1. Introduction

Here is a simple example of a system of linear equations. Suppose that recipes for regular and light ice cream call for the following amounts of cream and sugar:

<table>
<thead>
<tr>
<th>Type</th>
<th>Cream (cups)</th>
<th>Sugar (cups)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>2</td>
<td>( \frac{5}{4} )</td>
</tr>
<tr>
<td>Light</td>
<td>1</td>
<td>( \frac{3}{4} )</td>
</tr>
</tbody>
</table>

Suppose we have 10 cups cream and 6 cups sugar. How many pints of regular and light can we make without waste?

This problem can be solved both algebraically and geometrically. Let's do it algebraically first. Let \( l \) denote the number of pints of light, and \( r \) the number of pints of regular. To make \( r \) pints of regular we need \( 2r \) cups of cream and \( \frac{5}{4}r \) cups of sugar. To make \( l \) pints of light we need \( l \) cups of cream and \( \frac{3}{4}l \) cups of sugar. In order not to waste any of the ingredients, we must have

\[
\begin{align*}
2r + l &= 10, \\
\frac{5}{4}r + \frac{3}{4}l &= 6.
\end{align*}
\]

This is called a system of linear equations. System means there is more than one equation. Multiply the second equation by 4 to get

\[
3r + 3l = 24.
\]

Subtract three times the first equation from the second to get

\[
-3r = -6 \implies r = 2.
\]
Substitute $r = 2$ into the first equation to get
\[ l = 6 \]

What we have done is a simple case of elimination. We'll come back to this later.

Geometrically we can solve the ice-cream system using vectors. A 2-vector is a pair of numbers, called components.

**Example 1.1.** $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are 2-vectors. The components of $v$ are 2 and 1.

Geometrically, we represent a 2-vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ as an arrow in the plane. Draw horizontal and vertical axes on the plane, so that the origin is the intersection of the axes. The tail of the vector can be at any point. The head of the vector is drawn $v_1$ units to the right, and $v_2$ units above the tail of the vector. Here is the vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, with tails at the origin.

We add 2-vectors by adding their components:
\[ \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 + 3 \\ 1 + 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \]
The sum of two vectors can be drawn by putting the vectors head to tail. Say $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ are 2-vectors. Draw $v$ with tail at 0, and $w$ with tail at the head of $v$. Now draw the vector from tail (0,0) to the head of $w$. This is the vector $v + w$. Here the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is drawn head to tail with $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Subtraction is similar: We subtract each component.
\[ \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 - 3 \\ 1 - 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \]

The final operation on vectors is scalar multiplication. A scalar is another name for a number. We multiply a scalar times a vector by multiplying each component:

**Example 1.2.** $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

In linear algebra, we do not use the notation $\times$ for multiplication of numbers; the symbol $\times$ has a special meaning we will discuss later. Here is the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and its scalar multiple $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Geometrically scalar multiplication $c$ leaves the direction of the vector unchanged, but multiplies the length by $c$. The vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ forms the hypotenuse of a triangle with adjacent side $v_1$ and opposite side $v_2$. By the Pythagorean theorem, the length or magnitude of $v$ is
\[ \|v\| = \sqrt{v_1^2 + v_2^2} \]
This is the distance from the head to the tail. If the vector has magnitude 1, it is called a **unit vector**. For any non-zero vector \( \mathbf{v} \) there is a **unit vector in the direction of** \( \mathbf{v} \), given by the formula

\[
\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.
\]

**Example 1.3.** The unit vector in the direction of \( \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) is

\[
\frac{1}{\sqrt{2^2 + 1^2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.
\]

Two vectors are **equal** if they have the same components, in order. This means that any system of linear equations can be written as a single vector equation. For instance, the ice-cream system

\[
2r + l = 10 \\
\left(\frac{3}{4}\right)r + \left(\frac{3}{4}\right)l = 6.
\]

can be written as the single vector equation

\[
r \begin{bmatrix} 2 \\ 1 \end{bmatrix} + l \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}.
\]

A sum of scalar multiples of vectors is called a **linear combination**. To solve the ice cream system, we have to write \( \begin{bmatrix} 10 \\ 6 \end{bmatrix} \) as a linear combination of \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \). Figure 1 shows that two of \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and six of \( \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) give \( \begin{bmatrix} 10 \\ 6 \end{bmatrix} \). This is what we mean by solving the system geometrically.

**Figure 1. Two and Six**

We end this section with a miscellaneous remark about vectors. If we write \( \mathbf{v} \) horizontally, as above, \( \mathbf{v} \) is called a **row vector**. If we write \( \mathbf{v} \) vertically

\[
\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.
\]

\( \mathbf{v} \) is a **column vector**. It’s still the same vector, however we write it.

A 3-vector \( \mathbf{v} \) is a triple of numbers, called the components of \( \mathbf{v} \).

**Example 1.4.**

\[
\mathbf{v} = \begin{bmatrix} 2 \\ \frac{3}{4} \\ \frac{3}{4} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix}
\]
are 3-vectors. Scalar multiplication multiplies a number times each component:

\[
5 \begin{pmatrix} 2 \\ 2 \\ \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \\ 15/4 \end{pmatrix}, \quad 2 \begin{pmatrix} 1 \\ \frac{3}{4} \\ \frac{6}{4} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6/4 \end{pmatrix}.
\]

Add vectors by adding their components:

\[
5\mathbf{v} + 2\mathbf{w} = \begin{pmatrix} 10 + 2 \\ 10 + 3 \\ 15/4 + 6/4 \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \\ 21/4 \end{pmatrix}.
\]

Geometrically a 3-vector \( \mathbf{v} = [v_1 \ v_2 \ v_3] \) is drawn as an arrow on a plane with three axes. The head of the arrow is drawn \( v_1 \) units along the first axis, \( v_2 \) units along the second axis, and \( v_3 \) units along the third axis from the tail. The axes can be drawn in many different ways. Our convention is that the first axis is drawn horizontally on the plane, with positive direction to the right, the second diagonally, with positive direction heading down and left, and the third axis vertically, with positive direction up. Usually, we draw the vector with tail at the origin \((0, 0, 0)\).

The vector \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) with tail at \((0, 0, 0)\) is drawn below in purple. In the other colors it is shown how to arrive at the head: by starting at the origin, moving one unit along the first axis, two units along the second, and three units along the third.

An unavoidable problem with representing three-vectors on the page is that different vectors look the same when drawn. For instance,

\[
\begin{pmatrix} -1/2 \\ 0 \\ \frac{3}{2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]

are drawn the same on our representation of three-space. It’s not even possible to determine whether a vector is zero, just from how it is drawn \( \begin{pmatrix} 3 \\ 4 \\ \frac{3}{4} \end{pmatrix} \) looks the same as the zero vector! In order to really visualize the vectors you have to imagine that the vectors are pointing in or out of the page. For instance, if you imagine that the \( y \)-axis is pointing down, left, and out of the page, then the vector \( \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \) is pointing out of the page directly at the viewer. On the other hand, \( \begin{pmatrix} -3 \\ -4 \\ -3 \end{pmatrix} \) is pointing directly into the page. For emphasis: the length of a three-vector is not the length of the vector as it is drawn on the page.

Three-vectors are used to represent linear systems in which there are three unknowns. For instance, in the ice-cream problem suppose we also consider how many eggs are needed for each type, so that the table of ingredients becomes

<table>
<thead>
<tr>
<th></th>
<th>Eggs</th>
<th>Cream</th>
<th>Sugar</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>2</td>
<td>2</td>
<td>3/4</td>
</tr>
<tr>
<td>Light</td>
<td>1</td>
<td>3/2</td>
<td>3/4</td>
</tr>
</tbody>
</table>

The system of linear equations becomes

\[
(2) \quad 2r + l = 12
\]
\[
2r + (\frac{3}{2})l = 13
\]
\[
(\frac{3}{4})r + (\frac{3}{4})l = 5 \frac{1}{4}.
\]
Subtracting the first equation from the second gives
\[ c - e = (1/2)l = 1 \] so \( l = 2 \).

Plugging \( l = 2 \) into the first equation gives
\[ e = 2r + 2 = 12, \] so \( r = 5 \).

It’s easy to check that the last equation is also solved by \( r = 5 \). The unique solution is \( r = 5, \ l = 2 \).

In vector form, the ice cream system is now
\[
\begin{bmatrix} r \\ l \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{2}{4} \\ \frac{3}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix}.
\]

The vectors \( \begin{bmatrix} 2 \\ 2/4 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 12 \\ 13 \\ 21/4 \end{bmatrix} \) are shown below:

To solve the system geometrically, we need to find the number of blue arrows and the number of pink arrows which together give the red arrow. In the picture below, we see that we need 5 blue arrows and 2 pink arrows; hence \( r = 5 \) and \( l = 2 \) is the unique solution.

![Diagram showing vectors and solution](image-url)

What do we see geometrically that we didn’t see before algebraically?
The set of all linear combinations of the ingredient vectors (pink arrows and blue arrows) forms a plane inside space. The ice cream system is solvable because of the lucky accident that the vector of available ingredients (the red vector in the picture) lies inside this plane. If we accidentally dropped one of the eggs, for instance, and tried to solve the system
\[
\begin{bmatrix} r \\ l \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{2}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \\ 21/4 \end{bmatrix}.
\]
the head of the red vector would have moved a little to the left, which would make the system unsolvable.

1.1. Problems.

(1) Find another combination of ingredients (that is, not 12 eggs, 13 cream, 5 1/4 sugar) that can be used up without waste.
2. Operations on Vectors

An $n$-vector is an $n$-tuple of numbers

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$  

We say that $\mathbf{v}$ has size $n$. Add two $n$-vectors by adding their components:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}. $$

The product of a scalar $c$ and an $n$-vector is defined by

$$c \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$  

The length of an $n$-vector is

$$\Vert \mathbf{v} \Vert = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}. $$

This is the distance from the head to tail.

To find the distance between any two points $p_1, p_2$ in $\mathbb{R}^n$, take the length of the vector whose head is at $p_1$ and whose tail is at $p_2$.

**Example 2.1.** To find the distance between the points $(3, 1, 2, 4)$ and $(-1, 2, 1, 2)$ in $\mathbb{R}^4$, we take the length of the vector

$$\mathbf{v} = [-1 \ 2 \ 1 \ 2] - [3 \ 1 \ 2 \ 4] = [-4 \ 1 \ -1 \ -2]$$

which is

$$\Vert \mathbf{v} \Vert = \sqrt{(-4)^2 + 1^2 + (-1)^2 + (-2)^2} = \sqrt{22}. $$

If two vectors $\mathbf{v}$ and $\mathbf{w}$ are the same size, define the dot product of $\mathbf{v}$ and $\mathbf{w}$ by taking the sum of the product of their components:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1w_1 + v_2w_2 + \ldots + v_nw_n.$$

For instance,

$$[1 \ 2 \ 3] \cdot [4 \ 5 \ 6] = 1(4) + 2(5) + 3(6) = 32.$$

A 1-vector is just a number. In this case there is no sum in the dot product

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1$$

so dot product of 1-vectors is multiplication of numbers.

If we take the product of $\mathbf{u}$ with itself we get

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \ldots + u_n^2 = \Vert \mathbf{u} \Vert^2$$

which is the square of the length of $\mathbf{u}$. This is just like ordinary multiplication, since for any number $x$ we have

$$x \cdot x = |x|^2.$$

Geometrically, the dot product has to do with the angle $\theta$ between the two vectors:

(3) \hspace{2cm} \mathbf{u} \cdot \mathbf{v} = \Vert \mathbf{u} \Vert \Vert \mathbf{v} \Vert \cos(\theta).$

This formula can be proved using the law of cosines. Look at the triangle with edge vectors $\mathbf{u}, \mathbf{v},$ and $\mathbf{u} - \mathbf{v}$. The law of cosines says

$$\Vert \mathbf{u} - \mathbf{v} \Vert^2 = \Vert \mathbf{u} \Vert^2 + \Vert \mathbf{v} \Vert^2 - 2 \Vert \mathbf{u} \Vert \Vert \mathbf{v} \Vert \cos(\theta).$$

The left-hand-side is

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}.$$

Subtracting the quantity

$$\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \Vert \mathbf{u} \Vert^2 + \Vert \mathbf{v} \Vert^2.$$

from both sides leads to the formula.

The equation (3) can be used to compute the angle between vectors:

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\Vert \mathbf{u} \Vert \Vert \mathbf{v} \Vert}\right).$$
Example 2.2. The angle between the vectors $\mathbf{u} = [1 \ 1 \ 0], \ \mathbf{v} = [0 \ 1 \ 1]$ is

$$
\theta = \arccos \left( \frac{1(0) + 1(1) + 1(1)}{\sqrt{1^2 + 1^2 + 0^2} \sqrt{0^2 + 1^2 + 1^2}} \right)
= \arccos \left( \frac{1}{2} \right) = \frac{\pi}{3}.
$$

The formula (3) has an important special case. Two vectors are perpendicular or orthogonal if the angle between them is $\pi/2$, that is, 90 degrees. This is the case if and only if

$$
\theta = \frac{\pi}{2} \iff \cos(\theta) = 0 \iff \mathbf{u} \cdot \mathbf{v} = 0.
$$

Two vectors are perpendicular if and only if their dot product is zero.

Example 2.3. Suppose we want to find a vector $\mathbf{v} = [v_1 \ v_2 \ v_3]$ perpendicular to $\mathbf{u} = [1 \ 1 \ 1]$. The condition that the dot product is zero is

$$
1(v_1) + 1(v_2) + 1(v_3) = v_1 + v_2 + v_3 = 0.
$$

It’s easy to find solutions. For example, $v_1 = 1, v_2 = -1, v_3 = 0$, or $v_1 = 0, v_2 = 1, v_3 = -1$ are both solutions. These give the vectors

$$
\mathbf{v} = [1 \ -1 \ 0], \ \text{or} \ \mathbf{v} = [0 \ 1 \ -1].
$$

Here are some of the properties of vector operations. Later we will define new operations which do not satisfy some of these axioms.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Vector Addition is Commutative)
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (Vector Addition is Associative)
3. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (Scalar Mult. is Distributive)
4. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Dot Product is Commutative)
5. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Dot Product is Distributive)
6. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ (Scalar Mult. Commutes with Dot Product)
7. $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$ (Dot Self-Product is Square Length)
8. $||c\mathbf{v}|| = |c||\mathbf{v}||$ (Scalar Mult multiplies the Length)

The proofs of these properties use the corresponding properties for numbers. For example, here is the proof of (1). Let

$$
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.
$$

Then

$$
\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix} = \mathbf{v} + \mathbf{u}.
$$

End of proof. See Section 29 for more explanation.

2.1. Problems.

1. Find the $2\mathbf{v} + 3\mathbf{w}$ for the vectors (a) $\mathbf{v} = [5 \ 4], \ \mathbf{w} = [-1 \ -2]$. (b) $\mathbf{v} = [1 \ 0], \ \mathbf{w} = [0 \ 1]$. 9c) $\mathbf{v} = [5 \ 4 \ 1], \ \mathbf{w} = [1 \ -2 \ -3]$. (2) Find the angle between the vectors $[2 \ 2 \ 1]$ and $[2 \ -1 \ 2]$. (3) Find a unit vector $\mathbf{u}$ perpendicular to $\mathbf{w} = (2, 1, 2)$. (4) Prove $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for two-vectors $\mathbf{u}, \mathbf{v}$. (5) Prove $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for $n$-vectors $\mathbf{u}, \mathbf{v}$.

3. Matrices

In the last section, we wrote the ice-cream system in vector form

$$
\begin{bmatrix}
2 \\
2 \\
3/4
\end{bmatrix} + \begin{bmatrix}
1 \\
-2/3 \\
1
\end{bmatrix} = \begin{bmatrix}
12 \\
13 \\
21/4
\end{bmatrix}.
$$

It can be written even more simply, using matrices. A matrix is a table of numbers, called the entries of the matrix. If a matrix has $m$ rows and $n$ columns, it is called an $m \times n$ matrix. The entry in the $i$-th row and $j$-th column is called the $ij$-th entry.
For instance, the matrix for the ice-cream system is the $2 \times 3$ matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$  

One can think of a matrix as a collection of row vectors, or as a collection of column vectors. The row vectors for the matrix $A$ are

$$[2, 1], \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$  

The column vectors are

$$\begin{bmatrix} 2 \\ 2 \\ \frac{3}{4} \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ \frac{3}{4} \end{bmatrix}.$$  

Here are a few of the many special kinds of special matrices.

(1) The $m \times n$ zero matrix $0_{mn}$, whose entries are all zero. For example,

$$0_{32} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$  

If there is no confusion about the size, we drop the subscripts and write $0$ for the zero matrix.

(2) A matrix is square if it has the same number of rows as columns. For instance,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$  

is a square matrix, more precisely $2 \times 2$.

(3) A square matrix is diagonal if the only non-zero entries are on the diagonal, for instance,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$  

is diagonal.

(4) A square matrix $A$ is upper triangular if all of the entries below the diagonal are zero. For instance,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$  

is upper triangular. $A$ is lower triangular if all of the entries above the diagonal are zero. $A$ is strictly upper (or lower) triangular if it is upper (or lower) triangular and all of the diagonal entries are zero. For instance,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$  

is strictly lower triangular.

(5) The transpose of an $m \times n$ matrix $A$ is the $n \times m$ matrix $A^T$ whose columns are the rows of $A$, and whose rows are the columns of $A$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$  

If a matrix $A$ is equal to its own transpose $A^T$, it is called symmetric. For instance,

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$  

is a $3 \times 3$ symmetric matrix.

(6) A matrix is a permutation matrix if there is exactly one 1 in each row and each column, and otherwise the matrix is zero. For example,

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$  

is a permutation matrix.

**Example 3.1.** Show that if $P$ is a permutation matrix, then so is $P^T$.  


3.1. **Matrix addition, subtraction, and scalar Multiplication.** Matrices are added or subtracted in the same way as vectors, by adding or subtracting entries. For example,

\[
\begin{bmatrix}
0 & 1 \\
2 & 3 \\
4 & 5
\end{bmatrix} + \begin{bmatrix}
0 & 3 \\
1 & 4 \\
2 & 5
\end{bmatrix} = \begin{bmatrix}
0 & 4 \\
3 & 7 \\
6 & 10
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
2 & 3 \\
4 & 5
\end{bmatrix} - \begin{bmatrix}
0 & 3 \\
1 & 4 \\
2 & 5
\end{bmatrix} = \begin{bmatrix}
0 & -2 \\
1 & -1 \\
2 & 0
\end{bmatrix}
\]

Multiply a scalar times a matrix by multiplying each entry by the scalar.

\[
2 \begin{bmatrix}
0 & 1 \\
2 & 3 \\
4 & 5
\end{bmatrix} = \begin{bmatrix}
0 & 2 \\
4 & 6 \\
8 & 10
\end{bmatrix}
\]

3.2. **The product of a matrix and a column vector.** Suppose \( A \) is a \( 3 \times 2 \) matrix with column vectors \( v_1, v_2 \). Suppose \( x \) is the column vector

\[
x = \begin{bmatrix}
5 \\
2
\end{bmatrix}
\]

Then the product \( Ax \) is the sum

\[5v_1 + 2v_2.\]

That is, the product of a matrix times a vector is a sum of the column vectors of the matrix, with coefficients given by the components of the vector. For instance,

\[
\begin{bmatrix}
2 & 1 \\
2 & \frac{3}{4} \\
\frac{3}{4} & \frac{3}{4}
\end{bmatrix} \begin{bmatrix}
5 \\
2
\end{bmatrix} = 5 \begin{bmatrix}
2 \\
\frac{3}{4}
\end{bmatrix} + 2 \begin{bmatrix}
\frac{3}{4}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
12 \\
13 \\
21/4
\end{bmatrix}
\]

Here is an example with a \( 2 \times 2 \)-matrix:

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
3 \\
4
\end{bmatrix} = 3 \begin{bmatrix}
0 \\
1
\end{bmatrix} + 4 \begin{bmatrix}
1 \\
0
\end{bmatrix} = \begin{bmatrix}
4 \\
3
\end{bmatrix}
\]

Multiplying this matrix times a vector has the effect of switching the first and second components!

