ G)^{-1}(0) \subset X /\!\!/ G, \quad Z := \Phi^{-1}(0)$$

is a smooth subvariety. To simplify notation, we denote by $QH(Z/\!\!/G)$ the quantum cohomology defined over the Novikov ring Λ_X^G, that is, $QH(Z/\!\!/G) = H(Z/\!\!/G) \otimes \Lambda_X^G$. Let $r_{Z,G} : QH_G(X) \to QH_G(Z)$ and $r_{Z/\!\!/G} : QH(X/\!\!/G) \to QH(Z/\!\!/G)$ denote pull-backs.

The application is the following:

Theorem 1.8. *(*quantum Lefschetz for associated bundles) *Let X be a linearized projective or convex quasiprojective G-variety and $\Phi : X \to V$ a section as above with smooth zero set Z. The graph potentials for $Z/\!\!/G$ and $X/\!\!/G$ are related by*

$$\tau_{X/\!\!/G}^{V/\!\!/G} \circ \kappa_{X,G}^V = \tau_{X,G}^V = \tau_{Z/\!\!/G} \circ \kappa_{Z,G} \circ r_{Z,G} : QH_G(X) \to \Lambda_X^G.$$

Similarly for the qde solutions

$$\begin{aligned} r_{X/\!\!/G} \circ \tau_{X/\!\!/G,\pm}^{V/\!\!/G} \circ \kappa_{X,G}^V &= r_{X/\!\!/G}^{V/\!\!/G} \circ \tau_{X,G,\pm}^V \\ &= r_{Z/\!\!/G} \circ \tau_{Z/\!\!/G,\pm} \circ \kappa_{Z,G} \circ r_{Z,G} : QH_G(X) \to QH(X/\!\!/G). \end{aligned}$$

Combining this result with abelianization allows us to compute the graph potentials of certain holomorphic symplectic quotients. Suppose that X is equipped with an holomorphic moment map $\Phi : X \to \mathfrak{g}^\vee$ as well as a linearization $\tilde{X} \to X$. The holomorphic symplectic quotient is then the git quotient of the zero level set:

$$X /\!\!/\!\!/\!\!/ G := \Phi^{-1}(0) /\!\!/ G = \Phi^{-1}(0)^{\mathrm{ss}} / G.$$

Suppose that $Z = \Phi^{-1}(0)$ is smooth. Let $r_Z^X : Q\mathbb{H}_G(X) \to Q\mathbb{H}_G(Z)$ denote the restriction map. Let

$$\mu_{Z/\!\!/G}^{X/\!\!/T} : QH(X/\!\!/T) \to QH(Z/\!\!/G)$$

be the combination of pull-back $QH(X/\!\!/G) \to QH(Z/\!\!/G)$ with Martin's surjection $QH(X/\!\!/T) \to QH(X/\!\!/G)$ [48].

Theorem 1.9. (qde solutions for holomorphic symplectic quotients) *Let X be a linearized projective or convex quasiprojective G-variety and $\Phi : X \to \mathfrak{g}^\vee$ an equivariant map as above with zero set Z, and $T \subset G$ a maximal torus. The graph potential for $Z/\!\!/G$ satisfies*

$$\tau_{Z/\!\!/G} \circ \kappa_{Z,G} \circ r_{Z,G} = \tau_{X,T}^{\mathfrak{g} \oplus \mathfrak{g}/\mathfrak{t}} : Q\mathbb{H}_G(X) \to \Lambda_X^G.$$

Similarly the localized graph potentials are related by

$$\mu_{Z/\!\!/G}^{X/\!\!/T} \circ \tau_{X,T,\pm}^{\mathfrak{g} \oplus \mathfrak{g}/\mathfrak{t}} = \tau_{Z/\!\!/G,\pm} \circ \kappa_{Z,G} \circ r_{Z,G} : Q\mathbb{H}_G(X) \to QH(Z/\!\!/Q).$$

We apply this formula, at least in principle, to the moduli of framed sheaves on the projective plane. Let $\mathbb{P}^2$ denote the projective plane and let

$$\ell_\infty = \{[0, z_1, z_2]\} \subset \{[z_0, z_1, z_2]\} = \mathbb{P}^2$$

denote the divisor at infinity. Recall

$$\mathcal{M} = \left\{ (E,\Phi) \,\middle|\, \begin{array}{c} E : \text{torsion free sheaf on } \mathbb{P}^2 \\ \operatorname{rank}(E) = r, c_2(E) = k \\ \Phi : E|_{\ell_\infty} \to \mathcal{O}_{\ell_\infty}^{\oplus r} : \text{framing at infinity} \end{array} \right\} / \text{isomorphism}.$$

According to the Atiyah-Drinfeld-Hitchin-Manin description of the moduli space [2] there exists an isomorphism

$$\mathcal{M} \cong \left\{ (B_-, B_+, i_-, i_+) \,\middle|\, \begin{array}{c} [B_-, B_+] + i_- i_+ = 0 \\ \text{there exists no subspace} \\ S \subset \mathbb{C}^k \text{ such that } B_\pm(S) \subset S \\ \text{and } \operatorname{im}(i_-) \subset S \end{array} \right\} / G, \tag{11}$$

where

$$B_-, B_+ \in \operatorname{End}(\mathbb{C}^k), \quad i_- \in \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^k), \quad i_+ \in \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^r).$$

The action of $g \in G$ is given by

$$g \cdot (B_-, B_+, i_-, i_+) = (gB_- g^{-1}, gB_+ g^{-1}, gi_-, i_+ g^{-1}).$$

This moduli space is a special case of a Nakajima quiver variety.

The moduli space of gauged maps in the case of the quiver variety describing the moduli space is not proper. Instead one introduces an auxiliary torus action that acts with proper fixed point components so that the equivariant Gromov-Witten theory is defined by localization. The group $S = (\mathbb{C}^\times)^2$ acts equivariantly on

$$X := \operatorname{End}(\mathbb{C}^k)^{\oplus 2} \oplus \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^k) \oplus \operatorname{Hom}(\mathbb{C}^k, \mathbb{C}^r)$$

by

$$(s_-, s_+)(B_-, B_+, i_-, i_+) = (s_- B_-, s_+ B_+, i_-, s_- s_+ i_+).$$

For any $\lambda \in \mathbb{C}$ the action of S preserves the locus

$$Z = \{ (B_-, B_+, i_-, i_+) \,|\, [B_-, B_+] + i_- i_+ = \lambda \operatorname{Id} \} \subset X$$

and induces an S-action on the quotient $\mathcal{M}$. For any character $\chi \in \operatorname{Hom}(GL_k(\mathbb{C}), \mathbb{C}^\times) \cong \mathbb{Z}$, let $\tilde{\mathcal{M}}$ denote the shifted quotient

$$\tilde{\mathcal{M}} = Z /\!\!/_\chi G \subset X /\!\!/_\chi G$$

where $G = GL_k(\mathbb{C})$ and $/\!\!/_\chi$ denotes the χ-shifted geometric invariant theory quotient. We take λ, χ to be generic small values, so that $\tilde{\mathcal{M}}$ is a smooth variety.

To compute the qde solution, let $T \subset G$ denote the diagonal maximal torus. We consider the twisted Gromov-Witten theory of $X /\!\!/_\chi G$ corresponding to the relation defining Z, that is, twisted by the Euler class of the index bundle of $\operatorname{End}(\mathbb{C}^k)$. Define the factorial-like product for $\theta \in H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}^k$,

$$\Delta_{\underline{d}}(\theta, w) := \frac{\prod_{l=-\infty}^{\theta \cdot \underline{d}} (\theta + w + l\zeta)}{\prod_{l=-\infty}^{0} (\theta + w + l\zeta)}.$$

The twisted localized gauged potential for the T action on X has restriction to $QH_T^{\leq 2}(X)$ given by (cf. [45])

$$\pi_T^G \tau_{X,T,-} = \sum_{d\geq 0} q^d \sum_{\underline{d}:d_1+\ldots+d_k=d} e^{t_0+(t_1(\theta_1+d_1\zeta)+\ldots+t_k\theta_k+d_k\zeta)/\zeta}\tau_{X,T,-}(\underline{d})$$

where

$$\tau_{X,T,-}(\underline{d}) = \prod_{i\neq j} \frac{\Delta_{\underline{d}}(\theta_i-\theta_j,\xi_-+\xi_+)\Delta_{\underline{d}}(\theta_i-\theta_j,0)}{\Delta_{\underline{d}}(\theta_i-\theta_j,\xi_-)\Delta_{\underline{d}}(\theta_i-\theta_j,\xi_+)} \prod_{i=1}^k \frac{1}{\Delta_{\underline{d}}(\theta_i,0)^r\Delta_{\underline{d}}(-\theta_i,\xi_-+\xi_+)^r}$$

and ξ_-, ξ_+ are the equivariant parameters for S, that is, $H_S(\mathrm{pt}) \cong \mathbb{Q}[\xi_-,\xi_+]$.

Theorem 1.10. *The localized graph potential of the smoothed moduli space of framed sheaves on the projective plane on $QH_T^2(X)$ is given by*

$$\tau_{Z/\!\!/G,-} \circ r_{Z,G} = \mu_{Z/\!\!/G}^{X/\!\!/T} \exp(-\tau_{X,T,-}^{(1)}/\zeta)\tau_{X,T,-}$$

where $\tau_{X,T,-}^{(1)}$ is the ζ^{-1}-coefficient in

$$\tau_{X,T,-} = \sum_{d\geq 1} q^d \sum_{\underline{d}:d_1+\ldots+d_k=d} e^{t_0+(t_1(\theta_1+d_1\zeta)+\ldots+t_k(\theta_k+d_k\zeta))/\zeta} \prod_{i\neq j} \frac{\Delta_{\underline{d}}(\theta_i-\theta_j,\xi_-+\xi_+)\Delta_{\underline{d}}(\theta_i-\theta_j,0)}{\Delta_{\underline{d}}(\theta_i-\theta_j,\xi_-)\Delta_{\underline{d}}(\theta_i-\theta_j,\xi_+)} \prod_{i=1}^k \Delta_{\underline{d}}(\theta_i,0)^{-r}\Delta_{\underline{d}}(-\theta_i,\xi_-+\xi_+)^{-r}. \tag{12}$$

In the case $r=1$ the Theorem 1.10 reproduces an announced result of Ciocan-Fontanine-Maulik-Kim, see [45]. Results of Konvalinka-Ciocan-Fontanine-Pak [45] substantially simplify the "mirror map"

$$\exp(\tau_{X,T,-}^{(1)}/\zeta) = (1+q)^{k(\xi_-+\xi_+)/\zeta}.$$

In the case $r>1$ the factor $\exp(\tau_{X,T,-}^{(1)}/\zeta)$ is the identity. Formulas for quantum multiplication on these moduli spaces, and connections with integrable systems, are developed in Maulik-Okounkov [47].

2. Gauged Gromov-Witten invariants

In this section we review the construction of gauged Gromov-Witten invariants from the algebro-geometric and symplectic viewpoint.

2.1. **Mundet stability.** Mundet stability combines the slope conditions from Ramanathan stability for bundles and Hilbert-Mumford stability for points in the target. First we recall Mumford-Seshadri stability. Let C be a smooth projective curve and $E \to C$ a vector bundle of vanishing degree $\deg(E) = (c_1(E),[C])$. The bundle

$$E \text{ } \textit{semistable} \text{ resp. } \textit{stable} \iff (\deg(F)\leq 0 \quad \text{resp.} < 0, \quad \forall F\subset E)$$

for all holomorphic sub-bundles $F\subset E$ [56]. In the case of a rational curve the Birkhoff-Grothendieck theorem [34] shows that any bundle splits as a sum of line bundles. Semistability in degree zero on rational curves is simply the condition that the bundle is trivial and there are no stable bundles.

Ramanathan's stability [65] generalizes the Mumford-Seshadri condition to principal bundles as a condition on parabolic reductions. Let G be a connected reductive group with Lie algebra $\mathfrak{g}$. Let $T \subset G$ be a maximal torus, with Lie algebra $\mathfrak{t}$. Denote the integral resp. rational weights resp. coweights

$$\mathfrak{t}_{\mathbb{Z}} = \exp^{-1}(e), \quad \mathfrak{t}_{\mathbb{Z}}^{\vee} \subset \mathfrak{t}^{\vee} := \operatorname{Hom}(\mathfrak{t}, \mathbb{C}), \quad \mathfrak{t}_{\mathbb{Q}} = \mathfrak{t}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \quad \mathfrak{t}_{\mathbb{Q}}^{\vee} = \mathfrak{t}_{\mathbb{Z}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{Q}$$

As in (10) let $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_- \subset \mathfrak{t}_{\mathbb{Z}}^{\vee}$ denote a set of positive and negative roots so that

$$\mathfrak{g} \cong \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \mathcal{R}_+} \mathfrak{g}_\alpha.$$

A *parabolic subgroup* of G is a subgroup Q such that G/Q is complete. Up to conjugacy this means that the Lie algebra $\mathfrak{q}$ of Q is given by

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}_-} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \mathcal{R}_Q} \mathfrak{g}_\alpha$$

for some subset of the roots $\mathcal{R}_Q \subset \mathcal{R}_+$ such that $\mathfrak{q}$ is a Lie subalgebra of $\mathfrak{g}$. A *Levi subgroup* of Q is a maximal reductive subgroup $L(Q)$; again up to conjugacy the Lie algebra $l(\mathfrak{q})$ of $L(Q)$ resp. $\mathfrak{u}(q)$ of a maximal unipotent $U(Q)$ is

$$l(\mathfrak{q}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in -\mathcal{R}(Q)} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \mathcal{R}(Q)} \mathfrak{g}_\alpha, \quad \mathfrak{u}(\mathfrak{q}) = \bigoplus_{\alpha \in -\mathcal{R}(Q)} \mathfrak{g}_\alpha.$$

The parabolic subgroup and its Lie algebra admit decompositions into reductive and unipotent parts

$$\mathfrak{q} = l(\mathfrak{q}) \oplus \mathfrak{u}(\mathfrak{q}), \quad Q = L(Q)U(Q).$$

Taking the quotient by the maximal unipotent gives a projection

$$\pi_Q : Q \to Q/U(Q) \cong L(Q).$$

This projection has the following alternative description. A *dominant coweight* for Q is a coweight $\lambda \in \mathfrak{t}$ such that

$$(\alpha(\lambda) \geq 0, \quad \forall \alpha \in \mathcal{R}_+) \quad \text{and} \quad (\alpha(\lambda) = 0, \quad \forall \alpha \in \mathcal{R}(Q)).$$

Any rational coweight for Q determines a one-parameter subgroup

$$\phi_\lambda : \mathbb{C}^\times \to Q, \quad z \mapsto \phi_\lambda(z).$$

If $\lambda \in \mathfrak{q}$ is a dominant rational coweight then

$$\pi_Q(q) = \lim_{z \to 0} \operatorname{Ad}(\phi_{-\lambda}(z))q.$$

Choose an equivariant identification $\mathfrak{g} \to \mathfrak{g}^\vee$ that identifies the subspaces of rational weights and coweights $\mathfrak{t}_{\mathbb{Q}} \to \mathfrak{t}_{\mathbb{Q}}^\vee$. The identification $\lambda \in \mathfrak{t}$ determines a rational weight $\lambda^\vee \in \mathfrak{t}^\vee$. After finite cover λ defines a one-dimensional representation

$$\chi_{\lambda^\vee} : Q \to \mathbb{C}^\times, q \mapsto \chi_{\lambda^\vee}(q)$$

which factors through $L(Q)$.

