Quantum non-abelian localization: Conjectures and partial results

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joint with E. Gonzalez

Question: how are equivariant Gromov-Witten invariants of a Hamiltonian $G$-manifold related to those of its symplectic quotient?

thanks to C. Teleman
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\( \Phi : X \to \mathfrak{g}^* \) moment map

\( X/\!\!/G = \Phi^{-1}(0)/G \) symplectic quotient
$QH(X)$ counts holomorphic maps $u : \mathbb{P}^1 \to X$
Equivariant quantum cohomology

\( QH(X) \) counts holomorphic maps \( u : \mathbb{P}^1 \rightarrow X \)

\( QH_G(X)^{\text{Givental}} \) counts holomorphic maps \( u : \mathbb{P}^1 \rightarrow X \) equivariantly

\( QH_G(X) \) should count holomorphic maps 
\( u : \mathbb{P}^1 \rightarrow X_G := X \times_{G_{\mathbb{C}}} EG_{\mathbb{C}} \)
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i.e. maps $v : \mathbb{P}^1 \to B G_C$ plus lifts to $X_G$
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\( u : \mathbb{P}^1 \rightarrow P \times_{G_C} X \)

i.e. triples \((P, A, u)\) of principal \( G \)-bundle \( P \), connection \( A \),
and holomorphic \( u : \mathbb{P}^1 \rightarrow P \times_{G} X \)
Gauged holomorphic maps

Algebraic approach still on $X$ a point (moduli of bundles on a nodal curve? See e.g. Frenkel-Teleman to appear)
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$$\mathcal{A}(P, X) = \{(A, u), \bar{\partial}_A u = 0\}$$ space of gauged hol. maps
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$$\mathcal{A}(P, X) \rightarrow \Omega^2(\mathbb{P}^1, P(g)), (A, u) \mapsto F_A + \text{Vol } u^* \Phi$$
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moment map $\mathcal{A}(P, X) \to \Omega^2(\mathbb{P}^1, P(\mathfrak{g})), (A, u) \mapsto F_A + \text{Vol } u^* \Phi$

Depends on two-form $\text{Vol} \in \Omega^2(\mathbb{P}^1)$ and inner product $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. 
Symplectic vortices

\[ M(P, X) := \mathcal{A}(P, X) \sslash G(P) = \{ F_A + \epsilon^{-1} \text{Vol} \ u^* \Phi = 0 \} / G(P) \]
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Compactification \( \overline{M}(P, X) \) obtained by allowing sphere bubbles in fibers of \( P(X) \)

Question: how does \( \overline{M}(P, X) \) depend on \( \epsilon \)?
Finite Dimensional Analogy

\[ X_0, X_1 \text{ Hamiltonian } G\text{-manifolds} \]
$X_0, X_1$ Hamiltonian $G$-manifolds

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$X/G$ is a $X_0$ bundle over $X_1/G$ for $\epsilon$ small

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In between have wall-crossing formulas, for example Kalkman
Infinite volume limit

studied by Gaio-Salamon
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$\overline{M}(P, X)_0$ moduli space of 0-vortices
Infinite volume limit

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Conjecture: $\overline{M}(P, X)_0$ compactifies $\overline{M}(P, X)_\epsilon$ as $\epsilon \to 0$
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Very similar results by Ziltener, who conjectures a quantum Kirwan map

$$Q\kappa_G : QH_G(X) \to QH(X/\!/G)$$

counting stable vortices on $\mathbb{C}$. 
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\( \overline{M}(X)_{\infty} \) moduli space of stable \( \infty \)-vortices
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$G$ acts on $\overline{M}_{0,3}(X)$ and $\overline{M}(X)_{\infty}$ is the pre-symplectic quotient

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Theorem: $\overline{M}(X)_{\infty}$ is a compactification of $\overline{M}(P, X)_\epsilon$, $\epsilon \to \infty$. 
Say \((A_\alpha, u_\alpha)\) is a sequence of \(\epsilon_\alpha\) vortices with \(\epsilon_\alpha \to \infty\).
Proof of compactness