We can re-write the ice-cream equations a second time using this product. The equations (2) are written in matrix form

\[
\begin{bmatrix}
2 & 1 \\
2 & \frac{3}{4} \\
\frac{3}{4} & \frac{3}{4}
\end{bmatrix} \begin{bmatrix}
r \\
l
\end{bmatrix} = \begin{bmatrix}
12 \\
13 \\
21/4
\end{bmatrix}
\]

There is another way of looking at the product of a matrix \( A \) times a vector \( x \), using dot products. The components of the product are the dot products of the rows of \( A \) with the vector \( x \). For instance,

\[
\begin{bmatrix}
2 & 1 \\
2 & \frac{3}{4} \\
\frac{3}{4} & \frac{3}{4}
\end{bmatrix} \begin{bmatrix}
5 \\
2
\end{bmatrix} = 5 \begin{bmatrix}
2 \\
\frac{3}{4}
\end{bmatrix} + 2 \begin{bmatrix}
\frac{3}{4}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
5(2) + 2(1) \\
5(2) + 2(\frac{3}{4}) \\
5(\frac{3}{4}) + 2(\frac{3}{4})
\end{bmatrix}
\]

The first component is the dot product of \( \begin{bmatrix}
5 \\
2
\end{bmatrix} \) with \( \begin{bmatrix}
2 \\
1
\end{bmatrix} \), the second component is the dot product with \( \begin{bmatrix}
2 \\
\frac{3}{4}
\end{bmatrix} \) and so on.

In general notation, suppose \( A \) has column vectors \( v_1, \ldots, v_n \) and

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_n
\end{bmatrix}
\]

Define

\[Ax = x_1 v_1 + \ldots + x_n v_n.\]

If \( A \) has row-vectors \( w_1, \ldots, w_m \) then

\[Ax = \begin{bmatrix}
w_1 \cdot x \\
w_2 \cdot x \\
w_m \cdot x
\end{bmatrix}
\]

is the vector of dot products.

For each square size, there is a special matrix, called the *identity matrix* \( I \) which has 1's along the diagonal and 0's everywhere else. For
instance, the $3 \times 3$ identity is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

The identity matrix has the property that $I$ times any vector $v$ is itself:

$$Iv = v$$

For instance, using the definition we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

The matrix-vector products satisfy the following properties:

1. $A(v + w) = Av + Aw$ (Distributivity over vect. addition)
2. $A(cv) = c(Av)$ (Commutes with Scalars)
3. $(A + B)v = Av + Bv$ (Distrib. over matrix addition)

The proofs are omitted.

### 3.3. Matrix products

To define the product of two matrices $A$ and $B$, we take the product of the matrix $A$ times each column of $B$. These new vectors form the columns of the matrix, denoted $AB$. Because each column is made up of dot products of rows of $A$ with columns of $B$, the matrix product is the matrix of dot products. For example,

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} [12] [-1] & [12] [1] \\ [23] [-1] & [23] [1] \\ [34] [-1] & [34] [1] \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}.$$ 

Notice that the matrix product only makes sense if the dot products of the rows of $A$ with the columns of $B$ make sense. In other words, the rows of $A$ have to be the same size as the columns of $B$. To put it one more way, if $A$ is an $m \times n$ matrix and $B$ is a $p \times q$ matrix, the product $AB$ makes sense only if $n = p$. The result is an $m \times q$ matrix.

In general notation let $w_1, \ldots, w_m$ be the row-vectors of $A$, and $v_1, \ldots, v_q$ the column vectors of $B$. The matrix product $AB$ is the matrix whose $ij$-th entry is the dot product $v_i \cdot w_j$.

### 3.4. Matrix multiplication is counter-intuitive

Matrix multiplication is counter-intuitive in a number of different ways.

1. Matrix multiplication is not commutative, that is $AB$ is not necessarily the same matrix as $BA$, even if both are defined. For $AB$ to be defined $A$ must have the same number of columns as $B$ has rows. For $BA$ to be defined, $B$ has the same number of columns as $A$ has rows. Here is an example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix},$$

which is not equal to

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}.$$

2. Just because $AB = 0$ doesn’t mean that $A = 0$ or $B = 0$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

but neither of the matrices (which are equal) are zero. In fact, this property isn’t true for vectors either. Suppose that $v$ and $w$ are vectors of the same size, and $v \cdot w = 0$. As we said before, this means that $v$ is perpendicular to $w$, not that either $v$ or $w$ is zero.

3. If $AB = AC$ and $A$ is non-zero, then it is not necessarily true that $B = C$. We can’t just divide $A$ from both sides, since the expression $1/A$ doesn’t make sense - yet.

4. If $A$ is a square matrix we can define it’s matrix powers

$$A^2 = AA, A^3 = AAA,$$
et cetera. If $A^2 = 0$, this doesn’t mean that $A$ is zero. For instance,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has $A^2 = 0$, but $A$ is not zero. This property is true for vectors and dot product:

$$\mathbf{v} \cdot \mathbf{v} = 0 \implies \|\mathbf{v}\|^2 = 0 \implies \mathbf{v} = 0$$

since the only vector with length zero is the zero vector.

3.5. **Associativity of matrix products.** The most important property of matrix products is **associativity.** In its simplest version associativity says that if $A$ and $B$ are matrices, and $\mathbf{v}$ is a vector, then

$$B(A\mathbf{v}) = (BA)\mathbf{v}.$$  

That is, multiplying by $A$, then $B$, is the same as multiplying by $BA$. For example, if

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3/4 \\ 3/4 & 1/4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} 5 \\ 2/4 \end{bmatrix}$$

Then

$$A\mathbf{v} = \begin{bmatrix} 12 \\ 13 \\ 5\frac{1}{4} \end{bmatrix}$$

and

$$B(A\mathbf{v}) = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix} \begin{bmatrix} 12 \\ 13 \\ 5\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 10(12) + 80(13) + 10(5\frac{1}{4}) \\ 50(12) + 50(13) + 40(5\frac{1}{4}) \end{bmatrix} = \begin{bmatrix} 1212.5 \\ 1460 \end{bmatrix}$$

On the other hand,

$$BA = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3/4 \\ 3/4 & 1/4 \end{bmatrix} = \begin{bmatrix} 187.5 & 137.5 \\ 230 & 155 \end{bmatrix}$$

and

$$(BA)\mathbf{v} = \begin{bmatrix} 187.5 \times 5 + 137.5 \times 2 \\ 230(5) + 155(2) \end{bmatrix} = \begin{bmatrix} 1212.5 \\ 1460 \end{bmatrix}$$

Before we give a proof of the associativity property, let’s explain how it might come up in our ice-cream example. Suppose the cost and weight of the ingredients are

<table>
<thead>
<tr>
<th>ingredient</th>
<th>cost</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>eggs</td>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>cream</td>
<td>80</td>
<td>50</td>
</tr>
<tr>
<td>sugar</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

Let $B$ be the matrix

$$B = \begin{bmatrix} 10 & 80 & 10 \\ 50 & 50 & 40 \end{bmatrix}.$$  

The matrix $B$ transforms the ingredient vector into the cost/weight vector:

$$B \begin{bmatrix} e \\ c \\ s \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}$$

where $t$ is the number of cents that the ingredients cost and $w$ is their weight. Since

$$B \begin{bmatrix} 12 \\ 13 \\ 5\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1212.5 \\ 1512.5 \end{bmatrix}$$

the 12 eggs, 13 cups cream and $5\frac{1}{4}$ sugar costs 1212.5 and weighs 1512.5. We want a matrix that takes the regular/light vector to the cost/weight vector. This is the role of the matrix product:

$$BA \begin{bmatrix} r \\ l \end{bmatrix} = B \begin{bmatrix} e \\ c \\ s \end{bmatrix} = \begin{bmatrix} t \\ w \end{bmatrix}.$$  

Given the number of pints of regular and light we want to make, multiplying by $BA$ tells us how much it will cost and weigh.
Here is the proof of associativity. Let \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) be an \( n \)-vector, let \( A \) be a matrix with \( n \) columns \( \mathbf{w}_1, \ldots, \mathbf{w}_n \). The product \( A\mathbf{v} \) is

\[ A\mathbf{v} = v_1 \mathbf{w}_1 + \ldots + v_n \mathbf{w}_n. \]

So

\[ B(A\mathbf{v}) = B(v_1 \mathbf{w}_1 + \ldots + v_n \mathbf{w}_n) = v_1 (B \mathbf{w}_1) + \ldots + v_n (B \mathbf{w}_n) \]

by distributivity of matrix multiplication over addition. On the other hand, \( BA \) is the matrix with columns

\[ B \mathbf{w}_1, \ldots, B \mathbf{w}_n. \]

So

\[ (BA)\mathbf{v} = v_1 (B \mathbf{w}_1) + \ldots (B \mathbf{w}_n) \]

which equals \( B(A\mathbf{v}) \). End of proof.

Doing one operation after another is called “composition” of operations. The matrix associativity property says that the composition of matrix multiplications for \( A, B \) is given by the matrix multiplication by the product \( BA \).


1. \( A + B = B + A \) \hspace{2cm} (Commutativity of Addition)
2. \( A + (B + C) = (A + B) + C \) \hspace{2cm} (Assoc. of Addition)
3. \( A(BC) = (AB)C \) \hspace{2cm} (Assoc. of Matrix Product)
4. \( A(B + C) = AB + AC \) \hspace{2cm} (Distrib. of Left Matrix Product)
5. \( (A + B)C = AC + BC \) \hspace{2cm} (Distrib. of Right Matrix Product)
6. \( (A^T)^T = A \) \hspace{2cm} (Transpose is an Involution)
7. \( (AB)^T = B^T A^T \) \hspace{2cm} (Transpose of a Product Changes Order)

The proofs of (1)(2),(4),(5),(6) are omitted.

The associativity Property (3) follows from the associativity in the previous section: If \( \mathbf{u}_1, \ldots, \mathbf{u}_p \) are the columns of \( C \), then \( A(BC) \) is the matrix with columns

\[ A(B \mathbf{u}_1), \ldots, A(B \mathbf{u}_n) \]

and \( (AB)C \) is the matrix with columns

\[ (AB)\mathbf{u}_1, \ldots, (AB)\mathbf{u}_n. \]

By associativity of matrix-vector products, these are equal.

Property (7) is justified as follows:

\[ ij \)-th entry of \((AB)^T = ji \)-th entry of \( AB \)
\[ = (\text{row } j \text{ of } A) \cdot (\text{column } i \text{ of } B) \]
\[ = (\text{column } i \text{ of } A^T) \cdot (\text{row } i \text{ of } B^T) \]
\[ = (\text{row } i \text{ of } B^T) \cdot (\text{column } j \text{ of } A^T) \]
\[ = ij \)-th entry of \( B^T A^T \).

3.7. Problems.

(1) True or False? If true, explain; If false, give a counterexample.
   (a) If \( A \) and \( B \) are symmetric, then \( A + B \) is symmetric.
   (b) If \( A \) and \( B \) are symmetric matrices, then \( AB \) is also symmetric.
   (c) If \( A^2 = 0 \), then \( A = 0 \).
   (d) If \( AA^T = 0 \), then \( A = 0 \).

(2) Let \( A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 3 \end{bmatrix} \).
   (a) Compute \( AB \). (b) Compute \( A^T \) and \( B^T \).

(3) Compute the matrix product \( A^2 \) (\( A \) times \( A \)) for

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

(4) Construct a permutation matrix \( A \) not equal to \( I \), whose square \( A^2 \) is \( I \).

4. Elimination and Reduced Row-Echelon Form

4.1. Row operations. Elimination is the procedure by which we try to solve a system of linear equations by subtracting multiples of the equations from each other to eliminate the unknowns.
Example 4.1. To solve the ice-cream system we performed the following steps. Subtract the first equation from the second in

\[
\begin{align*}
2r + l &= 12 \\
2r + \left(\frac{3}{2}\right)l &= 13 \\
\left(\frac{3}{4}\right)r + \left(\frac{3}{4}\right)l &= 5 \frac{1}{4}.
\end{align*}
\]

to get

\[
\begin{align*}
2r + l &= 12 \\
\frac{1}{4}l &= 1 \\
\left(\frac{3}{4}\right)r + \left(\frac{3}{4}\right)l &= 5 \frac{1}{4}.
\end{align*}
\]

Multiply the second equation by two to get

\[
\begin{align*}
2r + l &= 12 \\
l &= 2 \\
\left(\frac{3}{4}\right)r + \left(\frac{3}{4}\right)l &= 5 \frac{1}{4}.
\end{align*}
\]

Now substitute \(l = 2\) into the first and third, and solve for \(r\).

We can do the same steps in matrix form, using a little less ink. The augmented matrix for the ice-cream system (2) is

\[
\begin{bmatrix}
2 & 1 & 12 \\
2 & \frac{3}{4} & 13 \\
\frac{3}{4} & \frac{3}{4} & 5 \frac{1}{4}.
\end{bmatrix}
\]

Subtract row one from row two to get

\[
\begin{bmatrix}
2 & 1 & 12 \\
0 & \frac{1}{2} & 1 \\
\frac{3}{4} & \frac{3}{4} & 5 \frac{1}{4}.
\end{bmatrix}
\]

Multiply the second equation by two to get

\[
\begin{bmatrix}
2 & 1 & 12 \\
0 & 1 & 2 \\
\frac{3}{4} & \frac{3}{4} & 5 \frac{1}{4}.
\end{bmatrix}
\]

The second equation says \(0r + 1l = 2\), that is, \(l = 2\). Substituting into the first equation gives \(r = 5\), as before.

Example 4.2. Suppose we want to solve the system of three equations with three unknowns

\[
\begin{bmatrix}
x - y &= 1 \\
y - z &= 2 \\
-x + z &= 3.
\end{bmatrix}
\]

The matrix form of the system is

\[
\begin{bmatrix}
1 & -1 & 0 & 1 \\
0 & 1 & -1 & 2 \\
-1 & 0 & 1 & 3.
\end{bmatrix}
\]

In the first step we add equation to equation 1 to equation 3 to get (also written in matrix form on the right)

\[
\begin{bmatrix}
x - y &= 1 \\
y - z &= 2 \\
- y + z &= 4.
\end{bmatrix}
\]

Then we add equations 2 and 3 to get

\[
\begin{bmatrix}
x - y &= 1 \\
y - z &= 2 \\
0 &= 6.
\end{bmatrix}
\]

The last equation \(0 = 6\) is a contradiction (obviously wrong). This means that the system has no solutions.

There are three possible row operations, or moves in elimination:

- (1) Add a multiple of one row (equation) to another.
- (2) Multiply a row (equation) by a non-zero number.
- (3) Switch two rows (equations).

Another example: Let’s solve the system corresponding to the augmented matrix

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
2 & 4 & 6 & 2 \\
3 & 6 & 8 & 1.
\end{bmatrix}
\]

We subtract 2 times row 1 from row 2, and 3 times row 1 from row 3 to get

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & -2.
\end{bmatrix}
\]

Now we switch rows 2 and 3 and multiply by \(-1\) to get

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0.
\end{bmatrix}
\]
The equations are

\[ \begin{align*}
    x + 2y + 3z &= 1 \\
    z &= 2 \\
    0 &= 0
\end{align*} \]

Substituting \( z = 2 \) into the first equation gives \( x + 2y = -5 \) or \( x = -5 - 2y \). The solution set to this system is therefore

\[ \begin{bmatrix}
    x \\
    y \\
    z
\end{bmatrix}, \quad z = 2 \quad \text{and} \quad x = -5 - 2y \]

In other words, the solution set is

\[ \begin{bmatrix}
    -5 - 2y \\
    y \\
    2
\end{bmatrix} \]

Instead of back substitution, we can do further row operations to get the matrix into reduced row echelon form. For instance, subtracting three times the second row from the first in

\[ \begin{bmatrix}
    1 & 2 & 3 & 1 \\
    0 & 0 & 1 & 2 \\
    0 & 0 & 0 & 0
\end{bmatrix} \]

gives

\[ \begin{bmatrix}
    1 & 2 & 0 & -5 \\
    0 & 0 & 1 & 2 \\
    0 & 0 & 0 & 0
\end{bmatrix} \]

The equations are

\[ x + 2y = -5 \quad z = 2 \]

so the solution set is

\[ \begin{bmatrix}
    -5 - 2y \\
    y \\
    2
\end{bmatrix} \]

as before.

It will be helpful to introduce short-hand for the row operations. Here are some examples:

\[ \begin{align*}
    &\text{(2) } \rightarrow \text{ (2) } - 2 \text{ (1)} \quad \text{subtract } 2 \text{ times row } 1 \text{ from row } 2. \\
    &\text{(3) } \rightarrow \text{ (3) } - 3 \text{ (1)} \quad \text{subtract } 3 \text{ times row } 1 \text{ from row } 3. \\
    &\text{(2) } \leftrightarrow \text{ (3)} \quad \text{switch rows } 2 \text{ and } 3
\end{align*} \]

\[ \text{(2) } \rightarrow -1 \text{ (2)} \quad \text{multiply row } 2 \text{ by } -1. \]

Note that the row numbers are always circled.

4.2. Reduced row-echelon form and the number of solutions to a linear system. Elimination can stop when the matrix is in row-echelon form:

**Definition 4.3.**

1. All rows of zeroes are at the bottom.
2. The first non-zero entry in any row is a 1, called a leading 1 or pivot.
3. The leading 1 in any row is to the right of the leading 1’s above it.

If, in addition, the entries above any leading 1 are zero, the matrix is said to be in reduced row-echelon form.

For instance, the matrix

\[ \begin{bmatrix}
    1 & 2 & 3 & 1 \\
    0 & 0 & 1 & 2 \\
    0 & 0 & 0 & 0
\end{bmatrix} \]

is in row echelon form. The matrix

\[ \begin{bmatrix}
    1 & 2 & 0 & -5 \\
    0 & 0 & 1 & 2 \\
    0 & 0 & 0 & 0
\end{bmatrix} \]

is in reduced row-echelon form. Since the equations are

\[ x + 2y = -5 \quad z = 2 \]

This system has an infinite number of solutions, one for each value of \( y \). The variables \( x, z \) that correspond to columns containing leading 1’s are called the bound variables; the other variables are called free variables. The bound variables can be expressed in terms of the free variables, using the equations corresponding to the rows of the row-echelon form system and back substitution.

The three examples we have done so far show the three possible outcomes of elimination:

1. There is a unique solution if the system is consistent and there is a leading 1 in every column (to the left of the bar).
(2) There are infinite solutions if the system is consistent and there are some columns without leading 1’s.

(3) The system is inconsistent if there is a row of zeroes, and a non-zero number to the right of the bar. In this case there are no solutions.

Let’s do another example with an infinite number of solutions.

**Example 4.4.** We solve the system
\[\begin{align*}
2x + 6y - 2z + 2w &= 6 \\
x + 3y + y + 10z &= 15
\end{align*}\]

The augmented matrix for this system is
\[
\begin{bmatrix}
2 & 6 & -1 & 2 & 6 \\
1 & 3 & 1 & 10 & 15
\end{bmatrix}.
\]

Switch the first and second rows to get a leading one in the first column:
\[
\begin{bmatrix}
1 & 3 & 1 & 10 & 15 \\
2 & 6 & -1 & 2 & 6
\end{bmatrix}.
\]

Subtract twice the first row from the second to get
\[
\begin{bmatrix}
1 & 3 & 1 & 10 & 15 \\
0 & 0 & -3 & -18 & -24
\end{bmatrix}.
\]

Multiply the second row by \(-1/3\) to create a leading one in the second row.
\[
\begin{bmatrix}
1 & 3 & 1 & 10 & 15 \\
0 & 0 & 1 & 6 & 8
\end{bmatrix}.
\]

The matrix is in row-echelon form. At this point we could write out the equations and do back-substitution to find the answer. Instead, we keep going to reduced row-echelon form. Subtract the second row from the first to get
\[
\begin{bmatrix}
1 & 3 & 0 & 4 & 7 \\
0 & 0 & 1 & 6 & 8
\end{bmatrix}.
\]

Now this is reduced row-echelon form. The equations are
\[\begin{align*}
x + 3y + 4w &= 7 \\
z + 6w &= 8
\end{align*}\]

The leading 1’s are in the first and third rows. So the bound variables are \(x\) and \(z\); the free variables are \(y\) and \(w\). Write the bound variables in terms of the free variables
\[\begin{align*}
x &= 7 - 3y - 4w, \\
z &= 8 - 6w.
\end{align*}\]

The solution set to the system is all vectors satisfying these equations. Since \(y\) and \(w\) can be anything, the solution set can be written
\[
\begin{bmatrix}
7 - 3y - 4w \\
y \\
8 - 6w \\
w
\end{bmatrix}.
\]

Here we have substituted the formulas for the bound variables. Since there are free variables, there are an infinite number of solutions.