The analog for principal bundles of the stability condition for sub-bundles is a condition for parabolic reductions together with dominant coweights. Let $P \to C$ be a principal G bundle on a curve C over a scheme S; bundles are by assumption locally trivial in the étale topology. A *parabolic reduction* is a section $\sigma : C \to P/Q$. Any parabolic reduction induces a reduction of structure group given by a sub-bundle $\sigma^*(P) \subset P$ with structure group Q, given by pull-back of the Q-bundle $P \to P/Q$. We denote by

$$\mathrm{Gr}(P) := \pi_{Q,*}\sigma^* P \to C$$

the corresponding $L(Q)$-bundle, called the *associated graded* bundle of P associated to the parabolic reduction σ. In case $G = GL(n)$, a parabolic reduction is equivalent to a partial flag of sub-bundles $E^{i_1} \subset E^{i_2} \subset \ldots \subset E^{i_l} = E$ in the associated vector bundle $E = P(\mathbb{C}^n)$; the corresponding parabolic reduction σ^*P is the bundle of frames whose first i_k-elements belong to E^{i_k} for $k = 1, \ldots, l$. The associated graded principal bundle is the principal bundle of frames of the associated graded vector bundle

$$\mathrm{Gr}(E) = \oplus_j (E^{i_{j+1}}/E^{i_j}), \quad \mathrm{Gr}(P) = \mathrm{Fr}(\mathrm{Gr}(E)).$$

The construction of the associated graded bundle also has an interpretation via degeneration. The family of elements $\phi_\lambda(z)$ defines a family of automorphism $\mathrm{Ad}(\phi_\lambda(z)) : G \to G$. This family determines a family of bundles $P^\lambda \to C \times \mathbb{C}^\times$ by conjugating the transition maps of σ^*P by $\phi_\lambda(z)^{-1}$. Then P^λ extends over the central fiber $C \times \{0\}$ as the bundle $\mathrm{Gr}(P)$. The *Ramanathan weight* of a principal bundle with respect to a parabolic reduction and dominant weight is the degree of the line bundle corresponding to the given dominant coweight:

$$\mu(\sigma, \lambda) = \deg(\pi_{Q,*}\sigma^*P \times_{L(Q)} \mathbb{C}_\lambda) = ([C], c_1 \pi_{Q,*}\sigma^*P \times_{L(Q)} \mathbb{C}_\lambda).$$

Then

$$P \quad \textit{semistable} \text{ resp. } \textit{stable} \iff \mu(\sigma, \lambda) \leq 0 \quad \text{resp.} < 0, \forall (\sigma, \lambda).$$

For rational curves Birkhoff-Grothendieck [34] again implies that a principal bundle with vanishing degree is semistable iff it is trivial. As for vector bundles, it suffices to check the condition for reduction to *maximal* parabolic subgroups Q. Ramanathan [65] shows the existence of a projective coarse moduli space for semistable principal bundles with reductive structure group and fixed numerical invariants.

Mundet semistability [52, 66] generalizes Ramanathan stability to the case of maps to a quotient stack. Let G be a connected reductive group acting on a smooth projective variety X. By a *gauged map* with domain a curve C we mean a map from C to the quotient stack X/G, given by a pair (P, u) of a G-bundle and section of the associated X-fiber bundle:

$$P \to C, \quad u : C \to P \times_G X.$$

Given a pair $(P \to C, u : C \to P(X))$, the section u defines a section u^λ of P^λ as follows: In any local trivialization $P(X)|U \cong U \times X$ the section u is given by a map $u|U : U \to X$, and the sections $\phi_\lambda(z)u$ patch together to a section of $P^\lambda(X)$. By Gromov compactness, u^λ extends over the central fiber $C \times \{0\}$ as a stable map denoted $\mathrm{Gr}(u) : \hat{C} \to \mathrm{Gr}(P)(X)$. Associated to this limit there is an associated *Hilbert-Mumford weight* defined as follows. The principal component C_0 of $\hat{C}$ is the irreducible component such that the restriction u_0 of u to C_0 maps isomorphically to C. The principal component $\mathrm{Gr}(u)_0$ of the associated graded section $\mathrm{Gr}(u)$ takes values in the fixed point set $(\mathrm{Gr}(P)(X))^\lambda = \mathrm{Gr}(P)(X^\lambda)$ of the infinitesimal automorphism of $\mathrm{Gr}(P)(X)$ induced by λ. The *Hilbert-Mumford weight*

$$\mu_H(\sigma, \lambda) \in \mathbb{Z} \tag{13}$$

determined by the linearization $\tilde{X}$, is the weight of the $\mathbb{C}^\times$-action generated by $-\lambda$ on the fiber of the bundle $(\mathrm{Gr}(P))(\tilde{X}) \to (\mathrm{Gr}(P))(X)$ over a generic value of $\mathrm{Gr}(u)_0$:

$$\phi_\lambda(z)\tilde{x} = z^{\mu_H(\sigma,\lambda)}\tilde{x}, \quad z \in \mathbb{C}^\times.$$

The *Mundet weight* is the sum of the Hilbert-Mumford and Ramanathan weights:

$$\mu_M(\sigma, \lambda) := \mu_H(\sigma, \lambda) + \mu_R(\sigma, \lambda).$$

Then

$$(P, u) \ \ \textit{semistable} \text{ resp. } \textit{stable} \iff \mu(\sigma, \lambda) \leq 0 \quad \text{resp.} < 0, \forall(\sigma, \lambda).$$

Mundet's original definition allowed possibly irrational λ, but this is unnecessary in the case that the symplectic class is rational by [76, Remark 5.8]. Mundet semistability is realized as a git stability condition in Schmitt [66, 67].

The moduli stack of Mundet-semistable morphisms admits a natural Kontsevich-style compactification that allows formation of bubbles in the fibers of the associated bundle: An *n-marked gauged map* from C to X over a scheme S is a datum $(\hat{C}, P, u, \underline{z})$ where $\hat{C} \to S$ is a proper flat morphism with reduced nodal curves as fibers, $P \to C \times S$ is a principal G-bundle; and

$$u : \hat{C} \to P(X) := (P \times X)/G$$

is a family of stable maps with base class $[C]$, that is, the composition of u with the projection $P(X) \to C$ has class $[C]$. A *morphism* between gauged maps $(S, \hat{C}, P, u)$ and $(S', \hat{C}', P', u')$ consists of a morphism $\beta : S \to S'$, a morphism $\phi : P \to (\beta \times 1)^*P'$, and a morphism $\psi : \hat{C} \to \hat{C}'$ such that the first diagram below is Cartesian and the second and third commute:

$$\begin{tikzcd} \hat{C} \arrow[r] \arrow[d, "\psi"'] & S \arrow[d, "\beta"] \\ \hat{C}' \arrow[r] & S' \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r] \arrow[d, "\phi"'] & S \times C \arrow[d, "\mathrm{id}"] \\ (\beta \times 1)^* P' \arrow[r] & S \times C \end{tikzcd} \qquad \begin{tikzcd} \hat{C} \arrow[r, "u"] \arrow[d, "\psi"'] & P(X) \arrow[d, "{[\phi \times \mathrm{id}_X]}"] \\ \hat{C}' \arrow[r, "u'"'] & P'(X). \end{tikzcd}$$

An *n-marked* nodal gauged map is equipped with an n-tuple $(z_1, \ldots, z_n) \in \hat{C}^n$ of distinct smooth points on $\hat{C}$. An n-marked nodal gauged map $(\hat{C}, P, \underline{z}, u)$ is *Mundet semistable* resp. *stable* if the principal component is Mundet semistable resp. stable and the section $u : \hat{C} \to P(X)$ is a stable section, in the sense that any component on which u is constant has at least three special (nodal or marked) points.

2.2. **Moduli stacks.** We introduce the following notations for moduli stacks. Denote by $\overline{\mathfrak{M}}_n^G(C,X,d)$ resp. $\overline{\mathcal{M}}_n^G(C,X,d)$ the category of gauged maps resp. Mundet semistable gauged maps from C to X/G of homology class d and n markings.

Theorem 2.1. *For any d,n, if stable=semistable then the stack $\overline{\mathcal{M}}_n^G(C,X,d)$ is a proper Deligne-Mumford stack equipped with evaluation morphisms*

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C,X,d) \to (X/G)^n, \quad (\hat{C},P,u) \mapsto (\underline{z}^*P, \underline{z}^*u)$$

and virtual fundamental class.

The properties of the moduli stacks in the above theorem were proved elsewhere. Properness is covered in detailed in [33, Theorem 1.1]. Virtual fundamental classes are [76, Example 6.6]. We sketch the construction for completeness. The proof of properness uses a simpler Grothendieck-style compactification obtained by allowing the maps to acquire base points, studied by Schmitt [66], [67, Section 2.7]. Suppose that $X \subset \mathbb{P}(V)$ is embedded in the projectivization $\mathbb{P}(V)$ of a G-representation V. A map $C \to P(\mathbb{P}(V))$ gives rise to a line sub-bundle $L \subset C \times P(V)$. By dualization such a sub-bundle gives rise to a quotient map $q : C \times P(V)^\vee \to L^\vee$. A *gauged quotient* is a datum $(P,L,q,\underline{z})$, called by Schmitt [66] a *bundle with map*. Denote by $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ the space of stable gauged quotients. The moduli stacks $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ only admit evaluation morphisms to the quotient stacks for the ambient vector spaces,

$$\mathrm{ev} : \overline{\mathcal{M}}_n^G(C,X,d) \to (V/(G\times\mathbb{C}^\times))^n, \quad (\hat{C},P,u) \mapsto (\underline{z}^*P, \underline{z}^*L, \underline{z}^*q).$$

The moduli stack of stable gauged quotients admits a construction as a geometric invariant theory quotient by Schmitt [66, 67]. Choose a faithful representation $G \to GL(V)$, so that $X \subset \mathbb{P}(V)$. A *k-level structure* for a stable gauged quotient is a collection of sections $s_1,\ldots,s_k : C \to P(V)$ generating $P(V)$. Equivalently, a level structure is a surjective morphism $\mathcal{O}_C^{\otimes k} \to P(V)^\vee$. The action of $GL(k)$ on $C \times \mathbb{C}^k$ induces an action on the set of level structures by composition. The stack $\overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C,X,d)$ of gauged quotients with level structure is naturally an Artin stack with an action of $GL(k)$ on the sections. Schmitt [66, Section 2.7] constructs a linearization $D(\tilde{X}) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C,X,d)$ giving rise to a projective embedding of the coarse moduli space, so that the git quotient is the stack of gauged quotients:

$$\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d) = \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C,X,d)/\!\!/GL(k).$$

In particular this construction implies that $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ has proper coarse moduli space. If stable=semistable then all stabilizers are finite, and since we are in characteristic zero, this implies that $\overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$ is Deligne-Mumford and proper. Now the Kontsevich-style compactification $\overline{\mathcal{M}}_n^G(C,X,d)$ admits a morphism by Popa-Roth [63, Theorem 7.1]

$$\overline{\mathcal{M}}_n^G(C,X,d) \to \overline{\mathcal{M}}_n^{G,\mathrm{quot}}(C,X,d)$$

and so is also proper. Denote by $\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d)$ the moduli stack of gauged maps with level structure on the associated vector bundle $P(V)$. The Givental construction on the moduli stack of maps with level structure gives a morphism $\pi : \overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev,quot}}(C,X,d)$. Then the moduli stack of gauged maps is also a stack-theoretic quotient

$$\overline{\mathcal{M}}_n^G(C,X,d) = \pi^{-1}(\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d)^{\mathrm{ss}})/G.$$

However, the pull-back of the linearization $D(\tilde{X})$ is not ample on $\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C,X,d)$. Thus this quotient cannot be considered a git quotient without further perturbation of the linearization.

Virtual fundamental classes are obtained from the construction of Behrend-Fantechi [9]. The argument uses the deformation theory from Olsson [57, Theorem 1.5] for morphisms to quotient stacks. The universal curve $\overline{\mathcal{C}}_n^G(C,X)$ is the stack whose objects are tuples $(\hat{C},P,u,\underline{z},z')$ where $(\hat{C},P,u,\underline{z})$ is a gauged map and $z' \in \hat{C}$ is a (possibly singular) point. Forgetting z' defines a projection

$$p : \overline{\mathcal{C}}_n^G(C,X) \to \overline{\mathcal{M}}_n^G(C,X)$$

while evaluating at z' defines a universal gauged map

$$e : \overline{\mathcal{C}}_n^G(C,X) \to X/G.$$

The relative obstruction theory has complex given by $R\pi_* e^* T(X/G)^\vee$ equipped with its canonical morphism to the cotangent complex of $\overline{\mathcal{M}}_n^G(C,X)$. If stable=semistable then the obstruction theory is perfect and $\overline{\mathcal{M}}_n^G(C,X,d)$ is a proper smooth Deligne-Mumford stack with perfect relative obstruction theory over the stack of *semistable* n-marked maps to C, see [76]. Denote by $[\overline{\mathcal{M}}_n^G(C,X,d)] \in H(\overline{\mathcal{M}}_n^G(C,X,d))$ the virtual fundamental classes constructed via Behrend-Fantechi machinery.

Using the virtual fundamental classes, gauged Gromov-Witten potentials are defined as follows. Suppose that stable=semistable for all gauged maps. The gauged potential $\tau_{X,G}$ is the formal map defined by

$$\tau_{X,G} : QH_G(X) \to \Lambda_X^G$$

$$h \mapsto \sum_{n\geq 0, d\in H_2^G(X,\mathbb{Z})/\,\mathrm{torsion}} (q^d/n!) \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]} \mathrm{ev}^*(h,\ldots,h)$$

for $h \in H_G(X)$.