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\( \mathbb{P}^1 \) simply connected, Uhlenbeck compactness implies \( \| A_\alpha \|_{L^1} < C\varepsilon_\alpha^{-1} \) after gauge.
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Vortex equation implies \(\epsilon_\alpha (dA_\alpha + A_\alpha \wedge A_\alpha) = u_\alpha^* \Phi \text{Vol}_{\mathbb{P}^1}\)
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Integrate to get \(\int_{\mathbb{P}^1} u_\alpha^* \Phi \text{ Vol}_{\mathbb{P}^1} \to 0\)
Definition of vortex invariants

\[ \overline{N}(P, X) \] moduli space of vortices with framings at 
\[ z_1, \ldots, z_n \in \mathbb{P}^1 \]
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\[ \overline{N}(P, X) \to X^n \] evaluation maps
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Compose

\[ H_G(X)^n \to H_G^n(\overline{N}(P, X)) \to H(\overline{M}(P, X)) \to \mathbb{Q} \] to get vortex invariants
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Foundations somewhat unelegant, unsatisfactory
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Conjectures

Conj 1: For $\epsilon \gg 0$, $\alpha \in H^*_G(X)^n$, $\langle \alpha \rangle_\epsilon = \langle \alpha \rangle^G_X$, the “invariant part” of Givental’s invariants.
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Conj 2: For $\epsilon^{-1} \gg 0$, $\langle \alpha \rangle_\epsilon = \langle Q\kappa_G(\alpha) \rangle_{X/G}$ where $Q\kappa_G$ is Ziltener’s (conjectural) quantum Kirwan map
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Conj 3: In between, have wall-crossing

\[
\sum_{\epsilon \in (\epsilon_0, \epsilon_1), \psi} \frac{\#W_\psi}{\#W \#W} \langle \alpha \rangle_{X, (g/g_\psi) \mathbb{C}, TX/TX_\psi, \epsilon}
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Conj 3: In between, have wall-crossing $\langle \alpha \rangle_{\epsilon_1} - \langle \alpha \rangle_{\epsilon_0} = \sum_{\epsilon \in (\epsilon_0, \epsilon_1), \psi \in g} (\#W_\psi / \#W) \langle \alpha \rangle^X_{X, (g/g_\psi)^C, TX/TX^\psi}, \epsilon$.

Wall-crossing terms count vortices in $X^\psi$, twisted by Euler classes of index bundles $(g/g_\psi)^C$ and $TX/TX^\psi$, and allowing sphere bubbling in $X$. 
Quantum Non-Abelian localization conjecture

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Not really Quantum, Non-Ab, or Local :)

\[ \langle \alpha \rangle^G_X \]
Quantum Non-Abelian localization conjecture

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**Theorem-in-progress:** \[ \langle \alpha \rangle_{X, \epsilon} = \langle \alpha \rangle^G_X + \sum_{\epsilon' > \epsilon} \text{wall-crossing terms for any } \epsilon > 0 \]
\[ \left\langle Q \kappa_G(\alpha) \right\rangle_{X/G} = \left\langle \alpha \right\rangle^G_X + \sum_{\epsilon \in (0, \infty)} \text{wall-crossing terms}. \]

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In other words, we believe we have proved Conjectures 1 and 3.
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Hori-Vafa: conjectured relation between GW invariants of Grassmannians and products of projective spaces

Bertram et al: prove Hori-Vafa and conjectured

\[ \langle \kappa_G(\alpha) \rangle_{X/G,d_G} = (\#W)^{-1} \sum_{d_T \mapsto d_G} \langle \kappa_T(\alpha') \rangle_{X/T,(g/t)_{\mathbb{C}}} \cdot \]
quantum Martin conjecture

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Theorem-in-progress: version of quantum Martin for any $\epsilon > 0$
quantum Martin conjecture: partial results

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Proof: Induction on $\dim(G)$ (for a slight generalization of the formula), Martin’s argument for $\epsilon = \infty$, and method of continuity
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Quantum non-abelian localization in general would imply genus zero quantum Martin with Kirwan replaced by quantum Kirwan