The fact that there are always 0,1 or infinite solutions is a special feature of linear systems of equations. Non-linear equations, can have other numbers of solutions, for example. For example \(x^2 = 1\), has two solutions \(x = \pm 1\).

**Note 4.5.**
1. It’s not possible to read off the number of solutions from the number of unknowns and the number of equations, without doing the elimination.
2. If the number of unknowns is greater than the number of equations, some of the unknowns must be free. So there are always infinite or no solutions.
3. The system is called homogeneous if the numbers to the right of the equals signs/bar are all zero. Since these systems are always consistent, the number of solutions is infinity or one.

4.3. **Problems.**

1. True or False? Explain your answer.
   (a) If the number of columns of \(A\) is greater than the number of rows, then \(Ax = b\) cannot have a unique solution.
   (b) If \(A\) has more rows than columns, then the number of solutions of \(Ax = b\) is either 0 or 1.
   (c) If \(A\) has the same number of rows as columns, then any linear system of equations \(Ax = b\) has a unique solution.
   (d) If \(A\) has more columns than rows, then the number of solutions of \(Ax = b\) is either 0 or \(\infty\).
4.4. Application to polynomial interpolation. There is a unique line passing through the points \((1, 1)\) and \((-1, 1)\). How many degree two polynomials are there passing through these points? Although this problem seems non-linear (a parabola is a graph of a quadratic function) in fact this is a system of linear equations. Suppose that the function is \(f(x) = ax^2 + bx + c\).

The unknowns here are the values of \(a, b, c\), since we are solving for the parabola, not \(x\). Each point gives us an equation for \(a, b, c\):

\[
\begin{align*}
    a(1)^2 + b(1) + c &= 1 \\
    a(-1)^2 + b(-1) + c &= 1.
\end{align*}
\]

The matrix form of this system is

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & -1 & 1 & 1
\end{bmatrix}.
\]

Since the number of unknowns is greater than the number of equations, from Theorem 4.5 we know that there are either zero or infinite solutions. To figure out which, we have to do elimination. There is already a leading 1 in the first row, so there’s nothing to do there. We subtract the first row from the second to get

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & -1 & 1 & 1
\end{bmatrix} - \begin{bmatrix}
    1 & 1 & 1 & 1 \\
    0 & 0 & 1 & 0
\end{bmatrix}.
\]

Divide the second row by \(-2\) to get a leading 1:

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    0 & 1 & 0 & 0
\end{bmatrix}.
\]

This matrix is now in row-echelon form. To get reduced row-echelon form, subtract the second row from the first:

\[
\begin{bmatrix}
    1 & 0 & 1 & 1 \\
    0 & 1 & 0 & 0
\end{bmatrix}.
\]

The equations are

\[
a + c = 1, \quad b = 0.
\]

The bound variables are \(a, b\); the free variable is \(c\). Expressing the bound variables in terms of the free variables gives

\[
a = 1 - c, \quad b = 0.
\]

The solution vectors are

\[
\begin{bmatrix}
    1 - c \\
    0 \\
    c
\end{bmatrix}.
\]

The solution functions are

\[
f(x) = (1 - c)x^2 + 0x + c = (1 - c)x^2 + c.
\]

It’s easy to check that these all satisfy \(f(\pm1) = 1\). Since there is one solution for each value of \(c\), there are infinite solutions.

**Example 4.6.** It’s easy to come up with a similar example which has no solutions: If the data points are \((1, 1)\) and \((1, -1)\), there is no function with these values because any function takes only one value at any value of \(x\). If we try to solve this system, we get the matrix form

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & 1 & 1 & -1
\end{bmatrix}
\]

which has rref (reduced row-echelon form)

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 \\
    0 & 0 & 0 & -2
\end{bmatrix}.
\]

This is inconsistent.

**Example 4.7.** Find all polynomials of degree 2 passing through the points \((-1, 1), (0, 2), (1, 1)\).

Pluggin in the data points into the polynomial \(ax^2 + bx + c\) gives the system of equations

\[
\begin{align*}
    a(−1)^2 + b(−1) + c &= 1 \\
    a(0)^2 + b(0) + c &= 2 \\
    a(1)^2 + b(1) + c &= 1.
\end{align*}
\]

This system has augmented matrix

\[
\begin{bmatrix}
    1 & -1 & 1 & 1 \\
    0 & 0 & 1 & 2 \\
    1 & 1 & 1 & 1
\end{bmatrix}.
\]

The matrix has rref equal to

\[
\begin{bmatrix}
    1 & 0 & 0 & -1 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 2
\end{bmatrix}.
\]

\[\text{If you want, imagine you have collected some experimental data and know for some reason that the quantities you are measuring are related by a degree two function.}\]
The equations are 
\[ a = -1, \quad b = 0, \quad c = 2. \]

There is a unique solution, \[ f(x) = -x^2 + 2. \]

Here is the general result, which we will prove later using determinants:

**Theorem 4.8.** Given \( n \) points \((x_1, y_1), \ldots, (x_n, y_n)\), with \( x_1, \ldots, x_n \) distinct, there is a unique polynomial of degree \( n + 1 \) passing through them. If \( d > n + 1 \), there are infinitely many polynomials passing through these points.

4.5. **Gaussian elimination.** Elimination can be done in many different ways. Gauss observed in the 1800s that there is a systematic way of doing elimination. Doing elimination by hand, Gauss’ method may not be the best way, for instance, it may involve fractions when a different choice of operations might avoid them.

Gauss’ algorithm depends on the following observation: if there is a non-zero entry in a column in a matrix, then that entry can be used to make all the other entries in that column zero, by subtracting multiples of that row from the other rows.

We followed Gauss’ method when we found the rref of the matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
3 & 6 & 8
\end{bmatrix}.
\]

The steps (in the short-hand we used above) were
\[
\begin{align*}
2 - 2 \cdot 1 & \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\
3 - 3 \cdot 1 & \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

This is called the forward pass. The backward pass has just one step:
\[
1 - 3 \cdot 2 \quad \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Guass’s algorithm is the following. First, the **forward pass**:

1. Look at the first column containing a non-zero value.
2. Find the first entry in that column that is non-zero. The place of this entry is a pivot.
3. Switch rows so the pivot is in the first row.
4. Make the first row have a leading one, by dividing the first row by the value in the pivot.
5. For each non-zero entry below the pivot, multiply the first row by its value, and subtract it to make the entry zero.
6. Repeat steps (b) - (e) for the next column containing a non-zero value, ignoring rows that already have leading ones.

After the forward pass, the matrix is in row-echelon form. Next do the **backward pass**.

1. Find the last non-zero row, that is, the last row containing a pivot.
2. For each non-zero entry above the leading one, subtract the value of that entry times the pivot row to make that entry zero.
3. Repeat step (b) for the row above.

After the backward pass, the matrix is in reduced row-echelon form.

4.6. **Elementary matrices.** Each row operation can be represented by multiplying by an elementary matrix on the left. The elementary matrix corresponding to the row operation is the result of performing the row operation on the identity matrix.

**Example 4.9.** The elementary matrix corresponding to \( 3 \leftrightarrow 3 - 3 \cdot 1 \) is
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3 & 0 \\ 1 \end{bmatrix}.
\]

The elementary matrix corresponding to \( 1 \leftrightarrow 2 \) is
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
\]

The elementary matrix corresponding to \( 2 \leftrightarrow -2 \) is
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}.
\]
Theorem 4.10. Doing a row operation on a matrix gives the same result as multiplying on the left by the corresponding elementary matrix.

Example 4.11.

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
3 & 6 & 8
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}.
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
0 & 0 & -1
\end{bmatrix}.
\]

4.7. Problems.

(1) True or False? Explain your answer.
   (a) The number of columns containing pivots in \( \text{rref}(A) \) is the same as the number of rows containing pivots.

(2) Find all solutions to the system
   \[
   \begin{align*}
x_1 + x_2 - 2x_3 + x_4 &= 6 \\
2x_1 - x_2 + x_3 - 3x_4 &= 0
   \end{align*}
   \]
   by finding the reduced row-echelon form (RREF) for the associated matrix. Circle the pivots in the RREF. Is this a homogeneous or a non-homogeneous system?

(3) (a) Let \( f(t) \) be a function of the form \( f(t) = c_0 + c_2t^2 + c_4t^4 \).
   Write the system of equations \( f(-1) = 0, \ f(0) = 1, \ f(1) = 0 \) in matrix form \( Ax = b \).
   (b) Using row reduction, find all solutions to this system (i.e. function satisfying these conditions.)
   (c) Find an \( LU \) factorization for \( A \).

(4) Write a 3 by 3 matrix that
   (a) subtracts \(-5\) times row 1 from row 3
   (b) exchanges rows 1 and 3
   (c) divides row 2 by \(-2\).

(5) Find all solutions to the system
   \[
   \begin{align*}
x_1 - x_2 + x_3 + 2x_4 &= 3 \\
2x_1 - 2x_2 + 4x_3 + 2x_4 &= 6 \\
x_1 - x_2 + 3x_3 &= 3.
   \end{align*}
   \]

What are the solutions to the corresponding homogeneous system?

(6) (a) a row echelon form and (b) the reduced row echelon form and (c) an \( LU \) factorization for the matrix
   \[
   \begin{bmatrix}
1 & -1 & 0 & -8 & -3 \\
-2 & 1 & 0 & 9 & 5 \\
3 & -3 & 0 & -2 & -11
\end{bmatrix}
   \]

(7) Using elimination find (a) a row echelon form (b) the reduced row echelon form
   \[
   A = \begin{bmatrix}
1 & 3 & -1 & 1 & 0 \\
2 & 6 & -1 & 4 & 15 \\
-1 & -3 & 2 & -1 & 15
\end{bmatrix}.
   \]

Find all solutions to the system whose augmented matrix is \( A \),
\[
\begin{align*}
x_1 + 3x_2 - x_3 + x_4 &= 0 \\
2x_1 + 6x_2 - x_3 + x_4 &= 0 \\
-x_1 - 3x_2 + 2x_3 - x_4 &= 0
\end{align*}
\]
that is, \( x_2 + 3x_2 - x_3 + x_4 = 15 \).

(8) Find the unique quadratic (degree 2) polynomial passing through the points \((0,1),(1,2),(-1,3)\).

5. Matrix inverses

We said before that \( AD = AE \) does not imply \( D = E \), even if \( A \) is non-zero. This is because it doesn’t make any sense to “divide by \( A \) on both sides”, for arbitrary matrices. The matrices for which it does make sense are called invertible.

5.1. The definition of the inverse.

Definition 5.1. A matrix \( A \) is left invertible if there is a matrix \( B \) such that \( BA = I \). A matrix \( A \) is right invertible if there is a matrix \( C \) such that \( AC = I \). A matrix is invertible if it is both left and right invertible.

Note 5.2. (1) If \( A \) is both left and right invertible then the left and right inverses are equal:
   \[
   C = IC = (BA)C = B(AC) = BI = B.
   \]

In this case the inverse is unique, by the same argument. The left/right inverse is called the inverse of \( A \) and denoted \( A^{-1} \).
(2) If $A$ is left invertible, then $AD = AE$ does imply that $D = E$, since we can multiply both sides by the left inverse $B$:

$$BAD = BAE \implies ID = IE \implies D = E.$$  

(3) Similarly, for right invertible matrices $DA = EA$ implies $D = E$.

**Example 5.3.** The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is invertible with inverse

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}.$$  

More generally, any diagonal matrix with diagonal entries $a_{11}, \ldots, a_{nn}$ is invertible with inverse the diagonal matrix with entries $1/a_{11}, \ldots, 1/a_{nn}$.

**Example 5.4.** The matrix $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is invertible with inverse

$$A^{-1} = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}.$$  

**Example 5.5.** The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ has right inverse $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

There is a simple formula for the inverse of a $2 \times 2$ matrix. The determinant of a $2 \times 2$ matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$  

The determinant is non-zero if and only if the matrix is invertible; the a formula for the inverse is (check!)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$  

Later we’ll generalize this formula to larger matrices.

If $A$ is invertible then so are its square $A^2$ and its transpose:

$$(A^2)^{-1} = (A^{-1})^2$$  

since $A^2(A^{-1})^2 = AAA^{-1}A^{-1} = AIA^{-1} = I$, and

$$(A^T)^{-1} = (A^{-1})^T.$$  

Similarly, the inverse of $A^n$ is $(A^{-1})^n$, for any $n > 0$. If $A$ and $B$ are invertible then

$$(AB)^{-1} = B^{-1}A^{-1}.$$  

You can think of the reason for this in the following silly way. Suppose at the beginning of the day you put on your shoes (call this operation $A$) and tie your shoe laces (call this operation $B$). What do you do when you come home?

5.2. **Finding inverses via elimination.** The most efficient way of finding the inverse of a square matrix $A$ is via elimination. Consider the vector equation $Ax = y$. The inverse matrix $A^{-1}$ solves the equation $x = A^{-1}y$. So if we can express $x$ in terms of $y$, we can read off the coefficients to get the matrix $A^{-1}$.

Writing out the equations for $Ax = y$ gives

- $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = y_1$
- $a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = y_2$
- $\vdots$
- $a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = y_n$.

We now have a system of linear equations with variables on the right-hand side. Reading off the coefficients we get the matrix $A^{-1}$.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix},$$

or, for short, $[A][I]$. To solve for $x$, we do elimination. If, at the end, we get the identity matrix on the left-hand side, then the right-hand side is the inverse is the matrix on the right.

**Example 5.6.** To find the inverse of $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ we do elimination on

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}.$$
We divide the first row by 2 and the second by 3 to get the rref
\[
\begin{bmatrix}
1 & 0 & 1/2 & 0 \\
0 & 1 & 0 & 1/3
\end{bmatrix}.
\]
The inverse is \(A^{-1} = \begin{bmatrix} 1/2 & 0 \\
0 & 1/3 \end{bmatrix} \).

**Example 5.7.** Find the inverse of \(A = \begin{bmatrix} 2 & 3 \\
4 & 6 \end{bmatrix} \). We do elimination on
\[
\begin{bmatrix}
2 & 3 & 1 & 0 \\
4 & 6 & 0 & 1
\end{bmatrix}.
\]
We subtract twice the first row from the second to get
\[
\begin{bmatrix}
2 & 3 & 1 & 0 \\
0 & 0 & -2 & 1
\end{bmatrix}.
\]
The second equation is inconsistent: this matrix has no inverse.

**Example 5.8.** Find the inverse of \(A = \begin{bmatrix} 1 & 1 & 1 \\
1 & 2 & 3 \\
0 & 1 & 1 \end{bmatrix} \). We do elimination on the augmented matrix
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 3 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
\]
We subtract the first row from the second to get
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
\]
We subtract the second from the first and third to get
\[
\begin{bmatrix}
1 & 0 & -1 & 2 & -1 & 0 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 & -1 & 1
\end{bmatrix}.
\]
Multiply the third by \(-1\) to create a leading 1:
\[
\begin{bmatrix}
1 & 0 & -1 & 2 & -1 & 0 \\
0 & 1 & 2 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 & -1
\end{bmatrix}.
\]
Now add the third to the first, and subtract twice the third from the second to get
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & -1 \\
0 & 1 & 0 & 1 & -1 & 2 \\
0 & 0 & 1 & -1 & 1 & -1
\end{bmatrix}.
\]
The inverse is
\[
A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\
1 & -1 & 2 \\
-1 & 1 & -1 \end{bmatrix}.
\]

Notice that if \(A\) is invertible, then \(Ax = y\) has a unique solution for every \(y\). This implies that \(\text{rref}(A) = I\), since (1) if there were a row of zeroes, the system would be inconsistent for some values of \(y\), and (2) if column did not contain a leading 1, there would be free variables, so any solution would not be unique. So any invertible matrix is automatically square. Let’s summarize what we’ve shown so far:

**Theorem 5.9.** A matrix \(A\) is invertible if and only if \(A\) is square and \(\text{rref}(A) = I\). In this case, the inverse is the right hand side of the matrix \(\text{rref}([A|I])\).

It’s easy to check that a 2 \times 2-matrix \(A\) has \(\text{rref}(A) = I\) if and only if the determinant is non-zero.

### 5.3. Application: A formula for the line between two points.

There is a unique line through any two points \((x_1,y_1), (x_2,y_2)\). Let’s find a formula for it, using matrices. (Think for a moment about you would find a formula another way.) The equation for a line is \(f(x) = ax + b\). The two data points give
\[
a x_1 + b = y_1, \quad a x_2 + b = y_2
\]
or in matrix form
\[
\begin{bmatrix} x_1 & 1 \\
x_2 & 1 \end{bmatrix} \begin{bmatrix} a \\
b \end{bmatrix} = \begin{bmatrix} y_1 \\
y_2 \end{bmatrix}.
\]

The solution is
\[
\begin{bmatrix} a \\
b \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\
x_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\
y_2 \end{bmatrix}.
\]
Let’s find the inverse using the formula for two by two inverses. The determinant is \( x_1 - x_2 \), so the inverse is
\[
\begin{bmatrix}
  x_1 & 1 \\
  x_2 & 1 
\end{bmatrix}^{-1} = \frac{1}{x_1 - x_2} \begin{bmatrix}
  x_2 & -1 \\
  -x_1 & 1 
\end{bmatrix}.
\]

The solution is
\[
\begin{bmatrix}
a \\
b 
\end{bmatrix} = \frac{1}{x_1 - x_2} \begin{bmatrix}
y_1 - y_2 \\
y_1 x_2 + x_1 y_2 
\end{bmatrix}
\]
or
\[
f(x) = \frac{y_1 - y_2}{x_1 - x_2} x + \frac{x_1 y_2 - y_1 x_2}{x_1 - x_2}.
\]
Check that the slope and \( y \)-intercept make sense.

5.4. **Inverse of Elementary Matrices**. You don’t have to do elimination to compute the inverse of an elementary matrix:

**Proposition 5.10.** If \( E \) is an elementary matrix corresponding to a row operation, then \( E^{-1} \) is the elementary matrix corresponding to the opposite operation.

**Example 5.11.** The elementary matrix corresponding to \( \overrightarrow{3} \mapsto \overrightarrow{3} - 3\overrightarrow{1} \) is \( E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \). The opposite operation is \( \overrightarrow{3} \mapsto \overrightarrow{3} + 3\overrightarrow{1} \). So the inverse of \( E \) is
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = E^{-1}.
\]

The elementary matrix corresponding to \( \overrightarrow{1} \mapsto -2\overrightarrow{2} \) is \( E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \). The opposite operation is \( \overrightarrow{2} \mapsto -2/(-2) \). So the inverse of \( E \) is
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}.
\]

**5.5. Problems.**

1. True/False:
   (a) If \( A \) is invertible then \( A^T \) is invertible.
   (b) If \( A \) and \( B \) are matrices such that \( AB = I \) then \( A \) and \( B \) are invertible.
   (c) The inverse of an invertible upper triangular matrix is upper triangular.
   (d) If \( A \) is invertible then ref(\( A \)) is invertible.
   (e) The identity matrix \( I \) is invertible.
   (f) The inverse of a symmetric matrix is symmetric.

2. Find the inverse of
   \[
   A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}
   \]

3. Find the inverse of
   \[
   A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.
   \]
   (Matrices of this form are called upper triangular.) You might want to check your answer by multiplying \( A \) by \( A^{-1} \).