Later we will need several variations on the gauged Gromov-Witten potential. We describe three variations which will be used later.

Definition 2.2. (a) (Gauged invariants with Deligne-Mumford classes) The first variation involves pull-back classes from the curve C. That is, let

$$f : \overline{\mathcal{M}}_n^G(C,X,d) \to \overline{\mathcal{M}}_n(C)$$

be the map obtained by projecting from $C \times X/G$ to C. For any class $\beta \in H(\overline{\mathcal{M}}_n(C))$ and $h \in H_G(X)^n$ define a gauged invariants with insertions

$$\tau_{X,G}^{n,d}(h,\beta) := \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]} \mathrm{ev}^*(h \otimes \ldots \otimes h) \cup f^*\beta. \tag{14}$$

In particular, by taking $n = 3$ and β the dual class of a point in $\overline{\mathcal{M}}_3(C)$ we obtain the gauged analog of three-point invariants.

(b) (Twisted invariants) The second variation gives Euler-twisted gauged Gromov-Witten invariants. For any G-equivariant bundle $E \to X$ we denote by

$$\mathrm{Ind}(E) := Rp_* e^*(E/G) \tag{15}$$

the index of the bundle $E/G \to X/G$. The index class $\mathrm{Ind}(E)$ lies in the bounded derived category of $\overline{\mathcal{M}}_n^G(C,X)$, since p is proper. Furthermore $\mathrm{Ind}(E)$ admits a resolution by vector bundles, since p is a local complete intersection morphism, see [20, Appendix]. It follows that the Euler class

$$\epsilon(E) := \mathrm{Eul}_{\mathbb{C}^\times}(\mathrm{Ind}(E)) \in H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n^G(C,X)) \tag{16}$$

is well-defined after passing to the equivariant cohomology of $\overline{\mathcal{M}}_n^G(C,X)$ for the trivial $\mathbb{C}^\times$-action corresponding to scalar multiplication on the fibers and inverting the equivariant parameter. The Euler-twisted gauged invariants are defined by

$$\begin{aligned} \tau_{X,G}^{n,d} &: QH_G(X)^n \times H(\overline{\mathcal{M}}_n(C)) \to \mathbb{Q} \\ &(h,\beta) \mapsto \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]} \mathrm{ev}^*(h,\ldots,h) \cup f^*\beta \cup \epsilon(E). \end{aligned} \tag{17}$$

(c) (Parabolic structures) A final variation involves adding a parabolic structure at a point on the curve as in Heinloth-Schmitt [39] and Beck [6]. This variation will be used later to shift the stability condition slightly so that certain stacks become Deligne-Mumford. Recall from e.g. [3] that a *quasi-Borel structure* on a G-bundle $P \to C$ at $z_0 \in C$ consists of a reduction of structure group $\rho_{z_0} \subset P_{z_0}/B$. In the case of GL_r, a quasi-Borel structure is a full flag in the associated vector bundle $E = P(\mathbb{C}^r)$ at the point z_0. A *Borel-structure* is a quasi-Borel structure ρ_{z_0} together with a *dominant weight* $\nu \in \mathfrak{t}_{\mathbb{Q}}^\vee$ for B that lies in the interior of the Weyl alcove; That is, we have $(\alpha_0,\nu) < 1$ with respect to the basic inner product on $\mathfrak{t}^\vee$. We consider here only generic small parabolic weights, see [3] for the general theory.

The definition of the Ramanathan weight extends to a Ramanathan weight for bundles with parabolic structure, with an additional term arising from the parabolic structure [3]. The parabolic weight defines a line bundle over the generalized flag variety $G \times_B \mathbb{C}_\nu \to G/B$. Given any point $gB \in G/B$, the limit of gB under $z^\lambda, z \to 0$ is determined by the Bruhat cell $BwB \ni gB$ containing g. We denote by $\mu_B(\lambda)$ the corresponding Hilbert-Mumford weight,

$$\mu_B(\lambda) = (w\nu,\lambda) \in \mathbb{Q}.$$

Now let $P^\lambda \to C \times \mathbb{C}^\times$ denote the family of bundles in Section 2.1. The reduction ρ_{z_0} defines a reduction ρ_{z_0} in P^λ and, since G/B is complete, extends over the central fiber to a reduction $\mathrm{Gr}(\rho)_{z_0} \in \mathrm{Gr}(P)_{z_0}/B$. The automorphism λ acts on $\mathrm{Gr}(P_{z_0}) \times_B \mathbb{C}_\mu$ by some integer $\mu_B(\sigma, \lambda)$. The Ramanathan weight becomes modified by the addition

$$\mu_R(\sigma, \lambda, \nu) = \mu_R(\sigma, \lambda) + \mu_B(\sigma, \lambda) = \mu_R(\sigma, \lambda) + (w\nu, \lambda)$$

where w is the Weyl group element corresponding to the Bruhat cell containing $\sigma(z_0)$.

We introduce the following notations for moduli stacks of gauged maps with parabolic structure. Let $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \nu)$ denote the Artin stack consisting of Mundet semistable pairs (P, u, σ) of a bundle $P \to C$, a section $u : C \to P(X)$, markings $\underline{z}$, and a parabolic structure (σ, ν) at z_0.

Proposition 2.3. *If stable=semistable then $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \nu)$ is a smooth, proper Deligne-Mumford stack with a perfect relative obstruction theory. For generic parabolic weights ν, stable=semistable and so the conclusion holds. If stable=semistable for $\overline{\mathcal{M}}_n^G(C, X, \tilde{X})$ and ν is sufficiently small, then the parabolic structure does not play a role in stability and the forgetful morphism*

$$\pi : \overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \nu) \to \overline{\mathcal{M}}_n^G(C, X, \tilde{X})$$

is a G/B-bundle.

Sketch of proof. By considering a principal bundle as a vector bundle together with section of the associated $GL(r)/G$ bundle as in [6], the construction of the quot-scheme compactification of the moduli stacks reduces to the construction of vector bundles with local and global decorations in [7]. On the other hand, the Kontsevich-style compactification $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \nu)$ is obtained from the quot-scheme compactification by taking stable sections of the associated fiber bundle. It follows that $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \nu)$ is an Artin stack, and Deligne-Mumford if stable=semistable. For generic weights ν equality cannot hold in the semistability inequality. It follows that stable=semistable. The construction of the moduli space of gauged maps with parabolic structure is a standard extension of the construction of moduli of parabolic bundles, and omitted. The final assertion follows from the fact that for small ν, the parabolic structure plays no role in the stability condition. Forgetting the parabolic structure gives a morphism to the moduli stack $\overline{\mathcal{M}}_n^G(C, X, \tilde{X})$ with fiber $P_{z_0}/B \cong G/B$. □

Remark 2.4. The following construction will be used later to avoid singularities in the "master space" used for wall-crossing. If stable=semistable then the integral of any class h over $\overline{\mathcal{M}}_n^G(C, X, \tilde{X})$ may be written as an integral over the moduli space of maps with parabolic structure: Let L_π be the relative cotangent complex for the projection $\pi : \overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \nu) \to \overline{\mathcal{M}}_n^G(C, X, \tilde{X})$. Then

$$\int_{[\overline{\mathcal{M}}_n^G(C,X,\tilde{X})]} h = \int_{[\overline{\mathcal{M}}_n^G(C,X,\tilde{X},\nu)]} \pi^* h \cup \mathrm{Eul}(L_\pi)/|W|. \tag{18}$$

Indeed the integral of $\mathrm{Eul}(L_\pi)/|W|$ over the fiber of π is

$$\int_{[G/B]} \mathrm{Eul}(T^\vee(G/B))/|W| = \chi(G/B)/|W| = 1.$$

2.3. **Localized graph potentials.** Restricting to the case of a rational curve, one may factorize the graph potential into *localized gauged potentials* corresponding to the two fixed points:

Theorem 2.5. [28], [77] *There exist* localized graph resp. gauged potentials

$$\tau_{X/\!\!/G,\pm} : QH(X/\!\!/G) \to QH_{\mathbb{C}^\times}(X/\!\!/G), \quad \tau_{X,G,\pm} : QH_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G),$$

that represent the contribution from the fixed points 0 *resp.* ∞*, such that the graph resp. gauged potential then admits a factorization*

$$\tau_{X/\!\!/G} = (\tau_{X/\!\!/G,\pm}, \tau_{X/\!\!/G,+}) : QH(X/\!\!/G) \to \Lambda_X^G \tag{19}$$
$$\tau_{X,G} = (\tau_{X,G,\pm}, \tau_{X,G,+}) : QH_G(X) \to \Lambda_X^G$$

where $(\cdot,\cdot)$ *denotes the pairing given by integration over* $I_{X/\!\!/G}$*.*

The proof is given elsewhere [77]. We sketch the construction for completeness. Let $\mathbb{C}^\times$ act on $C = \mathbb{P}^1$ via the standard action with weights $-1, 1$:

$$\mathbb{C}^\times \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (w, [z_-, z_+]) \mapsto w[z_-, z_+] = [w^{-1}z_-, wz_+].$$

The fixed points of the induced action on $\mathcal{M}_n(C, X/\!\!/G)$ correspond to configurations of a constant map to $X/\!\!/G$ together with bubble trees attached at the fixed points $[1,0],[0,1]$. That is, the fixed point locus is a union of fibered products

$$\mathcal{M}_n(C, X/\!\!/G)^{\mathbb{C}^\times} = \bigcup_{n_- + n_+ = n} \bigcup_{d_+ d_+ = d} \overline{\mathcal{M}}_{0,n_-+1}(X/\!\!/G, d_-) \times_{\overline{I}_{X/G}} \overline{\mathcal{M}}_{0,n_++1}(X/\!\!/G, d_+).$$

By pushing-forward the classes $\mathrm{ev}^*(h \otimes \ldots \otimes h)$ over the extra marked point in $\overline{\mathcal{M}}_{0,n_\pm+1}(X/\!\!/G, d_\pm)$ define

$$\tau_{X/\!\!/G,\pm} : QH(X/\!\!/G) \to QH_{\mathbb{C}^\times}(X/\!\!/G)$$

that represent the contribution to $\tau_{X/\!\!/G}$ from the fixed points 0 resp. ∞. More precisely,

$$\tau_{X/\!\!/G,\pm}(h) := 1 + h/\zeta + \sum_{n \ge 0} (1/n!) \tau^n_{X/\!\!/G,\pm}(h \otimes \ldots \otimes h)$$

where

$$\tau^n_{X/\!\!/G,\pm}(h_1, \ldots, h_n) = \sum_{d \in H_2(X/\!\!/G,\mathbb{Z})} q^d/n!\, \mathrm{ev}_{n+1,*}\left(\mp\zeta(\pm\zeta - \psi_{n+1})^{-1} \bigcup_{i=1}^n \mathrm{ev}_i^*\, h_i \right)$$

is the sum over n, d such that the moduli space of stable maps is non-empty (that is, either $n \ge 3$ or $d > 0$) and the class

$$\psi_{n+1} = c_1(L_{n+1}) \in H^2(\overline{\mathcal{M}}_{0,n+1}(X/\!\!/G, d))$$

is the first Chern class of the cotangent line at the $(n+1)$-st marked point,

$$L_{n+1} \to \overline{\mathcal{M}}_{0,n+1}(X/\!\!/G, d)), \quad (L_{n+1})_{(u:C \to X/\!\!/G, \underline{z})} = T_{z_{n+1}}C.$$

The graph potential then admits a factorization

$$\tau_{X/\!\!/G} = (\tau_{X/\!\!/G,\pm}, \tau_{X/\!\!/G,+}) : QH(X/\!\!/G) \to \Lambda_X^G$$

where $(\cdot,\cdot)$ denotes the pairing given by integration over $I_{X/\!\!/G}$ [28]. We denote, for later use, the map

$$\tau^{\infty}_{X/\!\!/G,\pm} : QH(X/\!\!/G)^2 \to \Lambda_X^G, \quad \tau^{\infty}_{X/\!\!/G,\pm}(h_1, h_2) = (\tau_{X/\!\!/G,\pm}(h_1), h_2) \tag{20}$$

obtained by dualizing one factor.

The fixed points for the circle action on the space of gauged maps over the projective line are described in [77]: Fixed maps are data $(\hat{C}, P, u, \underline{z})$ such that there exists a one-parameter family of bundle isomorphisms preserving the section:

$$\phi : \mathbb{C}^\times \to \mathrm{Hom}(P, m_w^* P) \quad P(\phi(w))^* m_w^* u = u, \forall w \in \mathbb{C}^\times$$

where $m_w : C \to C$ is multiplication by C. In local trivializations near the fixed points $[1,0], [0,1]$, the bundle automorphism ϕ is given by homomorphisms

$$\phi_\pm : \mathbb{C}^\times \to G.$$

In the corresponding trivializations of $P(X)$, the section u is given by maps

$$u_\pm : \mathbb{C} \to G, \quad \phi_\pm(w) u_\pm(z) = u_\pm(wz).$$

Furthermore, for $u : \hat{C} \to P(X)$ a $\mathbb{C}^\times$-fixed map sends the components of $\hat{C}$ map to either of the fixed points $[1,0], [0,1] \in (\mathbb{P}^1)^{\mathbb{C}^\times}$. Thus u consists of a pair of sections (u_-, u_+) with bubble trees attached at $[1,0], [0,1] \in \mathbb{P}^1$, glued together via a transition map

$$\mathbb{C}^\times \to G, \quad z \mapsto \phi_+(z)\phi_-(z^{-1})^{-1}.$$

One may therefore view the fixed point locus as a fiber product, as follows: A *framing* of a gauged map is a trivialization of P neighborhood of a point $z' \in C$. We consider the stack $\overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_\pm,\pm}(C, X, d)^{\mathbb{C}^\times}$ of $\mathbb{C}^\times$-fixed *framed gauged maps* at the point $z' = [1,0]$ resp. $z' = [0,1]$. In the case of large linearization, evaluation at the extra marked point defines maps

$$\mathrm{ev}' : \overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_\pm,\pm}(C, X, d) \to I_{X/\!\!/G}.$$

Then the fixed point locus factorizes

$$\overline{\mathcal{M}}^{G}_{n}(C, X, d)^{\mathbb{C}^\times} = \overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_-,-}(C, X, d_-)^{\mathbb{C}^\times} \times_{I_{X/G}} \overline{\mathcal{M}}^{G,\mathrm{fr}}_{n_+,+}(C, X, d_+)^{\mathbb{C}^\times}.$$