4. Let \( A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 0 \end{bmatrix} \). Compute (a) \( A^2 \) (b) \( A^{-1} \) and (c) \( A^T \).

5. Compute (a) \( A^2 \) (b) \( A^{-1} \) and (c) \( A^T \) for \( A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 1 & 0 \\ 1/3 & 0 & 0 \end{bmatrix} \).
(or \( A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix} \)).

(6) Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix} \). Find the inverse of \( A \), by row reduction.

(7) Prove that if a matrix \( A \) is invertible, then \( A^2 \) is also invertible.

6. LU Factorization

The row-echelon form of a matrix is always upper triangular, since the leading 1 in each row is to the right of the leading 1 above it. In this section, we call the row-echelon form \( U \), since it is upper triangular.

Suppose that \( A \) can be put into row-echelon form \( U \) by a sequence of row operations corresponding to elementary matrices \( E_1, \ldots, E_k \). Doing the row operation is the same as multiplying by the elementary matrix, by Theorem 4.10. So \( U \) is the matrix product

\[ U = E_k E_{k-1} \cdots E_1 A. \]

Example 6.1. Gaussian elimination on \( A = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix} \) is

\[
\begin{bmatrix} \frac{1}{2} \cdot 1 \\ -\frac{3}{1} \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The elementary matrices corresponding to these operations are

\[ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Therefore,

\[
\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

Check it for yourself.

A factorization of a matrix \( A \) is a formula for \( A \) as a product of matrices. We can take the inverses of the elementary matrices to get a factorization for \( A \):

(4) \[ A = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} U. \]

Example 6.2. A factorization for \( \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix} \) is

\[
\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 3 & 6 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

In general, it’s not that easy to take the product of the matrices \( E_1^{-1}, \ldots, E_k^{-1} \).

However, suppose that none of the operations are row switches. The row operations in the forward pass subtract rows from lower rows or multiply rows, so the elementary matrices \( E_1, \ldots, E_k \) in the forward pass, as well as their inverses, are lower triangular. Since the product of lower triangular matrices is lower triangular, the product \( L \) defined by

\[ L = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} \]

is also lower triangular. From this and (4) we get

\[ A = LU. \]

This is called an LU factorization of \( A \).

Proposition 6.3. If the row operations are in the order given by Gauss’s algorithm, then each entry in \( L \) is the unique non-zero entry in \( E_1^{-1} \ldots E_k^{-1} \).
Example 6.4. The row operations used to put $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & 6 & 9 \end{bmatrix}$ into reduced row-echelon form

$$\text{ref}(A) = U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

are order)

$\begin{array}{c} 2 \mapsto 2 - 2\overline{1} \\ 3 \mapsto 3 - 3\overline{1} \\ 2 \mapsto - \frac{\overline{2}}{2} \end{array}$.

The corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Their inverses are

$$E_{1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_{3}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Their product is

$$L = E_{1}^{-1} E_{2}^{-1} E_{3}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$ 

Let’s summarize the discussion in a theorem:

Theorem 6.5. If Gaussian elimination on $A$ does not involve any row switches, then there is an $LU$ factorization of $A$ where $U$ is the result of the forward pass in elimination. If elimination involves subtracting a scalar $c_{ij}$ times row $i$ from row $j$, and dividing row $i$ by a scalar $c_{ii}$, then $L$ is the lower triangular matrix whose $ij$-th entry is $c_{ij}$.

This simple way of getting $L$ only works if the elementary matrices are in order. For instance the product

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & -2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which is not equal to $L$. So in order to get the $LU$ factorization, you have to do the operations in the order that in Gauss’ algorithm.

On the other hand, if you are only trying to find the row echelon form by hand, it might be easier to use a different order of operations, for instance in order to avoid fractions.

More generally, if Gaussian elimination on a matrix $A$ involves row switches then all of the row switches can be done after the other row operations. This gives a factorization

$$A = LPU$$

where $P$ is the product of elementary matrices corresponding to the row switches.

Example 6.6. Gaussian elimination on $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix}$ with all the row switches at the end is

$\begin{array}{c} 2 \mapsto 2 - 2\overline{1} \\ 3 \mapsto 3 - 3\overline{1} \end{array}$

The upper triangular matrix $U$ is

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

The elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

Their inverses are

$$E_{1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_{3}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
\[ E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \]

To get \( L \) we take the product of the matrices corresponding to adding and multiplying row operations:

\[ L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}. \]

To get the \( P \) matrix we take the product of the matrices corresponding to the switches. In this case there is only one:

\[ P = E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \]

The \( LPU \) factorization of \( A \) is

\[ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}. \]

6.1. Problems.

(1) Find an \( LU \) factorization for \( A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \).

7. Determinants

Determinants are another way, besides finding the rref, of determining whether a matrix is invertible. The determinant is non-zero if and only if the matrix is invertible.

7.1. The definition of the determinant. Let \( A \) be a square \( n \times n \) matrix. A pattern in \( A \) is a choice of \( n \) entries from \( A \), so that one entry is chosen from each row and column. The product of the pattern is the product of chosen entries.

Example 7.1. The patterns in the matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) are

\[ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 6 \\ 3 \end{bmatrix}. \]

Each pair of entries in the pattern is either oriented southwest-northeast (SW-NE) or southeast-northwest (SE-NW). The pair is said to be an involution if it is oriented SW-NE. The sign of the pattern is

\[ \text{sign}(P) = (-1)^{\text{# involutions}}, \]

that is 1 if the number of involutions is even, and \(-1\) if the number of involutions is odd.

Example 7.2. \( ad \) is not inverted, \( bc \) is. The sign of \( ad \) is 1, the sign of \( bc \) is \(-1\).

The determinant of \( A \) is defined by

\[ \det(A) = \sum_{\text{patterns } P} (-1)^{\text{# involutions}(P)} \text{ product of entries}(P). \]

Example 7.3. The determinant of a \( 2 \times 2 \) matrix is \( \det(A) = ad - bc \).

We can ignore patterns that contain a zero, since these don’t contribute to the determinant.

Example 7.4. Find the determinant of \( A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} \). The non-zero patterns are

\[ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ (no involutions)}, \quad \begin{bmatrix} 4 \\ 5 \end{bmatrix} \text{ (3 involutions)} \quad \begin{bmatrix} 6 \end{bmatrix} \text{ (2 involutions)}. \]

So the determinant is

\[ \det(A) = (1)(4)(7) + (-1)^3(5)(4)(2) + (-1)^2(3)(6)(2) = 24. \]

Therefore, the matrix is invertible.

Example 7.5. Find the determinant of the upper triangular matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 \end{bmatrix}. \]

The only non-zero choice from the first row is 1. The only non-zero choice from the second row, that is not in the same column as 1, is 3.
In the same way, one sees that the only possible non-zero choices from the third and fourth rows are 6 and 10. There are no involutions in this pattern. Therefore, the determinant is
\[ \det(A) = (1)(3)(6)(10) = 180 \]
and the matrix is invertible. More generally, the same reasoning shows

**Theorem 7.6.** Let \( A \) be upper triangular, lower triangular, or diagonal. Then the determinant is the product of diagonal entries. Therefore, \( A \) is invertible if none of the diagonal entries are zero.

### 7.2. Determinants of Elementary Matrices

An elementary matrix \( E \) corresponding to a row operation which adds a row to another has 1’s along the diagonal, and is either upper or lower triangular. By Theorem 7.6, the determinant of \( E \) is 1.

If \( E \) is the elementary matrix corresponding to multiplying row \( i \) by \( c \), then \( E \) is diagonal so by the Theorem \( \det(E) \) is the product of diagonal entries
\[ \det(E) = c. \]

**Example 7.7.** Multiplying row 3 by 5 gives
\[
\begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{vmatrix}
\mapsto
\begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 5 \\
\end{vmatrix}
= E.
\]

which has determinant 5.

Suppose \( E \) is the elementary matrix corresponding to a switch of rows \( i \) and \( j \).

**Example 7.8.** Switching rows 2 and 4 gives
\[
\begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{vmatrix}
\leftrightarrow
\begin{vmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{vmatrix}
= E.
\]

\( E \) has exactly one non-zero entry in each column, and so a unique non-zero pattern. The inverted pairs are the \( ij \)-th pair, and the \( ik \)-th and \( jk \)-th pairs for \( k \) between \( i \) and \( j \). Therefore, the number of inverted pairs is
\[ \#2|i-j| + 1 \]
which implies that
\[ \det(E) = (-1)^{2|i-j|+1} = -1. \]

### 7.3. Properties of the determinant

1. (Transpose) Let \( A \) be a square matrix. Then \( \det(A) = \det(A^T) \).

For every pattern in \( A \) flips over into a transpose for \( A^T \), and vice-versa. For instance, \((2)(3)(6)\) is a pattern in both
\[
\begin{vmatrix}
1 & 0 & 2 \\
3 & 4 & 0 \\
5 & 6 & 7 \\
\end{vmatrix}
\text{ and } \begin{vmatrix}
3 & 4 & 0 \\
0 & 4 & 6 \\
2 & 0 & 7 \\
\end{vmatrix}.
\]

If a pair is oriented SW-NE before in \( A \), the flipped pair is also oriented SW-NE:
\[
\begin{vmatrix}
\circ & \circ \\
3 & 2 \\
\end{vmatrix}
\text{ and } \begin{vmatrix}
\circ & \circ \\
3 & 2 \\
\end{vmatrix}
\]

So the number of involutions in both patterns is the same. Since the determinant is the sum over patterns, with sign given by the number of involutions, this shows

2. (Switching Two Rows) Let \( B \) equal the matrix \( A \) with two rows switched. Then \( \det(B) = -\det(A) \).

**Example 7.9.**
\[
\begin{vmatrix}
1 & 0 & 2 \\
3 & 4 & 0 \\
5 & 6 & 7 \\
\end{vmatrix}
= -\det \begin{vmatrix}
3 & 4 & 0 \\
1 & 0 & 2 \\
5 & 6 & 7 \\
\end{vmatrix}.
\]

Every pattern in \( B \) corresponds to a pattern in \( A \) but the number of involutions is different. Say row \( i \) is switched with row \( j \). For every row \( k \) in between, the pair of entries in row \( i \) and row \( k \) switches from NE-SW to NW-SE or vice-versa. Similarly for the pair of entries in row \( j \) and row \( k \). The pair of entries in rows \( i,j \) also switches from NE-SW to NW-SE. As a result the number of involutions changes by 2 times the number of rows in
between, plus 1. So the sign \((-1)^{\#\text{involutions}}\) switches from + to −, or vice-versa.

(3) (Equal Rows) If \(A\) has two rows equal, \(\det(A) = 0\).

**Example 7.10.**

\[
\begin{vmatrix}
1 & 0 & 2 \\
3 & 4 & 0 \\
1 & 0 & 2
\end{vmatrix} = 0
\]

because the first and third rows are equal.

Let \(B\) be the matrix with the two rows switched, that is, \(B = A\). Then \(\det(A) = -\det(B) = \det(A)\) which can only happen if \(\det(A) = 0\).

(4) (Summing rows or columns) Let \(v, w\) be \(n\)-vectors. Let \(A, B, C\) be square matrices so that \(A, B, C\) are all equal except that one of the rows is \(v\) for \(A\), \(w\) for \(B\), and \(v + w\) for \(C\). Then \(\det(C) = \det(A) + \det(B)\).

Note it is *not* true that \(\det(A + B) = \det(A) + \det(B)\); it is *only true if rows or columns are added*.

This is best proved later, using cofactor expansion.

(5) If \(B\) is the matrix obtained by multiplying row \(i\) by \(c\), then \(\det(B) = c \det(A)\).

**Example 7.11.**

\[
\begin{vmatrix}
1 & 0 & 2 \\
9 & 12 & 0 \\
5 & 6 & 7
\end{vmatrix} = 3 \begin{vmatrix}
1 & 0 & 2 \\
3 & 4 & 0 \\
5 & 6 & 7
\end{vmatrix}
\]

because the second row has been multiplies by 3.

If \(B\) is obtained from \(A\) by multiplying every row by \(c\), then \(\det(B) = c^n \det(A)\). That is,

\[
\det(cA) = c^n \det(A).
\]

A common mistake is to forget the superscript \(n\). In particular, \(\det(-A)\) is *not equal to* \(-\det(A)\) unless the size \(n\) is odd.

(6) (Adding one row to another) If \(B\) is obtained from \(A\) by adding a multiple of one row to another, then \(\det(B) = \det(A)\).

This is a consequence of the previous two results: Say row \(i\) of \(A\) is \(v\), and row \(j\) is \(w\), and \(B\) has row \(i\) equal to \(v + cw\). Then \(\det(B) = \det(A) + c \det(C)\), where \(C\) is the matrix obtained by substituting \(w\) into row \(i\). But then \(C\) has two rows equal, so \(\det(C) = 0\).

**Example 7.12.** \(\det \begin{vmatrix}
1 & 0 & 2 \\
0 & 4 & -6 \\
5 & 6 & 7
\end{vmatrix} = \det \begin{vmatrix}
1 & 0 & 2 \\
3 & 4 & 0 \\
5 & 6 & 7
\end{vmatrix}\)

because three times the first row has been subtracted from the second.

Now we’re ready to show that the determinant determines whether the matrix is invertible.

**Theorem 7.13.** The following are equivalent for a square matrix \(A\):

1. \(\det(A) \neq 0\)
2. \(\text{rref}(A) = I\)
3. \(A\) is invertible.

**Proof.** We already proved (2) \(\iff\) (3). It remains to prove (1) \(\iff\) (2).

Suppose the in the course of reducing \(A\) to its reduced row-echelon form, we do a number of row operations, such as adding multiples of rows to other rows, multiplying rows by non-zero numbers \(c_1, \ldots, c_r\), and switching rows \(s\) times.

Under the first type of row operation, the determinant is unchanged. Under the second type, the determinant is multiplied by \(c_i\). Under the third, the determinant changes sign.

\[
\det(\text{rref}(A)) = (-1)^s c_1 \ldots c_r \det(A).
\]

This shows that \(\det(A)\) is non-zero if and only if \(\det(\text{rref}(A))\) is non-zero.

The \(\text{rref}(A)\) is upper triangular, by (3) in its definition (4.3). So its determinant is non-zero if and only if there are non-zero numbers along the diagonal. If \(\det(A)\) is non-zero, each of these must be a leading 1. But then the \(\text{rref}\) must equal the identity. \(\square\)

Using similar techniques we can show the following.

**Theorem 7.14.** For any square matrices \(A, B\), \(\det(AB) = \det(A) \det(B)\).
Proof. Look at the vector equation $Ax = ABy$. If $A$ is invertible, this equation has solution $x = By$. Suppose in the elimination $s$ rows get switched, and rows get multiplied by non-zero numbers $c_1, \ldots, c_r$. Then
\[ \det(A)(-1)^sc_1\ldots c_r = \det(I) = 1. \]
Since the same operations happen on the right,
\[ \det(AB)(-1)^sc_1\ldots c_r = \det(B). \]
But the left-hand side is $1/\det(A)$. □

Corollary 7.15. (1) If $A$ is invertible then $\det(A^{-1}) = 1/\det(A)$.

(2) $AB$ is invertible if and only if $A$ is invertible and $B$ is invertible.

(3) $A^n$ is invertible if and only if $A$ is invertible.

Proof. (a) $\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$. Now divide by $\det(A^{-1})$ on both sides. (a) $\det(AB) = \det(A)\det(B)$, so $\det(AB)$ is non-zero exactly if $\det(A)$ and $\det(B)$ are non-zero. (b) is similar. □

7.4. Elementary matrices and the determinant. (1) The determinant changes sign when rows are switched.

Proof: $\det(P_{ij}A) = \det(P_{ij})\det(A) = -\det(A)$, since $\det(P_{ij}) = -1$.

(2) The determinant is unchanged when a multiple of one row is added to another.

Proof: $\det(E_{ij}A) = \det(E_{ij})\det(A) = \det(A)$, since $\det(E_{ij})$ is the product of the diagonal entries which equals 1.

(3) The determinant is multiplied by $c$, if row $i$ is multiplied by $c$.

7.5. Problems.

(1) True or false:
   (a) For any square matrix $A$, $\det(-A) = -\det(A)$.
   (b) If $\det(A) = 0$, then $Ax = 0$ has infinite solutions.

(2) True/False:
   Suppose $A = [v_1|v_2|v_3]$ is the matrix with columns $v_1, v_2, v_3$. If $\det(A) = 2$, then
   (a) the determinant of the matrix $A' = [v_3 - v_1|v_2|v_1 - v_2]$ is $-2$.
   (b) the determinant of the matrix $A' = [v_2|v_3|v_1 - v_2]$ is $2$.
   (c) the determinant of the matrix $A' = [v_1 - v_2|v_2 - v_3|v_3 - v_1]$ is also $2$.

(3) Find the determinant of the matrix
\[
A = \begin{bmatrix}
  2 & -1 & 1 \\
  -1 & 0 & 3 \\
  2 & 1 & -4 \\
\end{bmatrix}
\]
   (a) by expanding along the second row;
   (b) by row-reducing $A$ to an upper triangular matrices and using the behavior of the determinant under elementary row operations.

(4) (a) Find the cofactor matrix for $A = \begin{bmatrix}
  1 & 2 & 3 \\
  1 & 0 & 1 \\
  1 & 1 & 0 \\
\end{bmatrix}$.
   (b) Find the determinant of $A$, by expanding along the third column.
   (c) Find the inverse of $A$, using parts (a),(b).
   (d) Find the inverse of $A$, by row reduction.

(5) (Strang) Compute the determinant of
\[
A = \begin{bmatrix}
  a & a & a \\
  a & b & b \\
  a & b & c \\
\end{bmatrix}
\]
   using row reduction.

(6) Compute the determinant of
\[
A = \begin{bmatrix}
  1 & 2 & 3 \\
  4 & 4 & 4 \\
  5 & 6 & 7 \\
\end{bmatrix}
\]
   using the cofactor formula.

(7) Let $A = \begin{bmatrix}
  1 & 2 & 3 & 0 \\
  4 & 5 & 6 & 0 \\
  7 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 \\
\end{bmatrix}$. Find the determinant of $A$, by expanding along the third column.

(8) Show that a square matrix $A$ is invertible, if and only if $\det(A) \neq 0$. 

8. Cofactors

In this section we want to explain how the formula for $2 \times 2$ matrices
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]
generalizes to bigger size.

8.1. Cofactors. Let $A$ be a square $n \times n$ matrix. Let $M_{ij}$ denote the matrix $A$ with row $i$ and column $j$ deleted. $M_{ij}$ is called the $ij$-th minor of $A$.

Example 8.1. The 13-th minor of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is $M_{13} = \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$.

The number
\[ A_{ij} = (-1)^{i+j} \det(M_{ij}) \]
is the $ij$-th cofactor of $A$. The signs $(-1)^{i+j}$ are given by the table of alternating signs, for example, for $3 \times 3$ the table is
\[
\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.
\]

Example 8.2. The 13 cofactor of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is
\[ A_{13} = (-1)^{1+3} \det(\begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}) = +(0-8) = -8. \]

Example 8.3. The cofactors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $d, -c, -b, a$.

Here is the reason we are interested in cofactors:

Theorem 8.4. For any $i$, the dot product of the $i$-th row of $A$ with the cofactors for the $i$-th row is $\det(A)$. If $i$ is not equal to $j$, the dot product of the $i$-th row of $A$ with the cofactors for the $j$-th row is equal to 0. That is,
\[ a_{i1}A_{11} + \ldots + a_{in}A_{in} = \det(A), \quad a_{i1}A_{j1} + \ldots + a_{jn}A_{jn} = 0. \]

Before we explain the theorem, here is an example.

Example 8.5. The cofactors for the first row of $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ are
\[ + (28-0) = 28, -(0-12) = 12, +(0-8) = -8. \]
The dot product of $[28 12 -8]$ with the rows of $A$ are
\[ 28(1)+12(3)-8(5) = 24, \quad 28(0)+12(4)-8(6) = 0, \quad 28(2)+12(0)-8(7) = 0. \]

Proof. Now we prove the theorem. Each pattern in $A$ contains exactly one element in row $i$, say $a_{ik}$. The remaining chosen entries form a pattern in $M_{ij}$. So the product of entries appears in the sum (5). Conversely, any pattern in $M_{ij}$ defines a pattern in $A$, by adding the entry $a_{ij}$. So the terms in the sum
\[ a_{i1} \det(M_{i1}) + \ldots + a_{in} \det(M_{in}) \]
are the same as those that appear in $\det(A)$.

It remains to explain the sign $(-1)^{i+j}$. The number of involutions in the pattern in $A_{ij}$ is the number of involutions in the pattern in $M_{ij}$, plus the number of involutions of pairs containing $a_{ij}$. Let’s compute $v$. The matrix $M_{ij}$ is naturally broken up into 4 parts: the entries that lie $NE, NW, SE, SW$ of $a_{ij}$. We have
\[ v = \#NE \text{ entries} + SW \text{ entries}. \]

Since there is only one chosen entries in each row and column
\[ \#NE \text{ entries} = (i-1)-\#NW \text{ entries}, \quad \#SW \text{ entries} = (j-1)-\#NW \text{ entries}, \]
So
\[ v = i+j - 2 - 2\#NW \text{ entries} \]
which implies
\[ (-1)^v = (-1)^{i+j}. \]
This proves the first part of (5).

Now suppose we take the dot product of row $j$ in $A$ with the cofactors for row $i$. Let $B$ be the matrix obtained from $A$ by replacing row $i$ with row $j$. The cofactors for row $i$ are the same for both $B$ and $A$. The dot product of the $j$-th row of $A$ with the cofactors, is the same as the dot product of the $i$-th row of $B$, with the cofactors. By the first part of
(5), applied to $B$, the result is $\det(B)$. But since $B$ has two rows equal, $\det(B) = 0$. \qed 

8.2. **Cofactor expansion of the determinant.** The first formula in (5) is called the **cofactor expansion** of the determinant along row $i$. For instance, suppose we want to compute the determinant of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}. $$

We choose a row along which to expand, say the second. We take each entry in the row, and multiply by the determinant of the corresponding minor, with the appropriate sign from the table of signs:

$$\det(A) = -0 \det \begin{bmatrix} 3 & 5 \\ 0 & 7 \end{bmatrix} + 4 \det \begin{bmatrix} 1 & 5 \\ 2 & 7 \end{bmatrix} - 6 \det \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = 24. $$

The same thing works for any column. For any $i$, the dot product of the $i$-th column of $A$ with the cofactors for the $j$-th column is $\det(A)$ if $i = j$, and 0 otherwise. These are the same as the entries of the matrix $\det(A)I$.

**Example 8.6.** Suppose we want to find the determinant of the $4 \times 4$ matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 0 \end{bmatrix}. $$

There is only one non-zero entry in the fourth column; therefore it’s best to expand along that column. We get

$$\det(A) = -0 + 4 \det \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 5 & 6 & 7 \end{bmatrix} = -0 + 0. $$

Now there is only one non-zero entry in the third column, so we expand along it:

$$\det(A) = 4(0 - 0 + 7 \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}) = 28(1(4) - 2(3)) = -56. $$

The **adjoint** of $A$ is the transpose of the matrix of cofactors. That is, the $ij$-the entry of $\text{adj}(A)$ is the $ji$-th cofactor $A_{ji}$.

**Example 8.7.** The adjoint of $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is the matrix

$$\text{adj}(A) = \begin{bmatrix} 28 & -21 & -2 \\ 12 & -3 & -6 \\ -8 & 6 & 4 \end{bmatrix}. $$

8.3. **The cofactor formula for the inverse.** Here is the promised formula for the inverse:

**Theorem 8.8.** $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$.

**Proof.** It suffices to show that $A \text{adj}(A) = \det(A)I$. The dot product of the $i$-th row of $A$ with the $j$-th column of $\text{adj}(A)$ is the dot product of the $i$-th row of $A$ with the cofactors for the $j$-th row. By (5), this equals $\det(A)$ if $i = j$, and 0 otherwise. These are the same as the entries of the matrix $\det(A)I$. \qed 

**Example 8.9.** The inverse of $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 4 & 6 \\ 2 & 0 & 7 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{24} \begin{bmatrix} 28 & -21 & -2 \\ 12 & -3 & -6 \\ -8 & 6 & 4 \end{bmatrix}. $$

The cofactor formula is particularly useful when there are unknowns in the matrix.

**Example 8.10.** Find the inverse of $A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$. Since $A$ is upper triangular, the determinant is the product of diagonal entries $\det(A) = a^3$. The adjoint is

$$\text{adj}(A) = \begin{bmatrix} a^2 & ab & b^2 - ac \\ 0 & a^2 & ab \\ 0 & 0 & a^2 \end{bmatrix}. $$

The inverse is

$$A^{-1} = \frac{1}{a^3} \begin{bmatrix} a^2 & ab & b^2 - ac \\ 0 & a^2 & ab \\ 0 & 0 & a^2 \end{bmatrix}. $$
Sometimes when the matrices contain unknowns it’s easier to find the determinant using the row operations.

Example 8.11. Suppose we want to the unique polynomial passing through \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\). We write \(f(x) = a + bx + cx^2\). The equations
\[
a + bx_1 + cx_1^2 = y_1, \quad a + bx_2 + cx_2^2 = y_2, \quad a + bx_3 + cx_3^2 = y_3
\]
can be written in matrix form
\[
\begin{bmatrix}
1 & x_1 & x_1^2 \\
1 & x_2 & x_2^2 \\
1 & x_3 & x_3^2
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= 
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}.
\]
The determinant of the matrix is
\[
\det(\begin{bmatrix}
1 & x_1 & x_1^2 \\
1 & x_2 & x_2^2 \\
1 & x_3 & x_3^2
\end{bmatrix}) = \det(\begin{bmatrix}
1 & x_1 & x_1^2 \\
0 & x_2 - x_1 & x_2^2 - x_1^2 \\
0 & x_3 - x_1 & x_3^2 - x_1^2
\end{bmatrix})
\]
since subtracting the first row from the second and third does not change the determinant. Multiplying the second by \(1/(x_2 - x_1)\) and subtracting \((x_3 - x_1)\) times the second row from the third gives
\[
(x_2 - x_1) \det(\begin{bmatrix}
1 & x_1 & x_1^2 \\
0 & 1 & x_1^2/(x_2 - x_1) \\
0 & 0 & (x_3^2 - x_1^2)/(x_2 - x_1)
\end{bmatrix}).
\]
So the determinant is
\[
(x_2 - x_1)((x_3^2 - x_1^2) - (x_3 - x_1)(x_2^2 - x_1^2))/(x_2 - x_1)
\]
\[
= (x_2 - x_1)((x_3 - x_1)(x_3 + x_1) - (x_3 - x_1)(x_2 + x_1))
\]
\[
= (x_2 - x_1)((x_3 - x_1)(x_3 + x_1) - (x_3 - x_1)(x_2 - x_1))
\]
\[
= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).
\]
This is called a Vandermonde determinant. You can easily guess how it generalizes to higher size.

9. Geometry of Determinants

A rotation matrix is a matrix of the form
\[
R = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}.
\]

Proposition 9.1. If \(v\) is any 2-vector, then \(Rv\) is the rotation of \(v\) around 0 by angle \(\theta\).

Proof. Write \(v = v_1e_1 + v_2e_2\). Linear combinations are preserved under rotation; so the rotation of \(v\) by \(\theta\) is \(v_1\) times the rotation of \(e_1\) by \(\theta\) plus \(v_2\) times the rotation of \(e_2\) by \(\theta\).

The rotation of \(e_1, e_2\) by \(\theta\) are the vectors
\[
Re_1 = \begin{bmatrix}
\cos(\theta) \\
\sin(\theta)
\end{bmatrix}, \quad Re_2 = \begin{bmatrix}
-sin(\theta) \\
\cos(\theta)
\end{bmatrix}
\]
so the rotation of \(v\) is
\[
v_1Re_1 + v_2Re_2 = Rv.
\]

Lemma 9.2. The determinant of \(R\) is 1.

Proof. \(\det(R) = \cos^2(\theta) + \sin^2(\theta) = 1\).

Let \(A\) be a 2 matrix with columns \(v_1\) and \(v_2\).

Proposition 9.3. The absolute value \(\det(A)\) of the determinant of \(A\) is the area of the parallelogram with edge vectors \(v_1\) and \(v_2\).
Proof. Choose a rotation matrix $R$ so that $R\mathbf{v}_1$ lies on the $x$-axis. Since $R$ is either a rotation or a reflection, the area of the parallelogram spanned by the columns of $RA$

\[ w_1 = R\mathbf{v}_1, \quad w_2 = R\mathbf{v}_2 \]

is the same as that of $\mathbf{v}_1, \mathbf{v}_2$. Since $w_1$ is on the $x$-axis,

\[ w_1 = \begin{bmatrix} a \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} b \\ c \end{bmatrix} \]

the area of the parallelogram is $|ac|$.

On the other hand, \( \det(RA) = ac \). Since $\det(RA) = \det(R) \det(A) = \det(A)$, this completes the proof. \( \square \)

**Example 9.4.** Find the area of the parallelogram $P$ with vertices at $[1 \ 0], [0 \ 2], [0 \ -1]$, and $[-1 \ 1]$.

The edge vectors are \( \mathbf{v}_1 = [1 \ 0] - [0 \ 2] = [1 \ -2], \quad \mathbf{v}_2 = [0 \ -1] - [0 \ 2] = [0 \ -3] \)

so

\[ A = \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} \]

which has determinant $-3$. Therefore the area of the parallelogram is $3$.

**Example 9.5.** Find the area of the triangle $T$ with vertices $[1 \ 0], [0 \ 2], [0 \ -1]$.

The area of the triangle is half that of the parallelogram, or $\text{area}(T) = 3/2$.

This formula generalizes to $n$-vectors (in particular, to $n = 3$) as follows. A parallelopiped in $\mathbb{R}^n$ is a set of vectors

\[ P = \{ \mathbf{v}_0 + c_1 \mathbf{v}_1 + \ldots + c_n \mathbf{v}_n, \ 0 \leq c_1, \ldots, c_n \leq 1 \} \]

**Example 9.6.** The parallelopiped with edge vectors $[1 \ 1 \ 0], [1 \ 0 \ 2], [0 \ 2 \ 1]$ is drawn.

**Proposition 9.7.** If $A$ is the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and $P$ is the parallopiped with edges $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then

\[ \det(A) = |(P)|. \]

**Proof.** We can rotate the vector so that the vector $\mathbf{v}_1$ lies on the $x$ axis, the vector $\mathbf{v}_2$ lies in the $xy$ plane, etc. (See the section on QR factorization.) This doesn’t change the volume, nor the determinant of $A$. Then $A$ is upper triangular, so the determinant is the product of the diagonal entries $a_1, \ldots, a_n$.

Let $P(j)$ denote the parallelopiped in $\mathbb{R}^j$ whose edge vectors are $\mathbf{v}_1, \ldots, \mathbf{v}_j$. Then $P(j)$ is the base of $P(j + 1)$ and

\[ \text{area}(P(j)) = \text{base} \cdot \text{height} = \text{area}(P(j - 1))|a_j| \]
so
\[ \text{area}(P(n)) = a_1 \ldots a_n = |\det(A)|. \]

Example 9.8. The parallelopiped with edge vectors \([1 \ 1 \ 0], [1 \ 0 \ 2], [0 \ 2 \ 1]\) has volume
\[ |\det(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix})| = | -4 - 1 | = 5. \]

A \(n\)-simplex is a set of vectors of the form
\[ S = \{ v_0 + c_1 v_1 + \ldots + c_n v_n, \quad 0 \leq c_1, \ldots, c_n \leq 1, \quad \sum_{i=1}^{n} c_i \leq 1 \}. \]

A 2-simplex is just a triangle.

Example 9.9. If \(v_0 = [1 \ 1], v_1 = [1 \ 0], v_2 = [0 \ 1]\) \(S\) is the triangle with vertices at \((1,1), (1,2), (2,1)\).

Proposition 9.10. The volume of an \(n\)-simplex \(S\) with edge vectors \(v_1, \ldots, v_n\) is
\[ (S) = \frac{1}{n!} |\det(A)| \]
where \(A\) is the matrix with columns \(v_1, \ldots, v_n\).

Proof. This is the same proof as 9.7, except that the volume of the simplex \(S(j)\) with edge vectors \(v_1, \ldots, v_j\) is related to the volume of \(S(j-1)\) by
\[ (S(j)) = \frac{\text{baseheight}}{\text{dimension}} = \frac{1}{j} (S(j-1)) a_j. \]

Example 9.11. The triangle with vertices \((1,1), (3,2), (2,3)\) has edge vectors
\(v_1 = [3 \ 2] - [1 \ 1] = [2 \ 1], \quad v_2 = [2 \ 3] - [1 \ 1] = [1 \ 2], \quad \text{area} \)
\[ (T) = 1 \frac{2 |\det(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix})| = 3/2. \]

9.1. Problems.

Find the area of the triangle with vertices (a) \([1 \ 1], [ -2 \ 0]\) and
\([ -2 \ -1]\) (b) \([1 \ 3], [2 \ 4]\) and \([4 \ 2]\)

10. Linear transformations

10.1. Definition of a linear transformation. A function from \(\mathbb{R}\) to \(\mathbb{R}\) assigns to any real number \(x\) another real number \(f(x)\).

A map \(T\) from \(\mathbb{R}^n\) to \(\mathbb{R}^m\) is similar, but it assigns to any vector \(x\) in \(\mathbb{R}^n\) a vector \(T(x)\) in \(\mathbb{R}^m\), called the value of \(T\) at \(x\).

For example,
\[ T_1[x_1 \ x_2] = [5x_1 + 2x_2 \ 3x_1 - x_2] \]
\[ T_2[x_1 \ x_2] = [x_1^2 - x_2^2 \ x_1^2 + x_2^2] \]
\[ T_3[x_1 \ x_2] = [x_1 + 2x_2 - 3] \]
are all maps from \(\mathbb{R}^2\) to \(\mathbb{R}^2\).

A map from \(\mathbb{R}^n\) to \(\mathbb{R}^m\) is a linear transformation if it preserves vector addition and scalar multiplication, that is, if
\begin{itemize}
  \item[(1)] \(T(x + y) = T(x) + T(y)\), for all \(x, y \in \mathbb{R}^n\);
  \item[(2)] \(T(cx) = cT(x)\), for all \(x \in \mathbb{R}^n, c \in \mathbb{R}\).
\end{itemize}

These conditions can be combined into a single condition, that \(T\) preserves linear combinations, that is,
\[ T(x + dy) = cT(x) + dT(y). \]

Example 10.1. Of the three maps \(\mathbb{R}^2\) to \(\mathbb{R}^2\) above, only the \(T_1\) is a linear transformation. In fact, if \(A\) is the matrix \(\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}\) then \(T_1(x) = Ax\). So
\[ T(cx + dy) = A(cx + dy) = cAx + dAy = cT(x) + dT(y). \]

More generally, any map \(T(x)\) of the form \(T(x) = Ax\) is a linear transformation.

The map \(T_2\) fails because, for example,
\[ T(2[3 \ 0]) = T([6 \ 0]) = [36 \ 36] \]
but

\[ 2T([3\ 0]) = 2[9\ 9] = [18\ 18]. \]

The map \( T_3 \) fails because, for example,

\[ T_3[1\ 0] + T_3[2\ 0] = [3\ 0] + [4\ 0] = [7\ 0] \]

but \( T_3[3\ 0] = [5\ 0] \).

### 10.2. Examples of linear transformations in two dimensions.

Let’s look at some examples of linear transformations in \( \mathbb{R}^2 \).

(1) Let \( L \) be a line in \( \mathbb{R}^2 \) passing through 0. For any \( x \in \mathbb{R}^2 \), define \( P(x) \) to be the vector whose head is the closest point to the head of \( x \) in \( L \).

\( P \) is orthogonal projection onto \( L \). Let’s check graphically that it is a linear transformation:

(Figure)

(2) Let \( L \) be a line in \( \mathbb{R}^2 \) passing through 0. For any vector \( x \in \mathbb{R}^2 \), define \( S(x) \) to be the reflection of \( x \) over \( L \). Then \( S \) is a linear transformation \( \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

10.3. Linear transformations are matrices.

**Theorem 10.2.** Any linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is of the form \( T(x) = Ax \) for some \( m \times n \) matrix \( A \), called the matrix for the linear transformation.

**Proof.** Define \( e_1 = [1\ 0\ldots\ 0], e_2 = [0\ 1\ 0\ldots\ 0], \ldots, e_n = [0\ 0\ldots\ 0\ 1] \). Define \( A \) to be the matrix whose columns are \( Ae_1, \ldots, Ae_n \). For any vector \( x \) we have

\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = \begin{pmatrix}
  x_1 \\
  0 \\
  \vdots \\
  0
\end{pmatrix} + \ldots + \begin{pmatrix}
  0 \\
  \vdots \\
  0
\end{pmatrix} = x_1 e_1 + \ldots + x_n e_n.
\]

Since \( T \) preserves linear combinations

\[
T(x) = T(x_1 e_1 + \ldots + x_n e_n) = x_1 T(e_1) + \ldots + x_n T(e_n) = [T(e_1) \ldots T(e_n)]x = Ax.
\]

10.4. Examples in two dimensions. Let \( L \) be the line with slope 1 in \( \mathbb{R}^2 \), passing through 0. Let’s find the matrix for projection onto \( L \).
The closest vector to \( e_1 \) in \( L \) is \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). The closest vector to \( e_2 \) is the same vector. Therefore,

\[
P(x) = Ax, \text{ where } A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.
\]

Suppose we want to now find the closest vector to \( [5 \ 2] \) in \( L \). We multiply by \( A \) to get

\[
A \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix}.
\]

The matrix for reflection is similar. The reflection of \( e_1 \) through \( L \) is \( e_2 \), and the reflection of \( e_2 \) through \( L \) is \( e_1 \). So the matrix for reflection is

\[
A = [e_2 \ e_1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

Now let \( R \) be rotation around 0 by angle \( \theta \). We have

\[
R(e_1) = \begin{bmatrix} \cos(\theta) \\ -\sin(\theta) \end{bmatrix}, \quad R(e_2) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.
\]

So the matrix for \( R \) is

\[
A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.
\]

Which of these matrices are invertible? Let’s compute their determinants.

\[
\det \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = 1/4 - 1/4 = 0.
\]

\[
\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0 - 1 = -1.
\]

\[
\det \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1.
\]

In general, reflections and rotations are invertible; projections are not. In fact, the inverse of a reflection is just the same reflection, since \( S(S(x)) = x \). The inverse of a rotation by \( \theta \) is rotation by \(-\theta\).

10.5. The matrix of a composition is the matrix product. If \( T_1 : \mathbb{R}^n \to \mathbb{R}^m \) and \( T_2 : \mathbb{R}^m \to \mathbb{R}^p \) are maps, the composition is the map

\[
T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^p, \quad x \to T_2(T_1(x)).
\]

If \( T_1 \) and \( T_2 \) are linear, then so is \( T_2 \circ T_1 \):

\[
T_2(T_1((cx + dy)) = T_2(cT_1(x) + dT_2(y)) = cT_2(T_1(x)) + dT_2(T_1(y)).
\]

Proposition 10.3. Let \( T_1 \) and \( T_2 \) be linear transformations from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) with matrices \( A_1 \) and \( A_2 \). Then the matrix for \( T_2 \circ T_1 \) is \( A_2A_1 \).

Example 10.4. Suppose that \( P \) is orthogonal projection onto a line \( L \). Since \( P(v) \) already has been projected, \( P(P(v)) = v \), that is, \( P \circ P = P \). If \( A \) is the matrix for \( P \), then \( A^2 = A \).

Example 10.5. Suppose \( S \) is reflection over a line \( L \) in \( \mathbb{R}^2 \). Then \( S(S(v)) = v \), that is, the reflection reflects back to the original vector. If \( A \) is the matrix for \( S \), then \( A^2 = I \).