By pushing-forward classes $\mathrm{ev}^*(h \otimes \ldots \otimes h)$ we obtain maps

$$\tau_{X,G,\pm} : QH_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G)$$

so that

$$\tau_{X,G} = (\tau_{X,G,+}, \tau_{X,G,-}) : QH_G(X) \to \Lambda_X^G.$$

Later we will need a variation on this construction that involves dualizing one of the factors. For any real parameter ρ let $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}^\rho, d)_- \subset \overline{\mathcal{M}}_n^G(C, X, \tilde{X}^\rho, d)$ denote the locus of the fixed point set where all markings $z_1, \ldots, z_n$ map to $0 \in C$; all bubble components $C_1, \ldots, C_k$ map to $0 \in C$; the one-parameter subgroup ϕ_+ vanishes. Evaluation at $z_1, \ldots, z_n$ and at $\infty \in C$ defines a map

$$\mathrm{ev}_- \times \mathrm{ev}_+ : \overline{\mathcal{M}}_n^G(C, X, \tilde{X}^\rho, d)_- \to (X/G)^n \times X/G.$$

By integration one obtains a map linear in the second variable

$$(21) \quad \tau^\rho_{X,G,-} : QH_G(X)^2 \to \Lambda_X^G[\zeta, \zeta^{-1}]],$$
$$(h_-, h_+) \int_{[\overline{\mathcal{M}}_n^G(C,X,\tilde{X}^\rho,d)_-]} \left(\mathrm{ev}_-^*(h_- \otimes \ldots \otimes h_-\right) \cup \mathrm{ev}_+^* h_+ \cup \mathrm{Eul}(\nu_-)$$

where ν_- is the virtual normal complex of $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}^\rho, d)_-$ in $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}^\rho, d)$. In the limit $\rho \to \infty$, the gauged map is generically semistable and triviality in a neighborhood of ∞ implies that in fact ev_+ takes values in $X/\!\!/G$. Hence in this case $\tau^\rho_{X,G,-}$ is the obtained from the dualization

$$\tau^\vee_{X,G,-} : QH_G(X) \times QH(X/\!\!/G) \to \Lambda_X^G$$

of $\tau_{X,G,-}$ by composition with pull-back $QH_G(X) \to QH(X/\!\!/G)$. Similar definitions requiring the bubbles and markings to map to ∞ give rise to a map $\tau^\rho_{X,G,+}$: $QH_G(X)^2 \to \Lambda_X^G$ related to $\tau_{X,G,+}$ in the limit $\rho \to \infty$ by dualization.

2.4. **Master space.** In this section we study the area-dependence of the gauged Gromov-Witten invariants, by which we mean the dependence on the choice of linearization. The basic strategy is the same as that outline in e.g. Thaddeus [71] for the case of variation of linearization in geometric invariant theory. The wall-crossing formula of is obtained from a *master space* construction as follows: Suppose that

$$\tilde{X}_\pm \to X$$

are linearizations, that is, ample G-line bundles on X. Associated to $\tilde{X}_\pm$ are linearizations

$$D(\tilde{X}_\pm) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, d)$$

over the moduli stacks of curves with level structure constructed in Schmitt [66, 67]. Consider the rank two bundle obtained from the direct sum:

$$D(\tilde{X}_-) \oplus D(\tilde{X}_+) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, d).$$

Taking the projectivization of the total space gives a $\mathbb{P}^1$-fibration

$$\mathbb{P}(D(\tilde{X}_-) \oplus \ D(\tilde{X}_+)) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, d).$$

The action of $GL(k)$ lifts to the fibration, since the bundles $D(\tilde{X}_\pm)$ are $GL(k)$-equivariant. The bundle

$$\mathcal{O}_{\mathbb{P}(D(\tilde{X}_-)\oplus \ D(\tilde{X}_+))}(1) \to \mathbb{P}(D(\tilde{X}_-) \oplus \ D(\tilde{X}_+))$$

is automatically ample on the coarse moduli space of $\mathbb{P}(D(\tilde{X}_-) \oplus D(\tilde{X}_+))$. Denote the git quotient with respect to this linearization

$$\overline{\mathcal{M}}_n^{G,\mathrm{lev},\mathrm{quot}}(C, X, \tilde{X}_-, \tilde{X}_+, d) = \mathbb{P}(D(\tilde{X}_-) \oplus D(\tilde{X}_+)) /\!\!/ GL(k).$$

Similarly, let

$$\pi^* \mathbb{P}(D(\tilde{X}_-) \oplus D(\tilde{X}_+)) \to \overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C, X, d)$$

denote the pull-back to the stack of stable gauged maps with level structure and

$$\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C, X, \tilde{X}_-, \tilde{X}_+, d) = \pi^{-1}(\mathbb{P}(D(\tilde{X}_-) \oplus D(\tilde{X}_+)))^{\mathrm{ss}} / GL(k)$$

the quotient of the pull-back of the semistable locus. The action of $\mathbb{C}^\times$ on $\mathbb{P}(D(\tilde{X}_-) \oplus D(\tilde{X}_+))$ induces an action of $\mathbb{C}^\times$ on $\overline{\mathcal{M}}_n^{G,\mathrm{lev}}(C, X, \tilde{X}_-, \tilde{X}_+, d)$.

The fixed point components for the natural circle action are of two types. First, there there are inclusions

$$\mathbb{P}(D(\tilde{X}_\pm) \oplus 0) \to \mathbb{P}(D(\tilde{X}_-) \oplus D(\tilde{X}_+))$$

and isomorphisms

$$\mathbb{P}(D(\tilde{X}_\pm) \oplus 0) \cong \overline{\mathcal{M}}_n^G(C, X, \tilde{X}_\pm).$$

These induce embeddings

$$\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_\pm) \to \overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+)^{\mathbb{C}^\times}$$

in the locus of fixed points of the $\mathbb{C}^\times$-action. On the other hand, there are fixed point components correspond to *reducible* gauged maps for some stability condition interpolating between those defined by $\tilde{X}_\pm$. Reducibility means that the fixed point components consist of maps $v = (P, u) : \hat{C} \to X/G$ that admit a one-parameter family of automorphisms $\phi : \mathbb{C}^\times \to \mathrm{Aut}(P)$; via evaluation at a point $\mathrm{Aut}(P) \to \mathrm{Aut}(P_z)$, any such one-parameter family may be identified with a one-parameter family of automorphisms of G generated by some element $\lambda \in \mathfrak{g}$. Euler-twisted integration over the fixed point components gives rise to fixed point contributions

$$\tau_{X,G,\zeta,t} : QH_G(X) \to \Lambda_X^G.$$

The fixed point contributions are bubble trees consisting of maps to the quotient stack with one-parameter automorphisms and stable maps fixed up to isomorphism by one-parameter subgroups. Suppose that a gauged map $(P \to C, u : \hat{C} \to P(X))$ is reducible, that is, has a one-parameter family of automorphism $\phi : \mathbb{C}^\times \to \mathrm{Aut}(P)$ covering the identity on the principal component so that the associated automorphism

$$\phi(X) : P(X) \to P(X), \quad \phi(X)^* u = u.$$

Evaluation at any fiber defines a homomorphism $\phi_z : \mathbb{C}^\times \to \mathrm{Aut}(P_z) \cong G$ and so identifies ϕ_z with a one-parameter subgroup of G. Let $\lambda \in \mathfrak{g}$ be a generator of ϕ_z and $G_\lambda \subset G$ the centralizer. The structure group of P reduces to the centralizer G_λ of λ. Furthermore the restriction $u|\hat{C}_0$ of u to the principal component $\hat{C}_0$ takes values in $P(X^\lambda)$ where $X^\lambda = \{x \in X | \lambda_X(x) = 0\}$. Any bubble tree attached at $z \in \hat{C}_0$ must be fixed, up to isomorphism, by the action of $\phi(z) \in \mathrm{Aut}(P_z(X))$. That is, there exists a one-parameter family of automorphisms $\psi : \mathbb{C}^\times \to \mathrm{Aut}(\hat{C})$ so that

$\psi^* u = \phi(X) \circ u$, where $\phi(X) : P(X) \to P(X)$ is the automorphism of the associated fiber bundle induced by ϕ.

We introduce notation for these fixed point stacks and their normal complexes as follows. For each $\lambda \in \mathfrak{g}$, let $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \lambda)$ denote the stack of Mundet-semistable morphisms from C to X/G_λ that are $\mathbb{C}_\lambda^\times$-fixed and take values in X^λ on the principal component. Via the inclusion $G_\lambda \to G$ the universal curve over $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \lambda)$ admits a morphism to X/G. Denote by ν_λ the virtual normal complex for the morphism $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \lambda) \to \overline{\mathcal{M}}_n^G(C, X, \tilde{X})$.

The virtual fundamental classes on these fixed point stacks lead to fixed point contributions appearing in the wall-crossing formula. Let $QH_{G,\mathrm{fin}}(X)$ denote the tensor product of $H_G(X)$ with the sub-ring

$$\Lambda_X^{G,\mathrm{fin}} = \left\{ \sum_{i=1}^n c_i q^{d_i}, d_i \in H_2^G(X), c_i \in \mathbb{Q} \right\} \subset \Lambda_X^G$$

of finite sums. Virtual integration over $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}, \lambda)$ defines a "fixed point contribution"

$$\begin{aligned} (22) \quad & \tau_{X,G,\lambda,\tilde{X}} : QH_{G,\mathrm{fin}}(X) \to \tilde{\Lambda}_X^G \otimes H(B\mathbb{C}^\times), \\ & h \mapsto \sum_{d \in H_G^2(X,\mathbb{Z})} \sum_{n \geq 0} \int_{[\overline{\mathcal{M}}_n^G(C,X,\tilde{X},\lambda,d)]} (q^d/n!) \operatorname{ev}^*(h, \ldots, h) \cup \operatorname{Eul}(\nu_\lambda)^{-1} \end{aligned}$$

for $h \in H_G(X)$. Here we omit the restriction map $H_{G,\mathrm{fin}}(X) \to H_{G_\lambda}(X)$ to simplify notation. The following is [31, Theorem 3.14].

Theorem 2.6 (Wall-crossing for gauged Gromov-Witten potentials)**.** *Let X be a smooth projective G-variety. Suppose that $\tilde{X}_\pm \to X$ are linearizations such that semistable=stable for the stack of polarized gauged maps in [31]. Then the gauged Gromov-Witten potentials are related by*

$$(23) \qquad \tau_{X,\tilde{X}_+,G} - \tau_{X,\tilde{X}_-,G} = \sum_{[\lambda], t \in (-1,1)} \frac{|W_\lambda|}{|W_{\mathbb{C}\lambda}|} \operatorname{Resid}_\xi \tau_{X,G,\lambda,t}$$

where the sum is over equivalence classes $[\lambda]$ of unparametrized one-parameter subgroups generated by $\lambda \in \mathfrak{g}$. Similarly the localized gauged Gromov-Witten potentials are related by

(24)

$$\tau_{X,\tilde{X}^+,G,\pm} - \tau_{X,\tilde{X}^-,G,\pm} = \sum_{[\lambda], t \in (-1,1)} \frac{|W_\lambda|}{|W_{\mathbb{C}\lambda}|} \operatorname{Resid}_\xi \tau_{X,\tilde{X}^t,G,\pm,\lambda} : QH_G^{\mathrm{fin}}(X)^2 \to \Lambda_X^G.$$

The fixed point contributions can be re-written as contributions from gauged Gromov-Witten invariants with structure group of smaller rank as follows. For $\lambda \in \mathfrak{g}$ let $\mathbb{C}_\lambda^\times \subset G_\lambda$ denote the one-parameter subgroup generated by λ, and $G_\lambda/\mathbb{C}_\lambda^\times$ the quotient. Let $X^\lambda \subset X$ denote the fixed point set of $\mathbb{C}_\lambda^\times$. Let $\overline{\mathcal{M}}_{0,n}(X)^{\mathbb{C}_\lambda^\times} \subset \overline{\mathcal{M}}_{0,n}(X)$

denote the $\mathbb{C}^\times_\lambda$-fixed point stack of stable maps to X. The evaluation map restricted to $\overline{\mathcal{M}}_{0,n}(X)^{\mathbb{C}^\times_\lambda}$ automatically takes values in the fixed point locus $X^\lambda \subset X$, that is, $\mathrm{ev} : \overline{\mathcal{M}}_{0,n}(X)^{\mathbb{C}^\times_\lambda} \to (X^\lambda)^n$. Push-pull over the moduli stack $\overline{\mathcal{M}}_{0,n+1}(X)^{\mathbb{C}^\times_\lambda}$ defines a *quantum restriction map*

$$\iota_\lambda : QH_{G_\lambda}(X) \to QH_{G_\lambda}(X^\lambda), \quad h \mapsto h|_{X^\lambda} + \sum_{n,d} (q^d/n!)\, \mathrm{ev}_{n+1,*}\, \mathrm{ev}_1^* h \cup \ldots \cup \mathrm{ev}_n^* h.$$

Let $\pi^G_{G_\lambda} : \Lambda^{G_\lambda}_X \to \Lambda^G_X$ be the canonical map of Novikov rings induced by $H_2^{G_\lambda}(X) \to H_2^G(X)$.

Lemma 2.7. *Suppose that stable=semistable for λ-fixed gauged maps. Then*

$$\tau_{X,\tilde{X}^t,G,\lambda} = \pi^G_{G_\lambda} \circ \tau_{X^\lambda,\tilde{X}^t|X^\lambda,G_\lambda/\mathbb{C}^\times_\lambda} \circ \iota_\lambda. \tag{25}$$

Proof. Decomposing the fixed point locus according to the number of markings on each bubble tree gives an isomorphism

$$\begin{aligned}\overline{\mathcal{M}}^G_n(C,X,\tilde{X},\lambda) &\cong \bigcup_{i_1+\ldots+i_r=n} \prod_{j=1}^r (\{\mathrm{pt}\} \cup \overline{\mathcal{M}}_{0,i_j+1}(X)^{\mathbb{C}^\times_\lambda}) \times_{(X^\lambda)^r} \overline{\mathcal{M}}^{G_\lambda,\mathrm{fr}}_r(C,X^\lambda)/G^r_\lambda \\ &\cong \bigcup_{i_1+\ldots+i_r=n} \prod_{j=1}^r (\{\mathrm{pt}\}/G_\lambda \cup \overline{\mathcal{M}}_{0,i_j+1}(X)^{\mathbb{C}^\times_\lambda}/G_\lambda) \times_{(X^\lambda/G_\lambda)^r} \mathcal{M}^{G_\lambda/\mathbb{C}^\times_\lambda}_r(C,X^\lambda)\end{aligned}$$

where $\{\mathrm{pt}\}$ represents a trivial bubble tree attached at the j-th node on the principal component. It follows that integration over $\overline{\mathcal{M}}^G_n(C,X,\tilde{X},\lambda)$ is given by push-forward of $\mathrm{ev}_1^* h \cup \ldots \mathrm{ev}_{i_j}^* h$ over each

$$\mathrm{ev}_{i_j+1} : \{\mathrm{pt}\} \cup \overline{\mathcal{M}}_{0,i_j+1}(X)^{\mathbb{C}^\times_\lambda}/G_\lambda \to X^\lambda/G_\lambda$$

followed by integration over $\overline{\mathcal{M}}^{G_\lambda}_r(C,X^\lambda)$, or more precisely, $\mathbb{C}^\times_\lambda$-equivariant integration over the Deligne-Mumford stack $\overline{\mathcal{M}}^{G_\lambda/\mathbb{C}^\times_\lambda}_r(C,X^\lambda)$ (for which stable=semistable). □

It will be important for our induction argument later that the rank of the structure group for the fixed point contributions is less than the rank of the original group. This allows an approach to results such as quantum abelianization and quantum Lefschetz by induction on the rank of the structure group. More precisely, there exists a canonical isomorphism

$$\mathcal{M}^{G_\lambda}(C,X^\lambda) \to \mathcal{M}^{G_\lambda/\mathbb{C}^\times}(C,X^\lambda).$$

Indeed via the projection map $G_\lambda \to G_\lambda/\mathbb{C}^\times_\lambda$ any gauged map to X^λ/G defines a gauged map to $X^\lambda/(G_\lambda/\mathbb{C}^\times_\lambda)$ and we obtain a map

$$\mathcal{M}^{G_\lambda}(C,X^\lambda) \to \mathcal{M}^{G_\lambda/\mathbb{C}^\times}(C,X^\lambda). \tag{26}$$

Up to finite cover the exact sequence

$$1 \to \mathbb{C}^\times_\lambda \to G_\lambda \to G_\lambda/\mathbb{C}^\times_\lambda \to 1$$

splits. Given a gauged map to $X^\lambda/(G_\lambda/\mathbb{C}^\times))$, let c denote the weight of the $C_\lambda^\times$-action on $\tilde{X}|X^\lambda$. Taking the bundle $\mathbb{C}_\lambda^\times$-bundle with first Chern class $-c$ defines the inverse map to (26).

Finally we remove the requirement that stable=semistable for linearized gauged maps. This requirement can be weakened to stable=semistable for gauged maps with respect to the two linearizations being compared, using moduli stacks of parabolic gauged maps:

Proposition 2.8. *If stable=semistable then the moduli stack $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$ of parabolic gauged maps is a smooth, proper Deligne-Mumford stack with a perfect relative obstruction theory. For generic parabolic weights μ, stable=semistable and so the conclusion holds.*

The following proposition is proved in exactly the same way as Proposition 2.3.

Corollary 2.9. *If stable=semistable for $\tilde{X}_\pm$ then for generic parabolic weights ν the stack $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$ of Mundet-semistable gauged maps with parabolic structure is a smooth, proper Deligne-Mumford stack with a perfect relative obstruction theory containing G/B-bundles over $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_\pm)$ as fixed point components.*

In other words, at the cost of adding a parabolic structure we may always obtain a master space that is a Deligne-Mumford stack with a perfect relative obstruction theory.

3. Quantum Witten localization

In this section we combine the area-dependence studied in the previous section with large and small area limit theorems from [77], [29] to obtain a proof of the quantum Witten localization formula (4). The gauged potential and the graph potential of the quotient are related by the *adiabatic limit theorem* of [77] (which is a generalization of an earlier result of Gaio-Salamon [27]).

3.1. **Quantum Kirwan map.** In this section we recall the quantum Kirwan map $\kappa_{X,G}$ of (2). The map $\kappa_{X,G}$ is defined by virtual integration over a moduli stack *scaled affine gauged maps* to X.

Definition 3.1. (Affine gauged maps) Let $n \geq 0$ be an integer. An n-marked affine gauged map is a tuple

$$(P \to C, u : C \to P(X), \lambda : C \to \mathbb{P}(\omega_C \oplus \mathbb{C}), \underline{z} = (z_0, \ldots, z_n))$$

where C is a twisted balanced curve as in orbifold Gromov-Witten theory [1], $P \to C$ is a principal G-bundle, ω_C is the dualizing sheaf on C, and λ is a section of its projectivization $\mathbb{P}(\omega_C \oplus \mathbb{C})$ which satisfies a certain *monotonicity condition*: on any maximal non-self-crossing path of components $C_0, C_1, \ldots, C_l$ of C starting with the component C_0 containing z_0, $\lambda|C_i$ is non-zero and finite on exactly one component

C_i, on which λ has a single double pole. Such a map is semistable if u takes values in $X/\!\!/G$ on the locus $\lambda^{-1}(\infty) \subset C$, the bundle P is trivial on the locus $\lambda^{-1}(0)$, $z_0 \in \lambda^{-1}(\infty)$ while $z_1, \ldots, z_n \in \lambda^{-1}(< \infty)$ and the datum admits no automorphisms: each component on which the scaling λ is finite and non-zero resp. zero or infinite and on which (P, u) is trivializable has at least two resp. three special point.

We introduce notation for moduli stacks and evaluation maps. Each component $\overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X, d)$ of homology class $d \in H^G_2(X, \mathbb{Z})/$ torsion and n markings has evaluation maps

$$\mathrm{ev}_\infty \times \mathrm{ev} : \overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X, d) \to (X/\!\!/G) \times (X/G)^n \quad (P, u, \lambda, \underline{z}) \mapsto (u(z_0), \ldots, u(z_n)).$$

The formula for $\kappa_{X,G}$ is

$$\kappa_{X,G}(h) = \sum_{n \geq 0, d} (q^d/n!)\, \mathrm{ev}_{\infty,*}\, \mathrm{ev}^*(h, \ldots, h).$$

We remark that if the target satisfies an equivariant Fano condition then the derivative of the quantum Kirwan map at zero is homomorphism of small quantum cohomologies. Here equivariantly Fano means that the equivariant first Chern class $c^G_1(TX)$ of X is the same as the first Chern class $c^G_1(\tilde{X})$ of the linearization. It follows that the first Chern class $c^G_1(TX)$ pairs positively with any curve class for which there is a generically semistable map to the quotient stack. In the Fano case, the moduli stacks $\overline{\mathcal{M}}^G_n(\mathbb{C}, X, d)$ have dimension

$$\dim \overline{\mathcal{M}}^G_n(\mathbb{C}, X, d) \geq \dim(X) + 2.$$

For $h \in H^{\leq 2}_G(X)$, the push-forwards to $H(X/\!\!/G)$ under ev_∞ have degree larger than $2\dim(X/\!\!/G)$ [77] for $d \neq 0$. Hence

$$\kappa_{X,G}(0) = 0, \quad D_0\kappa_{X,G} : T_0QH_G(X) \to T_0QH(X/\!\!/G). \tag{27}$$

For similar reasons, a lower bound on m the minimal Chern number $(d, c^G_1(X))$ for classes $d \in H^G_2(X)$ realized by stable affine gauged maps $u : \mathbb{P}(1, r) \to X/G$ implies that $D_0\kappa_{X,G}$ has no quantum corrections on classes of degree at most $2m$. This ends the remark.

Example 3.2. (Quantum Kirwan map for the scalar multiplication on affine space) Let $G = \mathbb{C}^\times$ act on $X = \mathbb{C}^k$ by scalar multiplication, so that $X/\!\!/G = \mathbb{P}^{k-1}$. We have

$$T_0QH_G(X) = \Lambda^G_X[\xi],$$

with ξ the equivariant parameter, while

$$T_0QH(X/\!\!/G) = \Lambda^G_X[\omega]/(\omega^k - q),$$

with $\omega \in H(\mathbb{P}^{k-1})$ the standard hyperplane class. By the previous remark $\kappa_{X,G}(0) = 0$ and

$$D_0\kappa_{X,G}(\xi^l) = \omega^l, \quad l < k.$$

A special case of the main result of [32] (quantum Stanley-Reisner relations) implies that

$$D_0\kappa_{X,G}(\xi^k) = q.$$

Hence $D_0\kappa_{X,G}$ is surjective and

$$T_0QH(X/\!\!/G) = T_0QH_G(X)/\ker D_0\kappa_{X,G} = \Lambda_X^G[\xi]/(\xi^k - q)$$

as expected.

3.2. **Adiabatic limit theorem.** The following theorem describes the relationship between the gauged potential and the graph potential of the quotient. Let $\rho \in (0,\infty)$ and the corresponding family of linearizations $\tilde{X}^\rho$ with $\rho \to \infty$.

Theorem 3.3. (Adiabatic limit theorem [77]) *If stable=semistable for the action of G on X then stable=semistable for gauged maps for ρ sufficiently large (more precisely, for any class $d \in H_2^G(X,\mathbb{Z})$ there exists an $r > 0$ such that $\rho > r$ implies stable=semistable) and*

$$\tau_{X/\!\!/G} \circ \kappa_{X,G} = \lim_{\rho\to\infty} \tau_{X,G} : QH_G(X) \to \Lambda_X^G.$$

If C is a genus zero curve equipped with a $\mathbb{C}^\times$-action, then the same equality holds for $\mathbb{C}^\times$-equivariant potentials $\tau_{X/\!\!/G}^{\mathbb{C}^\times}, \tau_{X,G}^{\mathbb{C}^\times}$. Similarly for the localized gauged potentials, consider the $\mathbb{C}^\times$-equivariant version of the quantum Kirwan map

$$\kappa_{X,G} : QH_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G)$$

Then [75, Theorem 1.6]

$$\tau_{X/\!\!/G,\pm} \circ \kappa_{X,G} = \tau_{X,G,\pm} : QH_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G). \tag{28}$$

In other words, the diagram

$$\begin{tikzcd} QH_G(X) \arrow[rr, "\kappa_{X,G}"] \arrow[dr, "\tau_{X,G}"'] & & QH(X/\!\!/G) \arrow[dl, "\tau_{X/\!\!/G}"] \\ & \Lambda_X^G & \end{tikzcd} \tag{29}$$

commutes in the limit $\rho \to \infty$.

Remark 3.4. In semi-Fano cases the localized adiabatic limit equation (28) allows to solve for the quantum Kirwan map on divisor classes. The constant term in $\tau_{X,G,\pm}$ (with respect to the equivariant parameter) arises from configurations with $\phi_\pm$, $d_\pm$ and $n_\pm$ vanishing, which are constant maps to $X/\!\!/G$ with no markings. It follows that the expansion of $\tau_{X,G,\pm}$ in powers of the inverted equivariant parameter $1/\zeta$ is

$$\tau_{X,G,\pm} = 1 + \tau_{X,G,\pm}^{(1)}/\zeta + \tau_{X,G,\pm}^{(2)}/\zeta^2 + \tau_{X,G,\pm}^{(3)}/\zeta^3 + \dots.$$

Similarly the constant contribution to $\tau_{X/\!\!/G,\pm}$ is 1, corresponding to configurations with no markings at 0 or ∞, while the contribution from a single marking again involves no bubbles, with a single marking at 0 or ∞ with Euler class ζ and so is the Id/ζ. It follows that the expansion of $\tau_{X/\!\!/G,\pm}$ in powers of ζ^{-1} is

$$\tau_{X/\!\!/G,\pm} = 1 + \mathrm{Id}/\zeta + \tau^{(2)}_{X/\!\!/G,\pm}/\zeta^2 + \tau^{(3)}_{X/\!\!/G,\pm}/\zeta^3 + \ldots : QH_G(X) \to QH_{\mathbb{C}^\times}(X/\!\!/G).$$

Similarly $\kappa_{X,G}$ admits an expansion in powers of ζ,

$$\kappa_{X,G} = \kappa^{(0)}_{X,G} + \kappa^{(1)}_{X,G}\zeta + \kappa^{(2)}_{X,G}\zeta^2 + \ldots.$$

Suppose that X is equivariantly semi-Fano in the sense that $c_1^G(X)$ is non-negative on the class of every affine gauged map. Then for reasons of degree

$$\kappa_{X,G}(QH^{\leq 2}_G(X)) \subset QH^{\leq 2}(X/\!\!/G), \quad \kappa^{(\geq 2)}_{X,G}(QH^{\leq 2}_G(X)) = 0.$$

The constant term in the relation

$$(1+\mathrm{Id}/\zeta++\tau^{(2)}_{X/\!\!/G,\pm}/\zeta^2)\circ(\kappa^{(0)}_{X,G}+\kappa^{(1)}_{X,G}\zeta)+O(\zeta^{-2}) = 1+\tau^{(1)}_{X,G,\pm}/\zeta+O(\zeta^{-2}) \quad \text{on } QH^{\leq 2}(X/\!\!/G)$$

implies that $\kappa^{(1)}_{X,G}$ vanishes on $QH^{\leq 2}_G(X)$. Hence

$$\kappa^{(0)}_{X,G} = \tau^{(1)}_{X,G,\pm} \quad \text{on} \quad QH^{\leq 2}_G(X). \tag{30}$$

This ends the remark.

The limit of the Mundet semistability condition in which the linearization goes to zero is studied in the paper [29]. In this limit, the bundle must be semistable and so the moduli stack of gauged maps is a quotient of the moduli space of parametrized stable maps to X. Theorem 1.1 now follows from Theorems 2.6 and Theorem 3.3.

3.3. **Localization for convex varieties.** A slightly modified version of the quantum Witten localization formula holds in quasiprojective cases under a *convexity* assumption.

Definition 3.5. A finite dimensional complex G-vector space V will be called *convex* if there exists a central one-parameter subgroup $\phi_\lambda : \mathbb{C}^\times \to G$ such that X has positive weights for the induced action of ϕ,

$$V = \bigoplus_\mu V_\mu, \quad (\mu, \lambda) > 0.$$

Given a convex G-vector space, the *projectivization* of V is the quotient

$$\overline{V} = ((V \times \mathbb{C})^\times - \{(0,0)\})/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts on $\mathbb{C}$ with weight one. Thus $\overline{V}$ is a weighted projective space and contains V as an open subset. A quasiprojective G-variety X is *convex* if there exists a projective morphism $\pi : X \to V$ to a convex G-vector space V.