Example 10.6. Suppose \( R_\theta \) is rotation by \( \theta \), and \( R_\varphi \) is rotation by \( \varphi \). The composition is rotation by \( \theta + \varphi \),

\[
R_\theta \circ R_\varphi = R_{\theta + \varphi}.
\]

We get

\[
\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} = \begin{bmatrix} \cos(\theta + \varphi) & -\sin(\theta + \varphi) \\ \sin(\theta + \varphi) & \cos(\theta + \varphi) \end{bmatrix}.
\]

The left hand side is

\[
\begin{bmatrix} \cos(\theta)\cos(\varphi) & -\sin(\theta)\sin(\varphi) \\ \sin(\theta)\cos(\varphi) & \cos(\theta)\sin(\varphi) \end{bmatrix}.
\]

Equating the entries of the matrices we get the angle-sum formulas.

\[
\cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)
\]

\[
\sin(\theta + \varphi) = \cos(\theta)\sin(\varphi) + \sin(\theta)\cos(\varphi).
\]

Example 10.7. Derive a formula for the cos of \( 3\theta \), using the same method.
10.6. Problems.

(1) Determine whether the following maps \( T \) are linear transformations. If \( T \) is linear, find the matrix \( A \) such that \( T[x] = Ax \). If \( T \) is not linear, explain why.

(a) \( T[x_1, x_2, x_3] = [x_1 - x_3, x_2 - x_3, x_1 - x_2] \).

(b) \( T[x_1, x_2, x_3, x_4] = [x_1 + 1, x_2 + x_1 - 3] \).

(c) \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is rotation counterclockwise by 45 degrees, about the point \((0, 0)\).

(d) \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), where \( T[x] \) is the cross-product of \( x \) with the vector \([-1, 0, 1] \).

(e) \( T[x_1, x_2] = [x_1, 1/x_2] \).

11. Subspaces

Recall that \( \mathbb{R}^n \) is the set of all \( n \)-vectors. A subspace \( V \) of \( \mathbb{R}^n \) is a subset that satisfies the following three properties:

1. \( V \) contains the zero vector 0.
2. \( V \) is closed under vector addition: if \( v, w \) are in \( V \) then \( v + w \) is also in \( V \).
3. \( V \) is closed under scalar multiplication: if \( v \) is in \( V \) and \( c \) is a scalar then \( cv \) is also in \( V \).

Properties (b) and (c) are equivalent to saying that \( V \) is closed under linear combination: if \( v, w \) are in \( V \) and \( c, d \) are scalars then \( cv + dw \) is also in \( V \).

Property (a) is equivalent to saying that \( V \) is non-empty. This is because if \( V \) contains at least one vector \( v \), then it also has to contain \(-v\), by (c) and so contain \( v + (-v) = 0 \), by (b). Usually, when we check that a subset \( V \) is a subspace, we will only verify properties (b) and (c), since (a) is usually obvious.

Example 11.1. The set \( V \) of all vectors of the form \([x \ x^2] \) is not a subspace, because it is not closed under scalar multiplication. For \( v = [2 \ 4] \) is in \( V \), but \( 4v = [1 \ 2] \) is not.

Example 11.2. The set \( V \) of all vectors of the form \([x \ 5x] \) is a subspace. In fact, if \( A \) is the matrix \([-5 \ 1] \), then \( V \) is the set of vectors such that \( Av = 0 \).

More generally,

Definition 11.3. For any matrix \( A \), the nullspace of \( A \) is the set of all vectors \( v \) such that \( Av = 0 \).

Lemma 11.4. For any matrix \( A \), the nullspace of \( A \) is a subspace.

Proof. We have to check that nullspace(\( A \)) is closed under linear combinations: Assume that \( v, w \) are in nullspace(\( A \)). By definition \( Av = Aw = 0 \), so

\[ A(cv + dw) = cAv + dAw = 0. \]

This implies that \( cv + dw \) is also in \( V \).

Example 11.5. The set \( V \) of all vectors of the form \([x \ 2x + 1] \) is not a subspace, even though there are no higher order terms. \( V \) is closed under neither scalar multiplication nor vector addition; for instance, \([0 \ 1] \) is in \( V \), but twice it, \([0 \ 2] \) is not.

Example 11.6. The set \( V \) of all vectors of the form \([x \ y] \) with \( x, y \geq 0 \) is closed under vector addition, but not scalar multiplication by negative numbers, so it is not a subspace.

Example 11.7. The set \( V \) of all vectors that are either in the \( x \)-axis or the \( y \)-axis, that is, \([x \ y] \) such that either \( x \) or \( y \) is zero, is closed under scalar multiplication but not vector additions, so it is not a subspace.

11.1. Properties of Subspaces.

Definition 11.8. If \( V \) and \( W \) are subspaces their intersection \( V \cap W \) is the set of all vectors \( v \) that are in both \( V \) and \( W \). The union \( V \cup W \) is the set of vectors that are in either \( V \) or in \( W \). The sum \( V + W \) is the set of vectors of the form \( v + w \), for some \( v \) in \( V \) and \( w \) in \( W \).

Example 11.9. If \( V \) is the \( x \)-axis and \( W \) is the \( y \)-axis in \( \mathbb{R}^3 \), then \( V \cap W \) is just the origin, a single point; \( V \cup W \) is the union of the two axis; \( V + W \) is the \( xy \)-plane.

Theorem 11.10. (1) Any subspace \( V \) must contain 0.

(2) The intersection \( V \cap W \) of two subspaces \( V, W \) is a subspace.

(3) The union \( V \cup W \) of two subspaces \( V, W \) is not in general a subspace.

(4) The sum \( V + W \) of two subspaces \( V, W \) is a subspace.
Proof. (1) Take any vector \( v \) in \( V \) and multiply by \( c = 0 \). Since \( V \) is closed under scalar multiplication, \( cv = 0v = 0 \) is also in \( V \). (2) If \( v, w \) are in both \( V \) and \( W \), then \( cv + dw \) is in \( V \) and in \( W \), and so in \( V \cap W \). (3) See the example above. (4) An element in \( V + W \) is of the form \( v + w \) for some \( v, w \). Any scalar multiple \( c(v + w) = cv + cw \) is also of this form, so \( V + W \) is closed under scalar multiplication. We have to show that if we take two elements of \( V + W \), they sum to another element. Suppose the second element is \( v' + w' \). Then \( v + w + (v' + w') = (v + v') + (w + w') \) which is also in \( V + W \). □

11.2. Problems.

(1) Construct a matrix whose nullspace consists of all combinations of \((1, 1, 1, 0)\) and \((-1, 1, 0, 1)\).

(2) Construct a 2x3 matrix whose column space contains \([1 \ 2]\) and whose null-space contains \([1 \ 0 \ 1]\).

12. Span and Linear Independence

12.1. Span. Let \( v_1, \ldots, v_r \) be vectors in \( \mathbb{R}^n \). The span of \( v_1, \ldots, v_r \) is the set of linear combinations

\[ c_1v_1 + \cdots + c_rv_r. \]

Example 12.1. The span of a single non-zero vector \( v \) is the set of all \( cv \), that is, the line through \( v \).

Example 12.2. The span of the vectors \([1 \ -1 \ 0]\) and \([0 \ -1 \ 1]\) is the set of all combinations

\[ a[1 \ -1 \ 0] + b[0 \ -1 \ 1] = [a - a - b b]. \]

Any vector of this form has \( x + y + z = 0 \). Conversely, any vector with \( x + y + z = 0 \) can be written as

\[ [x - x - y z] = a[1 \ -1 \ 0] + b[0 \ 1 \ -1] \]

where \( a = x \) and \( b = y \). So the span is the plane \( x + y + z = 0 \).

Proposition 12.3. The span of any set of vectors is a subspace.

Proof. Closed under +: \((c_1v_1 + \cdots + c_rv_r) + (d_1v_1 + \cdots + d_rv_r) = (c_1 + d_1)v_1 + \cdots (c_r + d_r)v_r\). Closer under \( : k(c_1v_1 + \cdots + c_rv_r) = (kc_1)v_1 + \cdots (kc_r)v_r\).

Here is the algorithm for checking whether a set \( v_1, \ldots, v_r \) spans \( \mathbb{R}^n \): Write the equation

\[ c_1v_1 + \cdots + c_nv_n = v \]

in matrix form. We want to know whether it always has a solution. This is equivalent to showing that the row-echelon form has no rows of zeros.

Example 12.4. Determine whether \([1 \ -1 \ 0],[1 \ 0 \ -1],[0 \ 1 \ -1]\) span \( \mathbb{R}^3 \).

Proposition 12.5. If \( v_1, \ldots, v_r \) spans \( \mathbb{R}^n \), then \( r \) must be at least \( n \).

12.2. Linear independence.

Definition 12.6. Vectors \( v_1, \ldots, v_r \) are linearly independent (or independent, for short) if no vector in the list is a combination of the others. If \( v_1, \ldots, v_r \) are not independent, they are dependent.

Example 12.7. Two vectors are independent if and only if they are not proportional. For example, \([-1 \ 0 \ 1] \) and \([-2 \ 0 \ 2]\) are dependent, because \([-2 \ 0 \ 2] = 2[-1 \ 0 \ 1]\). But \([-1 \ 0 \ 1]\) and \([2 \ 0 \ 2]\) are independent.

Example 12.8. Three vectors are independent of none of the vectors lies in the plane spanned by the other two. For example, \([1 \ -1 \ 0],[0 \ -1 \ 1],[1 \ 0 \ -1]\) is dependent, because \([1 \ 0 \ -1] = [1 \ -1 \ 0] - [0 \ -1 \ 1]\) lies the plane spanned by \([1 \ -1 \ 0],[0 \ -1 \ 1]\).

Here are some equivalent definitions:

Theorem 12.9. Vectors \( v_1, \ldots, v_r \) are dependent (that is, not independent) if and only if there is a subset that has the same span as \( v_1, \ldots, v_r \).

Definition 12.10. A dependence relation on \( v_1, \ldots, v_r \) is a collection of scalars \( c_1, \ldots, c_r \) not all zero such that \( c_1v_1 + \cdots + c_rv_r = 0 \).

Theorem 12.11. Vectors \( v_1, \ldots, v_r \) are independent if and only if there is no dependence relation on them.

Example 12.12. For what values of \( c \) are the vectors \([0 \ 0 \ -1],[1 \ 1 \ 2],[1 \ 1 \ c]\) independent?

Example 12.13. For what values of \( c \) are the vectors \([0 \ 1 \ -1],[1 \ 1 \ 2],[1 \ 1 \ c]\) independent?
Here is the algorithm for checking whether a set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) is linearly independent. Write the equation

\[
c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r = 0
\]

in matrix form. There are non-trivial solutions if and only if there is a column without a leading one. In this case, the vectors are dependent, since any non-trivial solution is a dependence relation.

12.3. Problems.

(1) True/False:
   (a) The vectors \( \mathbf{v}_1 = [1 2 3], \mathbf{v}_2 = [-1 0 1], \mathbf{v}_3 = [2 6 10] \) are independent.
   (b) The vectors \( \mathbf{v}_1 = [1 0 -1], \mathbf{v}_2 = [1 -1 0], \mathbf{v}_3 = [0 1 -1] \) are independent.
   (c) The vectors \( \mathbf{v}_1 = [1 0 1], \mathbf{v}_2 = [1 1 0], \mathbf{v}_3 = [0 1 1] \) are independent.

(2) Find a set of vectors that is as small as possible that has the same span as \( \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \).

(3) Prove that if \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent, then \( \mathbf{v}_1 \) and \( \mathbf{v}_1 + \mathbf{v}_2 \) are linearly independent.

(4) Prove that if \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) span a subset \( V \), then so do \( \mathbf{v}_1 \) and \( \mathbf{v}_1 + \mathbf{v}_2 \).

13. Basis and Dimension

Definition 13.1. A set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) is a basis for a vector space \( V \) if (1) \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) is linearly independent and (2) \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) spans \( V \).

Example 13.2. \( e_1 = [1 0 0 \ldots 0], e_2 = [0 1 0 \ldots 0], \ldots, e_n = [0 0 \ldots 0 1] \) is the standard basis for \( \mathbb{R}^n \). Linear independence: no \( e_i \) is a combination of the others, since \( e_i \) has a 1 in the \( i \)-th entry and the other vectors have \( i \)-th entry 0. Span:

\[
[x_1 \ x_2 \ldots \ x_n] = x_1[1 0 \ldots 0] + x_2[0 1 0 \ldots 0] + \ldots + x_n[0 0 \ldots 0 1] = x_1 e_1 + \ldots + x_n e_n.
\]

The general procedure for finding a basis is the following: Find an expression for the general element of the vector space. Then, express it as a combination of linearly independent elements.

Example 13.3. Find a basis for the subspace \( V = \{[a \ b \ c \ d], \ a = d, \ b = c\} \).

Example 13.4. Find a basis for the subspace \( V = \{A \in M_{33}, A = A^T\} \).

Example 13.5. Find a basis for the subspace \( V \) of polynomials \( p(x) \) of degree at most 6 such that \( p(x) = p(-x) \).

Theorem 13.6. A set \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) of vectors is a basis for a vector space \( V \) if and only if any vector in \( V \) can be written uniquely as a linear combination of these vectors: \( v = c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r \) where \( c_1, \ldots, c_r \) are unique.

Theorem 13.7. A set of vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) are a basis for \( \mathbb{R}^n \) if and only if the matrix \( A \) with columns \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) is invertible.

Example 13.8. Show that if \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) is a basis, and \( A \) is an invertible matrix, that \( A \mathbf{v}_1, \ldots, A \mathbf{v}_n \) is also a basis.

Definition 13.9. A vector space is finite dimensional if it has a basis with a finite number of elements.

Theorem 13.10. Any two bases \( \mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{w}_1, \ldots, \mathbf{w}_s \) for a finite dimensional vectors space \( V \) of \( \mathbb{R}^n \) have the same number of elements.

Definition 13.11. Let \( V \) be a vector space. The dimension of \( V \) is the number of elements in any basis.

Example 13.12. Find a basis for the space \( V \) of vectors perpendicular to \([1 1 1 1]\) and \([1 2 3 4]\).


(1) Find a basis for the subspace of \( R^4 \) defined by \( x_1 + 2x_3 + x_4 = 0 \).

(2) Find a basis for the space \( V \) of solutions to the equation \( x - z - w = 0 \) in \( R^4 \).

14. Rank

The following three subspaces are associated to a matrix \( A \).

Definition 13.11. The nullspace of \( A \) is the subspace of all vectors \( x \) such that \( Ax = 0 \), that is, the solution set to the homogeneous system corresponding to \( A \). The column space of \( A \) is the span of the columns of \( A \). The row space of \( A \) is the span of the rows of \( A \).
14.1. The null-space: an example from epidemiology. Let’s look at the following model of flu epidemic. Suppose that in a population of 80 students, at any point in time there are \( w \) well students, \( s \) sick students, and \( i \) students who have already been sick and developed immunity. Suppose each week 20 percent of the well students get sick, 50 percent of the sick students get better and develop immunity, but after one week the immunity wears off. Find the matrix \( A \) that expresses the change \( \Delta w, \Delta s, \Delta i \) in the numbers of well, sick, and immune students in terms of \( w, s, i \).

\[
\begin{bmatrix} \Delta w \\ \Delta s \\ \Delta i \end{bmatrix} = A \begin{bmatrix} w \\ s \\ i \end{bmatrix}, \quad A = \begin{bmatrix} -.2 & 0 & 1 \\ +.2 & -.5 & 0 \\ 0 & +.5 & -1 \end{bmatrix}.
\]

What is the practical meaning of the nullspace? It is the set of all vectors \([w \ s \ i]\) such that the change to the next week is zero. That is, the population stays the same. The vectors \([w \ s \ i]\) for which this happens are called equilibrium vectors. Let’s find the null-space, by elimination.

\[
\text{nullspace}(A) = \text{span} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.
\]

For instance, 50 well, 20 sick, and 10 immune is an equilibrium population.

Now back to mathematics.

**Proposition 14.2.** Any vector in the null-space gives a dependence relation on the columns of \( A \).

**Proof.** \( Ax = 0 \) means \( x_1v_1 + \ldots + x_nv_n = 0 \). \( \square \)

For instance, \([5 \ 2 \ 1]\) gives the relation \ldots\ldots. So the last column is a combination of the first two.

14.2. The column space. Now let’s look at the column space. Since the last column is dependent, the span is the span of the first two. Multiplying by scalars does not change the span, so the span is the span of \([-1 \ 1 \ 0], [0 \ -1 \ 1]\). We already saw that this is the plan of vectors whose components sum to zero, in this case, \( w + c + s = 0 \).

**Proposition 14.3.** The column space is the space of all vectors \( b \) for which \( Ax = b \) has a solution.

In this case, \( Ax = b \) has a solution only if the components of \( b \) add up to zero. This is because only changes that preserve the total number 90 of students are possible, since we are assuming that no students die from the flu.

14.3. Bases. In this section, we call a column of \( A \) bound (resp. free) if the corresponding column in \( \text{ref}(A) \) contains (resp. does not contain) a leading 1.

**Theorem 14.4.** Let \( A \) be any matrix.

1. A basis for the null-space is obtained by solving the homogeneous system \( Ax = 0 \). The dimension of the null-space is the number of free variables.
2. A basis for the column-space is given by the bound columns in \( A \). The dimension of the column-space is the number of leading 1’s.
3. A basis for the row-space is given by the non-zero rows in the \( \text{ref}(A) \).

**Example 14.5.** Find a basis for the nullspace, the row-space, and the column space of the matrix

\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 7 & 9 & 11 \\ 3 & 6 & 10 & 13 & 16 \end{bmatrix}.
\]

Gaussian elimination gives

\[
-2\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \
-3\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

which is the \( \text{rref} \) of \( A \). The pivots are in columns 1 and 3, so taking the first and third columns from the first matrix gives a basis for the column space.
space

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \begin{bmatrix}
3 \\
7 \\
10
\end{bmatrix}.
\]

Taking the non-zero rows in the rref (one could use the ref as well) gives a basis for the row-space

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 2 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}.
\]

To find the basis for the null-space, we write out the equations for the rref

\[
a = -2b - d - e, \quad c = -d - e
\]

which imply the solution set to \(Ax = 0\) is

\[
\begin{bmatrix}
-2b - d - e \\
-2b - d - e \\
-d - e \\
d \\
e
\end{bmatrix} = b \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} + d \begin{bmatrix}
-1 \\
0 \\
1 \\
0 \\
0
\end{bmatrix} + e \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}.
\]

So a basis for the null-space is

\[
\begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
-1 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
0 \\
1
\end{bmatrix}.
\]

**Proof of Theorem:** The null-space is the set of vectors \(x\) such that \(Ax = 0\). For any solution \(x\), we call an entry in \(x\) *bound* if it corresponds to a pivot column in \(A\), and otherwise we say it is *free*. By Section 4, the bound entries in \(x\) are linear functions of the free variables, and free entries in \(x\) are equal to the free variables. So \(x\) is linear combination of vectors, one for each free variable, and each vector has one 1 and the rest 0’s in its free entries.

The non-zero rows in the ref or rref are linearly independent, because the pivot columns contain exactly one non-zero entry, and span the row-space of \(A\) since the span of the rows is unchanged by row operations.

Each basis vector for the null-space gives a dependence relation on the columns of \(A\), containing just one free column. Therefore, the free columns can be expressed in terms of the bound columns, using the basis for the null-space. There are no dependence relations on the free columns in \(A\), since there are no null-space vectors with free variables all zero.

### 14.4. Rank.

**Definition 14.6.** The *rank* of a matrix is the dimension of the column space, which is the same by the theorem above as the dimension of the row-space, and the same as the number of leading 1’s. The *nullity* of a matrix is the dimension of the null-space, which by the theorem above is the same as the number of columns without leading 1’s.

An \(m \times n\) matrix has rank between 0 and the minimum of \(m\) and \(n\), since there can be at most one leading 1 in each row and column.