The following is a simple application of the technique called *symplectic cutting* in the literature [46]:

Lemma 3.6. *Any convex G-variety X admits a G-equivariant compactification $\overline{X}$ by adding single $\mathbb{C}^\times_\lambda$-fixed divisor.*

Proof. Let $\tilde{X} \to X$ denote the given linearization on X and $\tilde{X}(k)$ the linearization on $X \times \mathbb{C}$ obtained by twisting by the $\mathbb{C}^\times$-character with weight k. Consider the git quotient

$$\overline{X} = (X \times \mathbb{C}) /\!\!/ \mathbb{C}^\times .$$

The inverse image of $(0,0) \in V \times \mathbb{C}$ is unstable, for sufficiently large d. Thus the proper morphism $X \to V$ induces a proper morphism $\overline{X}$ to $\overline{V}$. In particular, the quotient $\overline{X}$ is also proper. The G action on $X \times \mathbb{C}$ given by $g(x,z) = (gx, z)$ descends to a G-action on $\overline{X}$, and restricts to the given action on the open subset $X \subset \overline{X}$. □

Corollary 3.7. *Let $d \in H^2_G(\overline{X})$ be a class that pairs trivially with the divisor class $[\overline{X} - X] \in H_2^G(\overline{X})$. Then for $k \gg 0$ the moduli stack $\overline{\mathcal{M}}_n^G(C, \overline{X}, d)$ consists of maps whose images are disjoint from $(\overline{X} - X)/G$. Similarly, if $\tilde{X}_\pm \to X$ are two different linearizations then for $k \gg 0$ the moduli stack $\overline{\mathcal{M}}_n^G(C, \overline{X}, d)$ consists of maps whose images are disjoint from $(\overline{X} - X)/G$.*

Proof. The intersection number of any curve $u : \mathbb{P}^1 \to \overline{V}$ contained in $\overline{V} - V$ with $\overline{V} - V$ is non-negative. Indeed $\overline{V} - V$ has ample normal bundle in $\overline{V}$ being a prime invariant divisor in a weighted projective space. On the other hand, there are no stable gauged maps $C \to X/G$ with image in $(\overline{V} - V)/G$ for sufficiently large d, Indeed the trivial reduction σ together with the generator λ of the one-parameter subgroup $\mathbb{C}^\times$ has weight $\mu(\sigma, \lambda) \to \infty$ as $d \to \infty$. Combining these observations let $v : \hat{C} \to \overline{V}/G$ be a stable gauged map intersecting $(\overline{V} - V)/G$. Then the intersection number $\# u^{-1}(P(\overline{V} - V)) > 0$ is positive. But the intersection number is equal to the pairing $(d, [\overline{V} - V]) \in \mathbb{Q}$ of $d \in H_2^G(X, \mathbb{Q})$ with $[\overline{V} - V] \in H^2_G(\overline{V}, \mathbb{Q})$. The latter vanishes by assumption, a contradiction. □

The corollary implies that the wall-crossing formula also holds for convex varieties by applying the formula to the compactified variety with compactifying divisor sufficiently far away at infinity. However, the quantum Witten localization formula does not hold because, eventually, the compactifying divisor will make a contribution in the localization formula. The following alternative argument gives a formula similar to that in quantum Witten localization. Let χ be a character of G that is negative on the one-parameter subgroup generated by ξ and $\underline{\mathbb{C}}_\chi$ the corresponding trivial line bundle over X. Consider the piecewise linear path of linearizations $\tilde{X}_\rho \to X$ obtained by shifting by multiples of the character χ:

$$\tilde{X}_\rho = \begin{cases} \tilde{X} \otimes \underline{\mathbb{C}}_\chi^{\rho^{-1}-1} & \rho \leq 1 \\ \tilde{X}^\rho & \rho \geq 1 \end{cases} . \tag{31}$$

Lemma 3.8. *For any homology class $d \in H_2^G(X, \mathbb{Z})$, the moduli stack $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_\rho, d)$ is empty for $\rho \gg 0$.*

Proof. Let $\sigma : C \to P/G$ be the trivial parabolic reduction, and λ the generator of the one-parameter subgroup in the definition of convexity. Given a gauged map

$v : \hat{C} \to X/G$, the associated graded pair $\mathrm{Gr}(P), \mathrm{Gr}(u)$ for (σ, λ) projects to the origin in V/G. The Mundet weight picks up a term $(\rho^{-1} - 1)(\chi, \lambda)$ which goes to infinity as $\rho \to 0$. Hence there are no Mundet-semistable gauged maps with class d, for ρ sufficiently small. □

Theorem 3.9. (Quantum Witten localization for convex varieties) *Let X be a convex G-variety, C a genus zero curve, and suppose that stable=semistable for the G-action on X, for gauged maps with linearization $\tilde{X}$, and for polarized gauged maps for the path $\tilde{X}_\rho$. Then*

$$\tau_{X /\!\!/ G} \circ \kappa_{X,G} = \sum_{[\lambda] \neq 0, \rho} \frac{|W_\lambda|}{|W_{\mathbb{C}\lambda}|} \tau_{X,G,\lambda,\rho}. \tag{32}$$

where the sum is over equivalence classes $[\lambda]$ of unparametrized one-parameter subgroups generated by $\lambda \in \mathfrak{g}$. Similarly for the localized graph potentials from (20)

$$\tau^\infty_{X /\!\!/ G, -} \circ (\kappa_{X,G} \times \kappa_{X /\!\!/ G}) = \sum_{[\lambda] \neq 0, \rho} \frac{|W_\lambda|}{|W_{\mathbb{C}\lambda}|} \tau^\rho_{X,G,\lambda,\pm}.$$

Example 3.10. (Quantum Witten localization for the scalar multiplication on affine space) To explain the notation we use (32) to compute the three-point Gromov-Witten invariants of projective space using quantum Witten localization. Suppose that $G = \mathbb{C}^\times$ acts diagonally on $X = \mathbb{C}^k$ so that

$$X /\!\!/ G = \mathbb{C}^k /\!\!/ \mathbb{C}^\times = \mathbb{P}^{k-1}.$$

We have

$$H_2^G(X, \mathbb{Z}) \cong H_2(X /\!\!/ G) = \mathbb{Z}[\mathbb{P}^1], \quad H^2_G(X, \mathbb{Z}) \cong H^2(X /\!\!/ G) = \mathbb{Z}\omega$$

where ω is the hyperplane class, the image of the equivariant generator $\xi \in H^2_G(X, \mathbb{Z})$ under the Kirwan map. We compute the class $d = 1$ three-point invariants using quantum Witten localization. Let $\beta \in H^6(\overline{\mathcal{M}}_3(C))$ be the fundamental class, whose insertion fixes the positions of the marked points in (14). We identify $QH_G(X) \cong \Lambda_X^G[\xi]$. Consider the three-point invariants with insertions $\xi^a, \xi^b, \xi^c \in QH_G(X) \cong S(\mathfrak{g})^\vee$. Since $c_1^G(X)$ is at least $2k$ on classes $d > 0$, the derivative $D_0\kappa_{X,G}$ of the quantum Kirwan map has no quantum corrections by (27). The image of ξ^a, ξ^b, ξ^c under $D_0\kappa_{X,G}$ is equal to $\omega^a, \omega^b, \omega^c$ respectively. We consider a path $\tilde{X}_\rho$ obtained by shifting by a negative character χ; this means that in the fixed point formula we take the residue with respect to $-\xi$, see [31]. By the formula (32),

$$\begin{aligned} \sum_{d \geq 0} q^d \langle \omega^a, \omega^b, \omega^c \rangle_{0,d} &= \tau^3_{X /\!\!/ G}(\omega^a, \omega^b, \omega^c, \beta) \\ &= -\sum_{\rho, [\lambda]} \tau^3_{X, \tilde{X}_\rho, G, \lambda}(\xi^a, \xi^b, \xi^c, \beta). \end{aligned}$$

There is a unique G-fixed point in X. The G-bundle P with first Chern class $d = 1$ together with the zero section $u \in H^0(C, P \times_G X)$ forms a Mundet semistable map for a unique value of the parameter ρ. For $d = 1$ the index bundle and its Euler class are

$$\operatorname{Ind}(T(X/G)) = H^0(\mathcal{O}(k)^\times \times_{\mathbb{C}^\times} \mathbb{C}^k) \cong \mathbb{C}^{2k}, \quad \epsilon_+(\operatorname{Ind}(T(X/G))) = \xi^{2k}.$$

The unique fixed point contribution

$$\begin{aligned} \tau^3_{X,\tilde{X}_\rho,G,\lambda}(\xi^a,\xi^b,\xi^c,\beta) &= q \operatorname{Resid}_{-\xi} \frac{\xi^{a+b+c}}{\xi^{2k}} \\ &= \begin{cases} q & a+b+c = 2k-1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We obtain

$$\langle \omega^a, \omega^b, \omega^c \rangle_{0,1} = \begin{cases} 1 & a+b+c = 2k-1, \\ 0 & \text{otherwise} \end{cases}$$

as expected.

4. Abelianization for Gromov-Witten invariants

In this section we prove Theorem 1.3 from the introduction. Abelianization is first proved for graph potentials, then deduced for qde solutions. The section ends with the examples of the moduli spaces of odd numbers of points on a projective line and the Grassmannian.

4.1. **Abelianization for graph potentials.**

Proof of Theorems 1.3 and 1.4. We first prove the Theorem in the case stable=semistable for linearized gauged maps. We take as the inductive hypothesis that Theorem 1.3 holds for any group of rank less than $\dim(G)$. We wish to compare the fixed point contributions in the quantum Witten localization formulas

$$\tau_X^G - \tau_{X/\!\!/G} \circ \kappa_{X,G} = \sum_{[\lambda] \neq 0,\rho} \tau_{X,\tilde{X}^\rho,G,\lambda} \tag{33}$$

and

$$\tau_X^{T,\mathfrak{g}/\mathfrak{t}} - \tau^{\mathfrak{g}/\mathfrak{t}}_{X/\!\!/T} \circ \kappa^{\mathfrak{g}/\mathfrak{t}}_{X,T} = \sum_{[\lambda] \neq 0,\rho} \tau^{\mathfrak{g}/\mathfrak{t}}_{X,\tilde{X}^\rho,T,\lambda}. \tag{34}$$

In the version for T, both the traces and quantum Kirwan maps have been twisted by the Euler class of the index of $\mathfrak{g}/\mathfrak{t}$. Now $\tau_X^{T,\mathfrak{g}/\mathfrak{t}}$ resp. τ_X^G is defined by integration over $\overline{\mathcal{M}}_n(C,X)/\!\!/T$ resp. $\overline{\mathcal{M}}_n(C,X)/\!\!/G$. This is essentially the setting considered by Martin [48]. In González-Woodward [29, Chapter 5] we show

$$\tau_X^G = |W|^{-1} \pi_T^G \circ \tau_X^{T,\mathfrak{g}/\mathfrak{t}}$$

either by Martin's argument, if the moduli spaces of stable maps are smooth and the virtual fundamental classes are the usual ones, or by a virtual version of Martin's argument if the moduli spaces of stable maps are only virtually smooth. We note that the virtual non-abelian localization formula used in [29] had a gap in the proof, which was fixed by Halpern-Leistner [36].

We review the argument for abelianization in the small-area limit briefly from [29]. Write χ_G for the Euler characteristic on the stack S/G and χ_T for the Euler characteristic on the stack S/T. By the Weyl character formula

$$\chi_G(F) = |W|^{-1} \chi_T(F \otimes \underline{\mathrm{Alt}}(\mathfrak{g}/\mathfrak{t})).$$

Here $\underline{\mathrm{Alt}}(\mathfrak{g}/\mathfrak{t})$, the trivial sheaf with values in the exterior algebra $\mathrm{Alt}(\mathfrak{g}/\mathfrak{t})$, is the K-theory Euler class given by the trivial bundle with fiber the T-representation $\mathrm{Alt}(\mathfrak{g}/\mathfrak{t})$ with character $\prod_{\alpha \in \mathcal{R}} (1 - t^\alpha)$. For sufficiently positive bundles we have

$$\chi(S /\!\!/ G, F /\!\!/ G) = |W|^{-1} \chi_T(S /\!\!/ T, F \otimes \underline{\mathrm{Alt}}(\mathfrak{g}/\mathfrak{t}) /\!\!/ T) \tag{35}$$

Using virtual Riemann-Roch in Tonita [72] and taking limits as we obtain

$$\int_{S /\!\!/ G} \mathrm{Ch}(F /\!\!/ G) = |W|^{-1} \int_{S /\!\!/ T} \mathrm{Ch}(F /\!\!/ T) \cup \mathrm{Eul}(\mathfrak{g}/\mathfrak{t}))$$

where $\mathrm{Ch}(F) \in H(S)$ is the Chern character and $\mathrm{Eul}(\mathfrak{g}/\mathfrak{t})$ the Euler class in cohomology. This implies the abelianization formula in the small-area limit for Chern characters:

$$\int_{S /\!\!/ G} \kappa_{X,G}(h) = |W|^{-1} \int_{S /\!\!/ T} \kappa_{X,T}(h \cup \mathrm{Eul}(\mathfrak{g}/\mathfrak{t})), \quad \forall h \in \mathbb{H}_G(X).$$

Using abelianization in the small-area limit to prove abelianization it suffices to show abelianization for the right-hand-sides in (33), (34). Each fixed point component X^λ for the G-action corresponds to $|W/W_{\mathbb{C}\lambda}|$ fixed point components $X^{w\lambda}, w \in W/W_{\mathbb{C}\lambda}$ for the T-action. The identity we wish to show is

$$\tau_{X,\tilde{X}^\rho,G,\lambda} = |W_\lambda|^{-1} \pi_T^{G_\lambda} \circ \tau^{\mathfrak{g}/\mathfrak{t}}_{X,\tilde{X}^\rho,T,\lambda}. \tag{36}$$

In the case G_λ is abelian the group W_λ is trivial and so the equality holds automatically. More generally the equation (25) gives

$$\tau_{X,\tilde{X}^t,G,\lambda} = \tau_{X^\lambda, G_\lambda/\mathbb{C}^\times_\lambda} \circ \iota_\lambda$$

and

$$|W_\lambda|^{-1} \tau_{X,\tilde{X}^t,T,\lambda} = |W_\lambda|^{-1} \tau_{X^\lambda, T/\mathbb{C}^\times_\lambda} \circ \iota_\lambda.$$

By the inductive hypothesis,

$$\tau_{X^\lambda, G_\lambda/\mathbb{C}^\times_\lambda} = |W_\lambda|^{-1} \pi_T^{G_\lambda} \circ \tau^{\mathfrak{g}_\lambda/\mathfrak{t}}_{X^\lambda, T/\mathbb{C}^\times_\lambda}.$$

Equation (36) follows. See Guillemin-Kalkman [35, Section 4] for similar arguments involving recursive applications of fixed point formulae. The equality with $\tau^{\mathfrak{g}/\mathfrak{t}}_{X /\!\!/ T} \circ \kappa^{\mathfrak{g}/\mathfrak{t}}_{X,T} \circ \mathrm{r}^G_T$ follows from by combining wall-crossing with Theorem 3.3. Theorem 1.4 is proved by the same argument applied to the $\mathbb{C}^\times$-fixed locus in $\overline{\mathcal{M}}^G(C, X, \tilde{X}_-, \tilde{X}_+)$, where $\mathbb{C}^\times$ acts on $C \cong P^1$ by the action with weights $-1, 1$.