**Example 14.7.** Find the rank and nullity of \[
\begin{bmatrix}
1 & 2 & 3 \\
-1 & -2 & 0 \\
4 & 6 & 1
\end{bmatrix}.
\]

**Theorem 14.8.** A matrix has rank \(0\) if and only if it is the zero matrix. A matrix \(A\) has rank \(n\) if and only if \(A\) is invertible.

*Proof.* If there are no leading 1’s, \(\text{ref}(A) = 0\), but then \(A = 0\). If every column has a leading 1, \(\text{ref}(A) = I\), so \(A\) is invertible.

**Theorem 14.9.** (Rank-Nullity Theorem) The dimension of the column space (the rank) plus the dimension of the null-space (the nullity) is equal to the number of columns.

For instance, if \(A\) is a \(5 \times 3\) matrix with column space 2 dimensional, then the null-space is one-dimensional, so there are homogeneous solutions.

**Corollary 14.10.** If \(A\) is an \(m \times n\) matrix, and \(m > n\) then the rows of \(A\) are dependent. If \(n < m\) then the columns are dependent.

*Proof.* If \(m > n\) then the rank is at most \(n\), so the dimension of the row-space is at most \(n\). Since there are \(m\) vectors in an \(n\)-dimensional space, they are dependent. Similar for the case \(n < m\).

**Theorem 14.11.** The rank of \(A\) is equal to the rank of \(A^T\).
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Proof. The rank of $A^T$ equals the dimension of the column-space of $A^T$ equals the dimension of the row-space of $A$, which equals the dimension of the column-space of $A$, which equals the rank of $A$. □

14.5. Uniqueness of Reduced Row-Echelon Form.

Theorem 14.12. The reduced row echelon form $\text{rref}(A)$ of a matrix $A$ is unique.

Proof. Let $W$ denote the row-space of $A$, $w_1, \ldots, w_r$ the non-zero rows of the $\text{rref}(A)$ and $i_j$ the column number of the leading 1 in $w_j$. Let $e_1, \ldots, e_n$ be the standard basis for $R^n$ and

$$V_n = \text{span}(e_n), V_{n-1} = \text{span}(e_{n-1}, e_n), \ldots, V_1 = \text{span}(e_1, \ldots, e_n).$$

By induction on $i = r-j+1$, we show that $w_j, \ldots, w_r$ is the unique basis for the intersection $V_{i_j} \cap W$ such that the matrix with rows $w_j, \ldots, w_r$ is in reduced row-echelon form.

Case $i = 1$: Then $j = r$. $w_r$ is the unique vector in the row-space in $V_{i_j} \cap W$ with leading coefficient 1.

Case $i$ implies $i + 1$: Assume $w_{j+1}, \ldots, w_r$ is unique. Then there is a unique choice of $w_j$ so that $w_j, \ldots, w_r$ is in reduced row-echelon form, since the entries above the leading 1’s must be zero. □


(1) True/False:

(a) The row-space of $\text{rref}(A)$ is the same as the row-space of $A$.

(b) The row space is the orthogonal complement of the nullspace.

(2) Find a basis for the column-space and the null-space of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 2 & -2 & 4 & 2 \\ 1 & -1 & 3 & 0 \end{bmatrix}.$$ 

What are the rank and nullity of $A$? Write down a dependence relation on the columns of $A$.

15. Orthogonality and Gram-Schmidt


Definition 15.1. Vectors $v_1, \ldots, v_r$ are orthogonal if any two vectors $v_i, v_j$ with $i \neq j$ are perpendicular, that is, $v_i \cdot v_j = 0$

Example 15.2. Find a basis for the plane $V$ in $R^4$ perpendicular to $[2 1 0 1]$ and $[0 1 1 0]$. The equations for $V$ are

$$[2 1 0 1] \cdot v = [0 1 1 0] \cdot v = 0$$

which is equivalent to

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} v = 0.$$ 

So we are trying to find the null-space of the matrix $\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. We do this by the nullspace algorithm:

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

which has equations

$$a = c - d, \quad b = -c$$

and solution set

$$\begin{bmatrix} c - d \\ -c \\ c \\ d \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so the space has basis

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$ 

Example 15.3. The standard basis $e_1, \ldots, e_n$ for $R^n$ is orthogonal.

Example 15.4. $[1 0 0], [0 2 0], [0 0 3]$ are orthogonal.

Example 15.5. $[1 1], [1 -1]$ are orthogonal.

Orthogonal vectors are particularly nice for a number of reasons. For instance,
**Theorem 15.6.** Any orthogonal set of vectors \(v_1, \ldots, v_r\) is linearly independent.

**Proof.** First proof: Suppose one vector, say \(v_r\), is a combination of the others:

\[
v_r = c_1 v_1 + \ldots + c_{r-1} v_{r-1}.
\]

Dot with \(v_r\) on both sides to get

\[
v_r \cdot v_r = 0 \implies v_r = 0
\]

which is a contradiction.

Second, more symmetric proof: Suppose that

\[
c_1 v_1 + \ldots + c_r v_r = 0
\]

Dot with \(v_1\) to get

\[
c_1 v_1 \cdot v_1 + c_1 v_2 \cdot v_1 + \ldots = c_1 v_1 \cdot v_1 = 0
\]

which implies \(c_1 = 0\). Dotting with \(v_2, v_3\) etc. gives \(c_2 = c_3 = 0\). So there are no dependence relations.

A basis is **orthogonal** if it consists of orthogonal vectors.

**Proposition 15.11.** Any orthogonal set of vectors \(v_1, \ldots, v_r\) can be made into an orthonormal set by dividing by the lengths

\[
u_1 = \frac{v_1}{\|v_1\|}, \ldots, u_r = \frac{v_r}{\|v_r\|}.
\]

**Example 15.8.** Suppose we want to express \([3 \, 2]\) as a combination of \([1 \, 1]\) and \([1 \, -1]\). One way would be to solve the system

\[
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix}
3 \\
2
\end{bmatrix}.
\]

But since \([1 \, 1], [1 \, -1]\) is an orthogonal basis, there is an easier way:

\[
c_1 = \frac{[3 \, 2] \cdot [1 \, 1]}{[1 \, 1] \cdot [1 \, 1]} = \frac{5}{2},
\]

\[
c_2 = \frac{[3 \, 2] \cdot [1 \, -1]}{[1 \, -1] \cdot [1 \, -1]} = \frac{1}{2}.
\]

**Example 15.9.** Express \([3 \, 2 \, 1]\) as a combination of

\[
v_1 = [1 \, 1 \, 1], v_2 = [1 \, -1 \, 0], v_3 = [1 \, 1 \, -2].
\]

Step 1: check that \(v_1, v_2, v_3\) forms an orthogonal basis. \(v_1 \cdot v_2 = 1 - 1 = 0, v_2 \cdot v_3 = 1 - 1 = 0, v_1 \cdot v_3 = 1 + 1 - 2 = 0\). Step 2: Compute the coefficients \(c_1, c_2, c_3:\)

\[
c_1 = (3 + 2 + 1)/(1 + 1 + 1) = 2, c_2 = (3 - 2)/(1 + 1) = \frac{1}{2}, c_3 = (3 + 2 - 2)/(1 + 1 + 4) = \frac{1}{2}.
\]

15.2. **Orthonormality.**

**Definition 15.10.** A set of vectors \(v_1, \ldots, v_r\) is **orthonormal** if (1) \(v_1, \ldots, v_r\) are orthogonal and (2) each of the vectors \(v_1, \ldots, v_r\) is a unit vector.

**Proposition 15.13.** Suppose \(u_1, \ldots, u_r\) is an orthonormal basis. Then any vector \(v\) can be written

\[
v = c_1 u_1 + \ldots + c_r u_r
\]

where

\[
c_j = v_j \cdot v.
\]

**Example 15.14.** (Silly example) Let \(e_1, \ldots, e_n\) be the standard basis for \(\mathbb{R}^n\). Then for any vector \(x\), the formula gives \(c_j = e_j \cdot x = x_j\) so that \(x = x_1 e_1 + \ldots + x_n e_n = [x_1 \, 0 \, 0 \ldots \, 0] + \ldots [0 \ldots \, 0 \, x_n] = [x_1 \ldots x_n] = x\).

**Example 15.15.** Let’s express \([3 \, 2 \, 1]\) in terms of \(u_1 = \frac{[1 \, 1 \, 1]}{\sqrt{3}}, u_2 = \frac{[1 \, -1 \, 0]}{\sqrt{2}}, u_3 = \frac{[1 \, 1 \, -2]}{\sqrt{6}}\). We get

\[
c_1 = 6/\sqrt{3}, \quad c_2 = 1/\sqrt{2}, \quad c_3 = 3/\sqrt{6}.
\]
15.3. **Gram-Schmidt.** Any basis can be made into an orthonormal basis, by a procedure call the *Gram-Schmidt* process. Let's start with just two vectors. We define \( u_1 \) by making \( v_1 \) into a unit vector:

\[
\begin{align*}
  u_1 &= \frac{v_1}{\|v_1\|}.
\end{align*}
\]

We want to define \( u_2 \) to be a unit vector perpendicular to \( u_1 \). It's easier to first construct a vector perpendicular to \( u_1 \), and then make it a unit vector, since changing the length doesn't change any angles. Let's try

\[
\begin{align*}
  w_2 &= v_2 - cu_1.
\end{align*}
\]

In order to get \( u_1 \cdot u_2 = 0 \), we need

\[
\begin{align*}
  (v_2 - cu_1) \cdot u_1 = 0 \implies v_2 \cdot u_1 = cu_1 \cdot u_1 \implies c = v_2 \cdot u_1.
\end{align*}
\]

Hence

\[
\begin{align*}
  w_2 &= v_2 - (v_2 \cdot u_1)u_1, \quad u_2 = \frac{w_2}{\|w_2\|}.
\end{align*}
\]

**Example 15.16.** Make the vector \([3 \ 2] \ [2 \ 3]\) into an orthonormal basis using Gram-Schmidt.

\[
\begin{align*}
  u_1 &= \frac{[3 \ 2]}{\sqrt{13}}, \\
  w_2 &= [2 \ 3] - \frac{12}{13}[3 \ 2] = [-10/13 \ 15/13], \\
  u_2 &= w_2/\|w_2\| = [-2 \ 3]/\sqrt{13}.
\end{align*}
\]

One can continue the process for more than two vectors:

\[
\begin{align*}
  w_3 &= w_3 - (w_3 \cdot u_1)u_1 - (w_3 \cdot u_2)u_2, \quad v_3 = \frac{w_3}{\|w_3\|}
\end{align*}
\]

and so on.

**Example 15.17.** Make the basis \([1 \ 1 \ 0] \ [0 \ 1 \ 1] \ [1 \ 0 \ 1]\) into an orthonormal basis using Gram-Schmidt.

\[
\begin{align*}
  u_1 &= v_1/\|v_1\| = [1 \ 1 \ 0]/\sqrt{2}, \\
  w_2 &= v_2 - (v_2 \cdot u_1)u_1 \\
  &= ([0 \ 1 \ 1] - \frac{1}{2}[1 \ 1 \ 0]) \\
  &= [-1 \ 1 \ 2]/\sqrt{6}, \\
  u_2 &= w_2/\|w_2\| = [-11 \ 2]/\sqrt{6}, \\
  w_3 &= v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 \\
  &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} -1 & 1 & 2 \end{bmatrix} \\
  &= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}, \\
  u_3 &= v_3/\|v_3\| = [1 - 1]/\sqrt{3}.
\end{align*}
\]

**Theorem 15.18.** Let \( v_1, \ldots, v_r \) be linearly independent. Then the formulas

\[
\begin{align*}
  u_1 &= v_1/\|v_1\|, \\
  w_2 &= v_2 - (v_2 \cdot u_1)u_1, \quad u_2 = w_2/\|w_2\|, \quad \ldots \\
  v'_r &= v_r - (v_r \cdot u_1)u_1 - \ldots - (v_r \cdot u_{r-1})u_{r-1}, \quad u_r = v_r/\|v'_r\|
\end{align*}
\]

define an orthonormal basis for the span of \( v_1, \ldots, v_r \).

**Proof.** By induction on \( r \). Step \( r = 1 \): Clearly \( u_1 \) is a unit vector, with the same span as \( v_1 \). Step \( r - 1 \implies r \). Suppose we have shown that \( u_1, \ldots, u_{r-1} \) are orthonormal with the same span as \( v_1, \ldots, v_{r-1} \). Since \( v_r \) is not a combination of \( v_1, \ldots, v_{r-1} \),

\[
\begin{align*}
  v'_r &= (v_r - (v_r \cdot u_1)u_1 - \ldots - (v_r \cdot u_{r-1})u_{r-1})
\end{align*}
\]

is non-zero. So \( u_r \) is also non-zero. Therefore, the formula makes sense, and it clearly defines a unit vector. It remains to check \( u_r \cdot u_j = 0, j < r \). This follows from the formula above, since

\[
\begin{align*}
  v'_r \cdot u_j &= (v_r \cdot u_j - \ldots - (v_r \cdot u_j)u_j \cdot u_j - \ldots) = v_r \cdot u_j - v_r \cdot u_j = 0.
\end{align*}
\]

\[ \square \]

15.4. **Problems.**

(1) True/False: (a) The rows of an orthogonal matrix form an orthonormal basis. (b) If \( Q_1 \) and \( Q_2 \) are orthogonal, then so is \( Q_1Q_2 \). (c) If \( Q \) is orthogonal, then \( \det(Q) = \pm 1 \).

(2) Find an orthonormal basis for the subspace \( V \) that is the span of the vectors \([1 \ 1 \ 1]\) and \([1 \ 0 \ 1]\). Find the matrix for orthogonal projection onto \( V \). Find the projection of the vector \([1 \ 0 \ 2]\) onto \( V \). Find the closest point to \([1 \ 0 \ 2]\) in \( V^\perp \).

(3) Find a non-zero vector perpendicular to \( v = [3 \ 1 \ 2 \ 4] \) in \( R^4 \).
(4) Find a basis for the plane $V$ in $\mathbb{R}^4$ \textit{perpendicular} to $[1 \ 0 \ 0 \ 1]$ and $[1 \ 1 \ 0 \ 0]$. (b) Make the basis you found in part (a) into an orthonormal basis, using Gram-Schmidt. (c) Find the matrix for projection onto $V$. Find the projection of $b = [0 \ -1 \ 0 \ 1]$ onto $V$.

(5) If $V$ is the plane of vectors in $\mathbb{R}^4$ satisfying $x_1 + 2x_2 + x_3 = 0$, find a basis for $V^\perp$.

(6) (a) Find the projection of the vector $v = [101]$ onto the line $V$ through $[-10 -1]$. (b) Find the matrix $P$ for projection onto $V$. Check your answer to (a) by computing the product $Pv$.

(7) Apply the Gram-Schmidt process to make $\{(1,1),(2,0)\}$ into an orthonormal basis.

(8) Find an orthonormal basis for the space $V$ of solutions to the equation $x-z-w = 0$ in $\mathbb{R}^4$. Find the projection of the vector $v = [0,0,0,1]$ onto $V$. Find the distance of $v$ from $V$. Find the matrix for the orthogonal projection of $V$.

16. ORTHOGONAL MATRICES AND QR FACTORIZATION

16.1. Orthogonal matrices. The definition of an orthonormal basis can be written in matrix form. Let $Q$ be the matrix with columns $v_1, \ldots, v_r$. Note that the rows of $Q^T$ are $v_1, \ldots, v_r$. So $Q^TQ$ is the matrix whose entries are $v_i \cdot v_j$, that is, the rows of $Q^T$ dotted with the columns of $Q$.

The following conditions are equivalent:

(1) $v_1, \ldots, v_r$ is orthonormal;
(2) $v_i \cdot v_j = 1$, if $i = j$, and 0 otherwise;
(3) The matrix whose entries are $v_i \cdot v_j$ is the identity matrix;
(4) $Q^TQ = I$.

This motivates the following definition:

Definition 16.1. A matrix $Q$ is orthogonal if and only if (1) $Q$ is square and (2) $Q^TQ = I$. Equivalently, $Q$ is orthogonal iff $Q$ is square and $Q^{-1} = Q^T$. Equivalently, $Q$ is orthogonal iff its columns form an orthonormal basis for $R^n$, where $n$ is the number of columns of $Q$.

This explains why the inverse has some of the same properties as the transpose: the two operations are the same for a large class of matrices.

Example 16.2. The identity matrix $I$ is orthogonal. Indeed, $I^{-1} = I = I^T$. The columns of $I$ form the standard basis for $R^n$, which is orthonormal.

Orthogonal matrices have a number of nice properties:

Proposition 16.3. \begin{enumerate}
\item If $Q$ is orthogonal, then so is $Q^{-1}$.
\item If $Q_1$ and $Q_2$ are orthogonal, then so is $Q_1Q_2$.
\item If $Q$ is orthogonal, then $\det(Q) = \pm 1$.
\end{enumerate}

Proof. \begin{enumerate}
\item $Q^T = Q^{-1}$ implies $(Q^{-1})^T = (Q^T)^{-1} = (Q^{-1})^{-1}$.
\item (2) is left to you.
\item If $Q^TQ = I$ then applying $\text{det}$ to both sides we get $1 = \det(I) = \det(Q^TQ) = \det(Q^T)\det(Q) = \det(Q)^2$ so $\det(Q) = \pm 1$. \hfill $\square$
\end{enumerate}

Example 16.4. Classify orthonormal bases for $R^2$. The first vector $u_1$ can be any unit vector. This means $u_1 = [\cos(\theta) \ \sin(\theta)]$ for some angle $\theta$. The vector $u_2$ must be a unit vector perpendicular to $u_1$. There are only two possibilities: $u_2 = \pm [-\sin(\theta) \ \cos(\theta)]$.

Classify orthogonal $2 \times 2$ matrices. By what we have just said, the only possibilities are

$$Q_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad Q_\theta' = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$  

The matrix $Q_\theta$ has determinant $\cos^2(\theta) + \sin^2(\theta) = 1$; it is the matrix for the linear transformation given by counter-clockwise rotation by angle $\theta$. The matrix $Q_\theta'$ has determinant $-\cos^2(\theta) - \sin^2(\theta) = -1$.

16.2. QR Factorization. Suppose that $A$ is the matrix with columns $v_1, \ldots, v_n$, and let $Q$ be the matrix whose columns are the result of Gram-Schmidt. Each of the column operations in Gram-Schmidt can be realized as multiplication on the right by an elementary matrix:

$$Q = AE_1E_2 \cdots E_k.$$  

Example 16.5. Let’s apply Gram-Schmidt to the three vectors

$v_1 = [1 \ 1 \ 0], \ v_2 = [1 \ 0 \ 1], \ v_3 = [0 \ 1 \ 1].$
Then
\[ u_1 = [1 1 0] / \sqrt{2}, \]
\[ w_2 = v_2 \rightarrow v_2 - (v_2 \cdot u_1)u_1 = [1 0 1] - \frac{1}{2}[1 1 0] = \frac{1}{2} \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} = [1 - 1 2] \]
\[ u_2 = [1 - 1 2] / \sqrt{6} \]
\[ w_3 = (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 = [0 1 1] - \frac{1}{2}[1 1 0] - \frac{1}{6}[1 - 1 2] = \frac{2}{3} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \rightarrow [-1 -1 1] / \sqrt{3}. \]
Each of these operations is equivalent to multiplication of the matrix

\[ A = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \]

by an elementary matrix on the right. The elementary matrices are
\[ E_1 = \begin{bmatrix} 1 / \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & -1 / \sqrt{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
\[ E_4 = \begin{bmatrix} 1 & 0 & -1 / \sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 / \sqrt{6} \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 / \sqrt{3} \end{bmatrix}. \]

Switching the elementary matrices over to the other side gives
\[ A = Q E_k^{-1} \ldots E_1^{-1}. \]
Define
\[ R = E_k^{-1} \ldots E_1^{-1} \]
so that
\[ A = QR. \]
Since the \( E \)'s are upper triangular, so is \( R \). The \( ij \)-th entry of \( R \) is \( u_i \cdot v_j \), and the \( ii \)-th entry of \( R \) is \( ||w_i|| \).