To prove the Theorems 1.3 and 1.4 in the general case when semistable $\neq$ stable, we find a master space for which stable=semistable by adding a parabolic structure as in Proposition 2.8. Given stability parameters $\rho_\pm$ and corresponding linearizations $\tilde{X}_\pm = \tilde{X}^{\rho_\pm}$, let $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$ of polarized gauged maps with parabolic structure. As in Proposition 2.3, for generic μ stable=semistable for $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$. So $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$ is also a proper Deligne-Mumford stack with perfect relative obstruction theory. Localization on the stack $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$ produces a wall-crossing formula, whose fixed point contributions are the $\mathbb{C}^\times$-fixed components in $\overline{\mathcal{M}}_n^G(C, X, \tilde{X}_-, \tilde{X}_+, \nu)$. By induction on the rank of G, we obtain the identity

$$\begin{aligned}(37)\quad \int_{[\overline{\mathcal{M}}_n^G(C,X,\tilde{X}^\rho,\nu,d)]} \operatorname{ev}^* h \cup \pi^* \operatorname{Eul}(L_\pi) \\ = \int_{[\overline{\mathcal{M}}_n^T(C,X,\tilde{X}^\rho,\nu,d)]} \operatorname{ev}^* h \cup \epsilon(\mathfrak{g}/\mathfrak{t}) \cup \pi^* \operatorname{Eul}(L_\pi)/|W|\end{aligned}$$

for any ρ for which stable=semistable for gauged maps. Hence using (18),

$$\int_{[\overline{\mathcal{M}}_n^G(C,X,\tilde{X}^\rho,d)]} \operatorname{ev}^* h = \int_{[\overline{\mathcal{M}}_n^T(C,X,\tilde{X}^\rho,d)]} \operatorname{ev}^* h \cup \epsilon(\mathfrak{g}/\mathfrak{t})/|W|$$

as claimed. □

4.2. **Abelianization for convex varieties.** Continuing Theorem 3.9, we extend the abelianization results to convex G-varieties. Replacing the quantum Witten localization formula (4) by the alternate formula (32) obtained from wall-crossing by shifting by a character, the same arguments go through and imply the formulas 1.3 and 1.4. This also gives alternative argument for abelianization for git quotients $X/\!\!/G$ of projective X by G in the case that G has a non-trivial center, but not in the case that G is simple and non-abelian. This allows us, finally, to give some applications.

Example 4.1. (Grassmannians) In this example we reproduce the results on Grassmannians from Bertram et al [13]. For positive integers $r < k$ let $X = \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^k)$ be the space of linear maps from $\mathbb{C}^r$ to $\mathbb{C}^k$. Let $G = GL(r)$ act on X by composition. For the linearization $\lambda = c_1^G(X) \in H_G^2(X) \cong \mathbb{Q}$, the semistable maps are those with full rank and so

$$\begin{aligned} X/\!\!/G &= \{x \in \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^k) |\operatorname{rank}(x) = k\}/GL(r) \\ &= \operatorname{Gr}(r, k)\end{aligned}$$

is the Grassmannian of r-dimensional subspaces in $\mathbb{C}^k$ with $H_2^G(X, \mathbb{Z}) \cong \mathbb{Z}$. There exist stable maps to $X/\!\!/G$ of class d only if $d \geq 0$. Although X is not compact, the weights for the central $\mathbb{C}^\times$ action are all one, and so the git quotients are proper. Since X is equivariantly Fano, the map $\kappa_{X,G}$ is trivial on $H_G^2(X)$ and may be ignored by (27). By abelianization the localized graph potential $\tau_{X/\!/G,\pm}$ for the Grassmannian has restriction to $H^2(X/\!/G)$ given by

$$\tau_{X/\!/G,\pm} = \mu_T^G \circ \tau_{X,T,\pm} \circ \mathrm{r}_T^G \tag{38}$$

Let $\theta_1, \ldots, \theta_r$ denote the standard basis of $H_T^2(X,\mathbb{Z}) \cong \mathbb{Z}^r$. The localized gauged potential $\tau_{X,T,\pm}$ is Givental's I-function and by [28] given by

(39)
$$\tau_{X,T,\pm}^{\mathfrak{g}/\mathfrak{t}}(t_0 + t_1\theta_1 + \ldots + t_r\theta_r) = \sum_{\underline{d}\geq 0} q^{\underline{d}} e^{t_0 + (t_1(\theta_1 + d_1\zeta) + \ldots + t_{2k+2}(\theta_{2k+2} + d_{2k+2}\zeta))/\zeta} \tau_{X,T,\pm}^{\mathfrak{g}/\mathfrak{t}}(\underline{d})$$

where

$$\begin{aligned} \tau_{X,T,\pm}^{\mathfrak{g}/\mathfrak{t}}(\underline{d}) &= \frac{(-1)^{(k-1)d}\prod_{i<j}((\theta_i - \theta_j) + (d_i - d_j)\zeta)}{\prod_{i<j}(\theta_i - \theta_j)\prod_{i=1}^r \prod_{l=1}^{d_i}(\theta_i + l\zeta)^n} \\ &= \prod_{i\neq j} \frac{\prod_{l\leq d_i - d_j}((\theta_i - \theta_j) + l\zeta)}{\prod_{l\leq 0}((\theta_i - \theta_j) + l\zeta)} \frac{1}{\prod_{i=1}^r \prod_{l=1}^{d_i}(\theta_i + l\zeta)^n}. \end{aligned} \tag{40}$$

The formula obtained by combining (38), (39), (40) was conjectured in Hori-Vafa [40, Appendix]. The formula was proved in Bertram et al [13] , [12] by different methods.

4.3. **Abelianization for quantum products.** In this section we prove the Theorem 1.6 relating the quantum products for the abelian and non-abelian products, using the abelianization formula for qde solutions from the previous section. As an example, we describe the small quantum cohomology ring of the Grassmannian following Schmäshcke [68].

We introduce the following differential operator: For any root α let

$$\alpha_0 = \kappa_{X,T}^{q=0}(\alpha) = c_1(\mathbb{C}_\alpha /\!/ T) \in H^2(X/\!/T)$$

denote the corresponding class on the symplectic quotient. Define

$$\mathcal{D}^{\mathfrak{g}/\mathfrak{t}} = \prod_{\alpha\in\mathcal{R}_+} \frac{\partial_\alpha}{\alpha_0} : \mathrm{End}(\mathrm{Map}(H_T(X), H_{\mathbb{C}^\times\times\mathbb{C}^\times}(X/\!/T)));$$

here the invertibility of α_0 may be achieved by adding an additional equivariant parameter, as in the definition of inverted Euler classes. In the equivariant cohomology $H_{\mathbb{C}^\times\times\mathbb{C}^\times}(X/\!/T)$ the first factor corresponds to the rotation action on $\mathbb{C}^\times$, while the second factor is that used to define the Euler-twisted potential in (16).

Proposition 4.2. *The $\mathfrak{g}/\mathfrak{t}$-twisted potential $\tau_{X,T,\pm}^{\mathfrak{g}/\mathfrak{t}}$ is related to the untwisted potential by*

$$\tau_{X,T,\pm}^{\mathfrak{g}/\mathfrak{t}} = \mathcal{D}^{\mathfrak{g}/\mathfrak{t}} \tau_{X,T,\pm} : QH_T(X) \to QH_{\mathbb{C}^\times\times\mathbb{C}^\times}(X/\!/T).$$

Proof. The addition of $\epsilon(g\mathfrak{t}) = \mathrm{Eul}(\mathrm{Ind}(\mathfrak{g}/\mathfrak{t}))$ into the definition of $\tau_{X,T,\pm}$ causes the appearance of the additional factors involving α_0:

$$\begin{aligned}
\tau^{\mathfrak{g}/\mathfrak{t}}_{X,T,\pm} &= \sum_{d\in H_2^T(X)} q^d \prod_{\alpha>0} \Delta_d(\alpha_0,0)\Delta_d(-\alpha_0,0)\tau_{X,T,\pm,d} \\
&= \sum_{d\in H_2^T(X)} q^d \prod_{\alpha>0} \frac{\prod_{l=-\infty}^{\alpha\cdot d}(\alpha_0+l\zeta)}{\prod_{l=-\infty}^{0}(\alpha_0+l\zeta)} \frac{\prod_{l=-\infty}^{-\alpha\cdot d}(\alpha_0+l\zeta)}{\prod_{l=-\infty}^{0}(-\alpha_0+l\zeta)} \tau_{X,T,\pm,d} \\
&= \sum_{d\in H_2^T(X)} q^d \prod_{\alpha>0} \frac{\alpha_0+(\alpha_0\cdot d)\zeta}{\alpha_0}\tau_{X,T,\pm,d} \\
&= \sum_{d\in H_2^T(X)} q^d \prod_{\alpha\in\mathcal{R}_+} \frac{\partial_\alpha}{\alpha_0}\tau_{X,T,\pm,d} = \mathcal{D}^{\mathfrak{g}/\mathfrak{t}}\tau_{X,T,\pm}
\end{aligned}$$

as claimed. □

Proof of Theorem 1.6. Let $\Box$ be a W-invariant constant-coefficient differential operator on $QH_2^T(X)$ with symbol $\sigma(\Box)$. Since $\tau_{X,G,\pm}$ is a fundamental solution,

$$\begin{aligned}
(\Box\tau_{X,G,\pm})(h) = 0 \quad &\iff \quad (\Box\tau^{\mathfrak{g}/\mathfrak{t}}_{X,T,\pm})(h) = 0 \\
&\iff \quad D_h\kappa_{X,G}(\sigma(\Box)) = 0 \in T_{\kappa_{X,G}(h)}QH(X/\!\!/G).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Box\tau^{\mathfrak{g}/\mathfrak{t}}_{X,T,\pm} = 0 \quad &\iff \quad \Box\mathcal{D}^{\mathfrak{g}/\mathfrak{t}}\tau_{X,T,\pm} = 0 \\
&\iff \quad D_h\kappa_{X,T}(\sigma(\Box\mathcal{D}^{\mathfrak{g}/\mathfrak{t}})) = 0 \\
&\iff \quad D_h\kappa_{X,T}(\sigma(\Box))D_h\kappa_{X,T}(\sigma(\mathcal{D}^{\mathfrak{g}/\mathfrak{t}})) = 0 \\
&\iff \quad D_h\kappa_{X,T}(\sigma(\Box)) \in \operatorname{ann} D_h\kappa_{X,T}(e_\pm)
\end{aligned}$$

as claimed. □

Example 4.3. (Grassmannians) The relations in $QH((\mathbb{P}^{n-1})^k)$ are $H_i^n = q, i = 1,\ldots,k$, where H_i is the hyperplane class on the i-th factor. The Weyl group $W = S_k$ is the k-symmetric group. The W-invariant part of the cohomology ring $QH((\mathbb{P}^{n-1})^k)$ is generated by the Schur polynomials

$$\chi_{\lambda^\vee}(H_1,\ldots,H_n) = \prod_{w\in W} \frac{(-1)^{l(w)}H^{w(\lambda+\rho)-\rho}}{\prod_{i<j}(H_j-H_i)}$$

where $H^\lambda = H_1^{\lambda_1}\ldots H_k^{\lambda_k}, \quad \rho = (1,\ldots,k)$. By the Fano condition $\kappa_{X,T}$ has no quantum corrections and so

$$D_0\kappa_{X,T}(e_\pm) = \pm\prod_{i<j}(H_j-H_i).$$

Hence for any $\mu\in\mathbb{Z}^k$, we have

$$\chi_{\lambda+n\mu}(H_1,\ldots,H_n) - \chi_{\lambda^\vee}(H_1,\ldots,H_n) \in \operatorname{ann}(\kappa_{X,T}(e_\pm)) \subset H(X/\!\!/T).$$

This implies the relations

$$\chi_{\lambda+n\mu} = q^{\mu_1+\ldots+\mu_k}\chi_{\lambda^\vee} \in QH(\mathrm{Gr}(k,n)), \quad \lambda,\mu \in \mathbb{Z}^k.$$

The leading order terms in these relations are

$$\chi_{\lambda+n\mu} = 0, \quad \mu \geq 0.$$

These are the usual relations in the cohomology of the Grassmannian describing the cohomology as a truncation of the polynomial representation ring of $GL(k)$ obtained by setting the Schur polynomials corresponding to Young diagrams not fitting in the $k \times (n-k)$ box to zero:

$$H(G(k,n)) = \mathrm{Rep}(GL(k))/\langle \chi_{\lambda+n\mu}, \quad \mu > 0\rangle.$$

This implies that the relations above generate the ideal of relations in the small quantum cohomology ring. So one obtains the standard presentation of the quantum cohomology as the Verlinde algebra as in Bertram-Ciocan-Fontanine-Fulton [11]:

$$QH(G(k,n)) = \mathrm{Rep}(GL(k))/(\chi_{\lambda+n\mu} - q^{\mu_1+\ldots+\mu_k}\chi_\lambda, \quad \lambda,\mu \in \mathbb{Z}^k_{\geq 0}).$$

5. Quantum Lefschetz for holomorphic symplectic quotients

In this section we prove a formula for the qde solution for hypersurfaces for bundles associated to the semistable locus, including hypersurfaces defined by the zero level sets of holomorphic moment maps. in which the standard techniques introduced for complete intersections defined by concavex bundles break down. Suppose that X is a smooth variety equipped with the action and an equivariant map

$$G \times X \to X, \quad \Phi : X \to V.$$

In our examples Φ will be a holomorphic moment map. Let $\tilde{X} \to X$ be a linearization, that is, an ample G-line bundle. We denote by $Z = \Phi^{-1}(0)$ the zero level set. In this section we give a formula for the graph potential of the git quotient $Z/\!\!/G$. Over X we have a natural bundle $X \times V \to X$; we wish to compare the potentials

$$\tau_Z \circ r_{Z,G} : QH_G(X) \to \Lambda_X^G, \quad \tau_X^V : QH_G(X) \to \Lambda_X^G$$

where τ_X^V denotes the potential twisted by the index of the Euler class of V and $r_{Z,G}$ as before pull-back.

Proof of Theorem 1.8. We apply the wall-crossing formula in Theorem 2.6 to both sides: Consider the family of linearizations $\tilde{X} \otimes \mathbb{C}_{\rho t}$, where ρ is the weight corresponding to the one-parameter subgroup. The wall-crossing formulas give a relationship between the potentials for $\tilde{X}_\pm = \tilde{X}_{t_\pm}$ at $t = t_\pm$

$$\tau^V_{X,G,1} - \tau^V_{X,G,-1} = \sum_{[\lambda], t\in(-1,1)} \frac{|W_\lambda|}{|W_{\mathbb{C}\lambda}|} \mathrm{Resid}_\xi\, \tau^V_{X,G,\lambda,t} \tag{41}$$

$$\tau_{Z,G,1} - \tau_{Z,G,-1} = \sum_{[\lambda], t\in(-1,1)} \frac{|W_\lambda|}{|W_{\mathbb{C}\lambda}|} \mathrm{Resid}_\xi\, \tau_{Z,G,\lambda,t} \tag{42}$$

As in Proposition 2.8, at the cost of adding a parabolic structure we may assume stable=semistable for linearized gauged maps. By induction on the dimension of the group and (25) we may assume

$$\tau_{Z,G,\lambda,t} = \tau^V_{X,G,\lambda,t}, \quad \forall t \in (-1,1). \tag{43}$$

In the small-area chamber the equality $\tau_Z^G = \tau_X^{G,V}$ holds by standard properties of the Euler class: since V is a G-representation its index bundle

$$\mathrm{Ind}(V)_{[u:C\to X]} := H^0(u^*TV) \cong V.$$

The locus $\overline{\mathcal{M}}_n(C,Z,d)$ is the zero set of the map $\overline{\mathcal{M}}_n(C,X,d) \to V$ defined by $[u : C \to X] \mapsto u^*\Psi$. Hence

$$\int_{\overline{\mathcal{M}}_n(C,X,d)/\!\!/G} \mathrm{ev}^* h \cup \mathrm{Eul}(V) = \int_{\overline{\mathcal{M}}_n(C,Z,d)/\!\!/G} \mathrm{ev}^* r_{Z,G} h.$$

In the convex case, for sufficiently large linearization the moduli space is empty; hence the result follows from the equality of the wall-crossing terms (44) and the formula (41), (42). The proof for the localized graph potentials is similar. □

We apply Theorem 1.8 to give a formula for a twisted localized graph potential of moduli spaces of framed sheaves on the projective plane. Note that these results were already announced by Ciocan-Fontanine-Diaconescu-Kim-Maulik, see [45]. The quantum cohomology of these moduli spaces is also the subject of work by Maulik-Okounkov [47]. We recall the construction of the moduli space of framed sheaves from Nakajima's lectures [55, Chapter 3]. Recall the notation from (11). The data (B_-, B_+, i_-, i_+) forms a representation of the ADHM quiver. The group $S = (\mathbb{C}^\times)^2$ acts equivariantly on

$$X := \mathrm{End}(\mathbb{C}^k)^{\oplus 2} \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^k) \oplus \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^r)$$

by

$$(s_-, s_+)(B_-, B_+, i_-, i_+) = (s_- B_-, s_+ B_+, i_-, s_- s_+ i_+).$$

This preserves the locus

$$Z_\lambda = \{(B_-, B_+, i_-, i_+) \,|\, [B_-, B_+] + i_- i_+ = \lambda \, \mathrm{Id}\} \subset X$$

and induces an S-action on the quotient $\mathcal{M}$. In the case $k = 1$, $\mathcal{M}$ is a deformation of the Hilbert scheme $\mathrm{Hilb}_k(\mathbb{C}^2)$ and the S-action is the one induced from the S-action on $\mathbb{C}^2$. For any character $\chi \in \mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathbb{Z}$, let $\tilde{\mathcal{M}}$ denote Nakajima's desingularization [55] of $\mathcal{M}$ given by

$$\tilde{\mathcal{M}} = Z_\lambda /\!\!/_\chi G \subset X /\!\!/_\chi G$$

where $/\!\!/_\chi$ denotes the χ-shifted geometric invariant theory quotient. Let $T \subset G$ denote the diagonal maximal torus. We consider the twisted Gromov-Witten theory of $X/\!\!/_\chi G$ corresponding to the relation defining Z_λ, that is, twisted by the Euler class of the index bundle of $\mathrm{End}(\mathbb{C}^k)$. Although $X/\!\!/_\chi G$ is non-compact, the fixed points of the S-action are compact:

Lemma 5.1. (Properness of S-fixed loci)

(a) *The S-action on the moduli of parametrized stable maps $\overline{\mathcal{M}}_n(C, X/\!\!/_\chi G, d)$ has proper S-fixed loci for any homology class $d \in H_2^G(X)$.*

(b) *The S-action on the moduli space of gauged maps $\overline{\mathcal{M}}_n^G(C, X, d)$ has proper S-fixed loci for any homology class $d \in H_2^G(X)$.*

(c) *The S-action on the moduli space of scaled gauged maps $\overline{\mathcal{M}}_{n,1}^G(C, X, d)$ from [77] has proper S-fixed loci for any homology class $d \in H_2^{GL_n}(X)$.*

Proof. (a) By the theory of symplectic resolutions discussed in [26], any stable map defines a parametrized stable map to the corresponding affine quotient $X/\!\!/G$ by composition with the proper morphism $X/\!\!/_\chi G \to X/\!\!/G$. The latter is affine with compact S-fixed loci, hence any S-fixed stable map in $X/\!\!/_\chi G$ projects to an S-fixed point in $X/\!\!/G$. Since the inverse image is proper, the claim follows. (b) is Proposition 3.5 in Diaconescu [22]. However for the purposes of proving (c) we give a different proof. Let (P, u) be an object of the fixed point substack $\overline{\mathcal{M}}^G(C, X, d)^S$. Thus $P \to C$ is a G-bundle, $u : C \to P \times_G X$ is a section, and there exists a homomorphism

$$\varphi : S \to \mathrm{Aut}(P)$$

such that $su = \varphi(s)u$. After trivializing φ at a base point φ defines a homomorphism still denoted φ from S to G. Let G_φ denote the centralizer of φ. Then P admits a reduction of structure group $P_\varphi \subset P$ to $G_\varphi \subset G$. Each $\varphi(s)$ defines an automorphism of the associated fiber bundle $P(X)$, so that u takes values in the fixed point locus $P(X)^\varphi = P(X^\varphi)$. The fixed point locus of S on X defined by the homomorphism φ is

$$X^\varphi = \left\{ (B_-, B_+, i_-, i_+) \;\middle|\; \begin{array}{l} \mathrm{Ad}(\varphi(s_-, s_+))B_\pm = s_\pm B_\pm, \\ \varphi(s_-, s_+)i_- = i_-, \\ i_+\varphi(s_-, s_+)^{-1} = s_+ s_- i_+ \end{array} \right\}.$$

Under the action of $S \times \mathbb{C}^\times$ (where $\mathbb{C}^\times \subset G$ is the subgroup of diagonal matrices) the subspace X^φ splits into a sum of subspaces with weights $(1,0,0), (0,1,0), (0,0,1), (1,1,-1)$. This shows the existence of a central abelian three-parameter subgroup whose action on X^φ has weights contained in an open half-space. It follows that X^φ is convex (take for example the one-parameter subgroup of $S \times \mathbb{C}^\times$ generated by $(1,1,1)$). So $\overline{\mathcal{M}}_n^{G_\varphi}(C, X^\varphi, d)$ is compact for any class $d \in H_2^{G_\varphi}(X)$. Since any fixed point component arises in this way, $\overline{\mathcal{M}}_n^G(C, X, d)^S$ is compact. The argument for (c) is similar, using that any continuous family of S-fixed vortices with varying vortex parameter takes values in X^φ for any homomorphism φ. $\square$

By the properness results in Lemma 5.1, the abelianization argument for gauged potentials defined via localization at fixed point loci of the S-action implies

$$\tau_{Z^\varphi, G_\varphi, \lambda, t} = \tau^V_{X, G_\varphi, \lambda, t}, \quad \forall t \in (-1, 1). \tag{44}$$

Hence

$$\tau_{Z/\!\!/G,-} \circ \kappa_{Z,G} = \pi_T^G \tau^{V/\!\!/T, \mathfrak{g}/\mathfrak{t}}_{X,T,-} \circ \kappa^{V,\mathfrak{g}/\mathfrak{t}}_{X,T}.$$

This formula can be made explicit as follows. For any n-tuple of non-negative integers $\underline{d} = (d_1, \ldots, d_k)$, $\theta = \sum c_i \theta_i$ with $c_i \in \mathbb{Z}$, define

$$\Delta_{\underline{d}}(\theta, w) := \frac{\prod_{l=-\infty}^{\theta \cdot \underline{d}} (\theta + w + l\zeta)}{\prod_{l=-\infty}^{0} (\theta + w + l\zeta)}.$$

The twisted localized gauged potential for the T action on X has restriction to $QH_T^{\leq 2}(X)$ given by (cf. [45])

$$\pi_T^G \tau_{X,T,-}^{V,\mathfrak{g}/\mathfrak{t}} = \sum_{d \geq 1} q^d \sum_{\underline{d}: d_1 + \ldots + d_k = d} e^{t_0 + (t_1(\theta_1 + d_1\zeta) + \ldots + t_k(\theta_k + d_k\zeta))/\zeta} \tau_{X,T,\pm}(\underline{d}) \tag{45}$$

where

$$\tau_{X,T,\pm}^{V,\mathfrak{g}/\mathfrak{t}}(\underline{d}) = \prod_{i \neq j} \frac{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_- + \xi_+)\Delta_{\underline{d}}(\theta_i - \theta_j, 0)}{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_-)\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_+)} \prod_{i=1}^{k} \frac{1}{\Delta_{\underline{d}}(\theta_i, 0)^r \Delta_{\underline{d}}(-\theta_i, \xi_- + \xi_+)^r}$$

and ξ_-, ξ_+ are the equivariant parameters for S, that is, $H_S(\mathrm{pt}) = \mathbb{Q}[\xi_-, \xi_+]$. The quantity $\tau_{X,T,\pm}^{(1)}$ is studied in Konvalinka-Ciocan-Fontanine-Pak [45]. The product

$$\prod_{i=1}^{k} \frac{1}{\Delta_{\underline{d}}(\theta_i, 0)^r \Delta_{\underline{d}}(-\theta_i, \xi_- + \xi_+)^r}$$

has leading order term ζ^{-1} only if a single d_i is non-zero and $r = 1$. Furthermore, in the case $r = 1$ the constant term in the first product is

$$\prod_{i \neq j} \frac{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_- + \xi_+)\Delta_{\underline{d}}(\theta_i - \theta_j, 0)}{\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_-)\Delta_{\underline{d}}(\theta_i - \theta_j, \xi_+)} = \prod_{j \neq i} \frac{(\theta_i - \theta_j - \xi_-)(\theta_i - \theta_j - \xi_+)}{(\theta_i - \theta_j)(\theta_i - \theta_j - (\xi_- + \xi_+))}$$

It follows that for $r = 1$

(46)

$$\begin{aligned} \tau_{X,T,\pm}^{(1)}|_{t=0} = \sum_{d \geq 0} \frac{(-1)^d}{d} q^d \sum_{i=1}^{k} (\theta_i - (\xi_- + \xi_+)) \prod_{j \neq i} \frac{(\theta_i - \theta_j - \xi_-)(\theta_i - \theta_j - \xi_+)}{(\theta_i - \theta_j)(\theta_i - \theta_j - (\xi_- + \xi_+))} \\ \ln(1+q) \sum_{i=1}^{k} (\theta_i - (\xi_- + \xi_+)) \prod_{j \neq i} \frac{(\theta_i - \theta_j - \xi_-)(\theta_i - \theta_j - \xi_+)}{(\theta_i - \theta_j)(\theta_i - \theta_j - (\xi_- + \xi_+))}. \end{aligned}$$

In Konvalinka-Ciocan-Fontanine-Pak [45] this quantity is equated with

$$\exp(-\tau_{X,T,\pm}^{V,\mathfrak{g}/\mathfrak{t},(1)}|_{t=0}) = (1+q)^{k(\xi_- + \xi_+)}$$

via combinatorics of Young diagrams. It follows by (30) that

$$\exp(\kappa_{\hat{Z},G}|_{t=0}/\zeta) = \exp(\tau_{X,T,\pm}^{V,\mathfrak{g}/\mathfrak{t},(1)}|_{t=0}/\zeta).$$

For the classes $h = t_1 h_1 + \ldots t_n h_n \in H_T^2(X)$ the value if the potential is determined by the divisor equation, with the result

$$\kappa_{\hat{Z},G} = \kappa_{\hat{Z},G}|_{t=0}(q \mapsto qe^t).$$

Since the classes involved in $\tau_{X,T,\pm}^{V,\mathfrak{g}/\mathfrak{t}}$ pair trivially with the degrees of gauged maps, the exponential in (45) distributes out of the sum so that

$$\tau_{Z/\!\!/G,\pm} = \mu_T^G \exp(-\tau^{(1)}_{X,T,\pm}/\zeta)\tau_{X,T,\pm}.$$

That is, letting δ_{r-1} denote the Kronecker delta function at $r = 1$, the twisted localized graph potential of the smoothed moduli space of framed sheaves $Z/\!\!/G$ restricted to $QH^{\leq 2}(X/\!\!/G) \cong QH^{\leq 2}_G(X)$ is

$$\tau_{Z/\!\!/G,-} \circ r_{Z,G} = \mu_{Z/\!\!/G}^{X/\!\!/T}(1+qe^t)^{k\delta_{r-1}(\xi_- + \xi_+)/\zeta}\tau^{V,\mathfrak{g}/\mathfrak{t}}_{X,T,-}.$$

This completes the proof of Theorem 1.10.

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Department of Mathematics University of Massachusetts Boston 100 William T. Morrissey Boulevard Boston, MA 02125

E-mail address: eduardo@math.umb.edu

Mathematics-Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, U.S.A.

E-mail address: ctw@math.rutgers.edu