**Example 16.6.** In the example above
\[ R = E_6^{-1} E_5^{-1} E_4^{-1} E_3^{-1} E_2^{-1} E_1^{-1} = \begin{bmatrix} \sqrt{2} & 1 / \sqrt{2} & 1 / \sqrt{2} \\ 0 & \sqrt{2} & 1 / \sqrt{6} \\ 0 & 0 & 2 / \sqrt{3} / 2 \end{bmatrix}. \]

Therefore, the \( QR \) factorization of \( A \) is
\[ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 / \sqrt{2} & 1 / \sqrt{6} & -2 / 3 \\ 1 / \sqrt{2} & -1 / \sqrt{6} & 2 / 3 \\ 0 & 2 / \sqrt{6} & 1 / 3 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 / \sqrt{2} & 1 / \sqrt{2} \\ 0 & \sqrt{2} & 1 / \sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix}. \]

Let's summarize the discussion in a theorem:

**Theorem 16.7.** Any matrix \( A \) can be factored \( A = QR \), where the columns of \( Q \) are orthonormal vectors and \( R \) is the upper triangular matrix whose \( ij \)-th entry is \( u_i \cdot v_j \) and whose \( ii \)-th entry is \( ||w_i|| \).

**16.3. Problems.**

(1) True or false:
   a) If \( Q \) is orthogonal, then so is \( Q^T \).
   b) If \( Q \) is an orthogonal matrix, then any eigenvalue of \( Q \) has \( |\lambda| = 1 \).
   c) 

17. Orthogonal Projections

**Definition 17.1.** Let \( V \) be a subspace of \( R^n \). The orthogonal complement \( V^\perp \) of \( V \) is the set of all vectors \( w \) such that \( w \) is perpendicular to \( v \) for all vectors \( v \) in \( V \). Equivalently, \( w \cdot v \) for all \( v \) in \( V \).

Note that

**Lemma 17.2.** If \( v_1, \ldots, v_r \) is a basis for \( V \), then \( w \) is in \( V^\perp \) if and only if \( w \) is perpendicular to \( v_1, \ldots, v_r \).

**Proof.** \( wv_j = 0 \) for \( j = 1, \ldots, r \) implies \( w(c_1v_1 + \ldots + c_rv_r) = 0 \) for any scalars \( c_j \), which implies \( wv = 0 \) for all \( v \) in \( V \). \( \square \)

**Example 17.3.** Let \( v = [1 2 3] \) in \( R^3 \) and let \( V \) be the span of \( V \). The orthogonal complement is the set of all vectors perpendicular to \( v \), that is the set of \( w = [x y z] \) such that \( x + 2y + 3z = 0 \). \( V^\perp \) is the plane perpendicular to (or with normal equal to) \( [1 2 3] \).
Example 17.4. Let \( V \) be the span of \( \mathbf{v}_1 = [1 \ 2 \ 3] \) and \( \mathbf{v}_2 = [3 \ 2 \ 1] \). Then \( V \) is a plane and \( V^\perp \) is the perpendicular to this plane, and so is a line. To compute the equation for \( V^\perp \), we do elimination:

\[
V^\perp = \{ [x \ y \ z], \ x + 2y + 3z = 0 \} = \{ [-2y - 3z \ y \ z] \} = \text{span}([-2 \ 1 \ 0], [-3 \ 0 \ 1]).
\]

Note we have been using the null-space algorithm to find a basis for \( V^\perp \). We can always do this because of the following:

**Proposition 17.5.** For any matrix \( A \), the nullspace of \( A \) is the orthogonal complement of the rowspace of \( A \).

**Proof.** \( w \) is in the null-space of \( A \) if and only if \( A w = 0 \) if and only if each row \( \mathbf{v}_j \) dotted with \( w \) gives 0.

\[ \Box \]


**Theorem 17.6.**

1. \( V \) and \( V^\perp \) intersect in the zero vector.
2. If \( V \) has dimension \( r \), then \( V^\perp \) is a subspace of dimension \( n - r \).
3. Any vector \( u \) in \( \mathbb{R}^n \) may be written uniquely as a combination of a vector \( v \) in \( V \) and a vector \( w \) in \( V^\perp \).
4. For any subspace \( V \), \( (V^\perp)^\perp = V \).

**Proof.**

1. If \( u \) is in \( V \) and \( v \) is in \( V^\perp \), then \( u \cdot v = 0 \), so \( u = 0 \).

2. Since \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) are linearly independent, there is a leading one in every row. So there are \( r \) leading 1’s. Therefore, \( \dim V^\perp \) is the number of free variables, which is the number of columns without leading ones, which is \( n - r \).

3. Pick a basis \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) for \( V \), and a basis \( \mathbf{w}_1, \ldots, \mathbf{w}_{n-r} \) for \( V^\perp \). Then \( \mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{w}_1, \ldots, \mathbf{w}_{n-r} \) is orthonormal, so linearly independent, so a basis for \( \mathbb{R}^n \). Hence any vector can be written uniquely

\[
u = c_1 \mathbf{v}_1 + \ldots + c_r \mathbf{v}_r + c_{r+1} \mathbf{w}_1 + \ldots + c_n \mathbf{w}_{n-r}.
\]

Let \( v = c_1 \mathbf{v}_1 + \ldots + c_r \mathbf{v}_r \) and \( w = c_{r+1} \mathbf{w}_1 + \ldots + c_n \mathbf{w}_{n-r} \). Then \( v \) is in \( V \) and \( w \) is in \( W \).

We prove that \( v \) and \( w \) are unique. Suppose \( u = \mathbf{v}' + \mathbf{w}' \) with \( \mathbf{v}' \in V, \mathbf{w}' \in W \). Then

\[
v + w = v' + w' \implies v - v' = w' - w.
\]

So \( v - v' \in W \) and \( w' - w \in V \). But this is a contradiction, by (1).

(4) \( (V^\perp)^\perp \) is the set of vectors \( u \) such that \( u \) is perpendicular to any vector in \( V^\perp \). Given any such vector, we may write it \( u = v + w \) by (3). But then \( u \) is perpendicular to \( w \) so \( u \cdot w = 0 + w \cdot w = 0 \) which implies \( w = 0 \). Hence \( u \) is in \( V \). Conversely, any vector \( v \) in \( V \) is perpendicular to \( V^\perp \), and so lies in \((V^\perp)^\perp \). We have shown that \( V \) is contained in \((V^\perp)^\perp \) and vice-versa, so the two subspaces must be equal.

\[ \Box \]

17.2. Orthogonal Projections.

**Definition 17.7.** Let \( V \) be a subspace of \( \mathbb{R}^n \), and \( u \) a vector, and \( u = v + w \) the decomposition given by (3) above. The **orthogonal projection** of \( u \) onto \( V \) is the vector \( v \).

**Example 17.8.** Suppose \( V \) is the \( xy \)-plane and \( u = [1 \ 2 \ 3] \). Then \( V^\perp \) is the \( z \)-axis and the decomposition of \( u \) is

\[
[1 \ 2 \ 3] = [1 \ 2 \ 0] + [0 \ 0 \ 3].
\]

So \( v = [1 \ 2 \ 0] \) is the projection of \( u \) onto \( V \) and \( w = [0 \ 0 \ 3] \) is the projection of \( u \) onto \( V^\perp \).

**Theorem 17.9.** Let \( V \) be the span of a single vector \( \mathbf{v}_1 \). Then the **projection of \( u \) onto \( V \)** is \( v = \frac{u \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 \) and the projection of \( u \) onto \( V^\perp \) is

\[
w = u - \frac{u \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1.
\]

**Proof.** We write

\[
u = c \mathbf{v}_1 + (u - c \mathbf{v}_1)
\]

and solve for \( c \) so that

\[
c \mathbf{v}_1 \cdot (u - c \mathbf{v}_1) = 0.
\]

We get

\[
c \mathbf{v}_1 \cdot u = c^2 \mathbf{v}_1 \cdot \mathbf{v}_1 \implies c = \frac{\mathbf{v}_1 \cdot u}{\mathbf{v}_1 \cdot \mathbf{v}_1}.
\]

\[ \Box \]

**Example 17.10.** Find the projection of the vector \( u = [1 \ 0 \ 0] \) onto the span \( V \) of \( \mathbf{v}_1 = [1 \ 2 \ 3] \). Find the projection of \( u \) onto \( V^\perp \).
Theorem 17.11. Suppose that $V$ is a subspace with orthogonal basis $v_1, \ldots, v_r$. Then the projection of $u$ onto $V$ is

$$v = (u \cdot v_1)v_1 + \ldots (u \cdot v_r)v_r$$

and the projection of $u$ onto $V^\perp$ is

$$w = v - (u \cdot v_1)v_1 + \ldots (u \cdot v_r)v_r.$$ 

Proof. We write

$$v = c_1v_1 + \ldots + c_rv_r, \quad w = u - c_1v_1 + \ldots + c_rv_r$$

and solve for $c_1, \ldots, c_r$ so that $w \cdot v_j = 0$ for $j = 1, \ldots, r$. □

Example 17.12. Find the projection of the vector $u = [1 2 3]$ onto the subspace $V$ spanned by $v_1 = [1 1 0]$ and $v_2 = [0 1 1]$.

17.3. Projection Matrices.

Theorem 17.13. The map $T$ that sends $u$ to its projection $v$ is a linear transformation. If $v_1, \ldots, v_r$ is an orthonormal basis, the matrix $P$ for $T$ is

$$P = v_1v_1^T + \ldots + v_rv_r^T.$$ 

If $v_1, \ldots, v_r$ is an orthogonal basis, the formula for the matrix $T$ is

$$P = \frac{v_1v_1^T + \ldots + v_rv_r^T}{v_1^Tv_1 \cdot \ldots \cdot v_r^Tv_r}.$$ 

If $v_1, \ldots, v_r$ is an arbitrary basis, the formula for the matrix is

$$P = A(A^TA)^{-1}A^T$$ 

where $A$ is the matrix with columns $v_1, \ldots, v_r$.

Example 17.14. Find the matrix for projection onto the $xy$-plane.

Example 17.15. Find the matrix for projection onto the span of $[1 1 0]$ and $[0 1 1]$.

17.4. Problems.

(1) True/False:
   (a) If $P$ is the matrix for orthogonal projection onto a subspace $V$, then $P^3 = P$.
   (b) Show that if two vectors are perpendicular, then they are linearly independent.
   (c) Show that the two eigenvectors for a symmetric matrix are orthogonal, if they are from different eigenspaces.

18. Least Squares Approximation

Suppose we want to find the line that best fits the data points $(0, 0), (1, 0)$, and $(2, 3)$. Before we saw how to set this problem up as a system of linear equations: We write $f(x) = c_1x + c_0$ and solve for $c_1, c_0$

$$c_1(0) + c_0 = 0$$
$$c_1(1) + c_0 = 0$$
$$c_1(2) + c_0 = 3.$$ 

Since the three points are not colinear, there is no solution. The problem is that the vector $b = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ is not in the column space of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$. To fix this problem, we project the vector $b$ onto the column space of $A$. This gives a vector $Pb$ which is a close to $b$ as possible, yet now has a solution. The equation

$$Ax = Pb$$

is called the least square equation. Since $b - Pb$ is in the perp of the column space,

$$A^T(b - Pb) = 0.$$ 

Hence $A^T(Ax - Pb) = 0$ which implies that

$$A^TAx = A^Tb.$$
Any solution is called a least square solution.

**Example 18.1.** In our case, this equation is

\[
\begin{bmatrix}
0 & 1 \\
1 & 1 \\
2 & 1
\end{bmatrix}
T
\begin{bmatrix}
0 & 1 \\
1 & 1 \\
2 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
1 & 1 \\
2 & 1
\end{bmatrix}
T
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

which becomes

\[
\begin{bmatrix}
5 & 3 \\
3 & 3
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_0
\end{bmatrix}
= \begin{bmatrix}
6 \\
3
\end{bmatrix}.
\]

The solution is

\[
c_1 = \frac{3}{\bar{r}}, \quad c_0 = -\frac{1}{\bar{r}}.
\]

**Example 18.2.** Apply the least squares method to find the closest line(s) to the data points \((0,0),(2,0),(2,3)\).

**Example 18.3.** Apply the least squares method to find the curve of the form \(c_0 + c_1x + c_2x^2\) best fitting the points \((-1,1),(0,0),(1,1),(2,1)\).

**Example 18.4.** Find all the functions of the form \(f(t) = c_0 + c_1 \cos(\pi t) + c_2 \cos(2\pi t)\) that are best fits for the data points \((-\frac{1}{2},1),(0,0),(\frac{1}{2},0)\).

### 18.1. Problems.

1. Find the closest line, through the origin, to the points \((0,8),(1,8),(3,5),(4,6)\).
2. Using the least squares method, find the function of the form
   (a) \(f(x) = c_0 + c_2x^2\) that best fits the data points \((-1,1),(0,0),(1,2)\).
   (b) \(f(x) = a + bx^2\) that best fits the data points \((-1,1),(0,0),(1,3)\).
   (c) \(f(x) = ax + bx^2\) that best fits the data points \((-1,0),(0,1),(2,2)\).
3. Using least-squares approximation find all the functions of the form \(f(t) = c_0 + c_1t\). which are best fits for the data points \((-1,0),(0,0),(0,2)\).

### 19. Eigenvectors and eigenvalues

Consider the following mathematical model for the market for cola. Suppose \(c(t)\) (resp. \(p(t)\)) is the number of Coke (resp. Pepsi) drinkers at time \(t\) months. Suppose each month, 10 percent of the Coke drinkers switch to become Pepsi drinkers, and 20 percent of the Pepsi drinkers switch to Coke. If we start with 100 Pepsi drinkers and no Coke drinkers, what happens as \(t\) goes to infinity?

\[
\begin{array}{c|c|c}
\hline
\text{Time} & \text{Coke} & \text{Pepsi} \\
\hline
0 & 100 & 0 \\
1 & 20 & 80 \\
2 & 34 & 66 \\
\hline
\end{array}
\]

To set this up as a linear algebra problem we write

\[
c(t + 1) = .9c(t) + .2p(t)
\]

\[
p(t + 1) = .1c(t) + .8p(t)
\]

or in matrix form

\[
x(t + 1) = Ax(t)\]

where

\[
A = \begin{bmatrix}
.9 & .2 \\
.1 & .8
\end{bmatrix}
\]

\[
x(t) = A^t x(0)
\]

for any time \(t\). The best method for solving this for large \(t\) is eigenvectors/eigenvalues.

**Definition 19.1.** An eigenvector of a square matrix \(A\) is a vector \(x\) such that \(Ax = \lambda x\) for some number \(\lambda\), called the eigenvalue of \(x\). An eigenvalue of a square matrix \(A\) is a number \(\lambda\) such that \(Ax = \lambda x\) for some vector \(x\), called an eigenvector for \(\lambda\).

Geometrically, an eigenvector is a vector \(x\) such that \(Ax\) lies in the same direction (or opposite direction) as the original vector. The eigenvalue \(\lambda\) is the “stretch factor”.

**Example 19.2.** Say \(A = \begin{bmatrix}
.9 & .2 \\
.1 & .8
\end{bmatrix}\) as above. Then \(x_1 = \begin{bmatrix}
1 \\
-1
\end{bmatrix}\) is an eigenvector with eigenvalue \(.7\). Also \(x_2 = \begin{bmatrix}
2 \\
1
\end{bmatrix}\) is an eigenvector with eigenvalue 1.

Let’s use these eigenvectors to solve the coke/pepsi problem described above. To begin, we write the initial state vector \(x_0 = \begin{bmatrix}
0 \\
100
\end{bmatrix}\) in terms...
of the eigenvectors:
\[ \mathbf{x}_0 = \begin{bmatrix} 0 \\ 100 \end{bmatrix} = \left( \frac{200}{3} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left( \frac{100}{3} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

Then
\[ \mathbf{x}_t = \ldots A^t \mathbf{x}_0 = \left( \frac{200}{3} \right) A^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \left( \frac{100}{3} \right) A^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \left( \frac{200}{3} \right) \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^t + \left( \frac{100}{3} \right) \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^t. \]

For \( t \) very large, \( \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^t \) is approximately zero. So
\[ \mathbf{x}_t \cong \left( \frac{100}{3} \right) \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 66.7 \\ 33.3 \end{bmatrix}. \]

That is, in the long run \( \frac{2}{3} \) of the customers are with Coke, and \( \frac{1}{3} \) with Pepsi.

19.1. **Finding eigenvalues.** First we find the eigenvalues. The following are equivalent:

1. \( \lambda \) is an eigenvalue of \( A \).
2. \( A \mathbf{v} = \lambda \mathbf{v} \) for some vector \( \mathbf{v} \neq 0 \).
3. \( (A - \lambda I) \mathbf{v} = 0 \) for some vector \( \mathbf{v} \neq 0 \).
4. \( \text{nullspace}(A - \lambda I) \neq 0 \).
5. \( A - \lambda I \) is not invertible.
6. \( \det(A - \lambda I) = 0 \).

So to find the eigenvalues we have to solve \( \det(A - \lambda I) = 0 \). The polynomial \( \det(A - \lambda I) \) is the **characteristic polynomial** of \( A \).

**Example 19.3.** If \( A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \) then
\[ 0 = \det(A - \lambda I) = \det\left( \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = (.9 - \lambda)(.8 - \lambda) - .02 = .7 - 1.7\lambda + \lambda^2. \]

Solving such an equation is equivalent to factoring
\[ .7 - 1.7\lambda + \lambda^2 = (\lambda - 1)(\lambda - .7). \]

We want to find numbers \( \lambda_1, \lambda_2 \) so that \( \lambda_1 + \lambda_2 = 1.7 \) and \( \lambda_1\lambda_2 = .7 \). The solution is
\[ \lambda_1 = 1, \ \lambda_2 = .7. \]

This gives the eigenvalues above.

**Example 19.4.** Let \( A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \). To find the eigenvalues we set
\[ 0 = \det(A - \lambda I) = \begin{bmatrix} 2 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 2 - \lambda \end{bmatrix}. \]

By expanding along the first row this equals
\[ (2 - \lambda)((2 - \lambda)^2 - 1) - (-1)(-1)(2 - \lambda) = (2 - \lambda)(\lambda^2 - 4\lambda + 3) - (2 - \lambda) \]
\[ = (2 - \lambda)(\lambda^2 - 4\lambda + 2) = (2 - \lambda)(\lambda - (2 + \sqrt{2}))(\lambda - (2 - \sqrt{2})). \]

So the eigenvalues are
\[ \lambda_1 = 2, \ \lambda_2 = 2 + \sqrt{2}, \ \lambda_3 = 2 - \sqrt{2}. \]

**Example 19.5.** Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \). To find the eigenvalues we set
\[ 0 = \det(A - \lambda I) = (1 - \lambda)(1 - \lambda)(1 - \lambda). \]

So the eigenvalues are \( \lambda = 1, 1, 1 \).

More generally, if \( A \) is upper or lower triangular or diagonal, then \( A - \lambda I \) is also upper or lower triangular, so that \( \det(A - \lambda I) \) is the product \( (a_{11} - \lambda) \ldots (a_{nn} - \lambda) \). This shows

**Theorem 19.6.** If \( A \) is upper or lower triangular or diagonal then the eigenvalues of \( A \) are the diagonal entries \( a_{11}, \ldots, a_{nn} \).

19.2. **Finding eigenvectors.** Once we have found the eigenvalues, we can find the eigenvectors. The following are equivalent:

1. \( \mathbf{v} \) is an eigenvector of \( A \) with eigenvalues \( \lambda \);
2. \( A \mathbf{v} = \lambda \mathbf{v} \);
3. \( (A - \lambda I) \mathbf{v} = 0 \);
4. \( \mathbf{v} \) is in the nullspace of \( A - \lambda I \).

So to find the eigenvectors we have to find the nullspace of \( A - \lambda I \), for each eigenvalue \( \lambda \).
Example 19.7. Let \( A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \). The eigenvalues are \( \lambda = .7 \) and \( \lambda = 1 \). We compute
\[
\text{nullspace } A - .7I = \text{nullspace } \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} - \begin{bmatrix} .7 & 0 \\ 0 & .7 \end{bmatrix} = \text{nullspace } \begin{bmatrix} .2 & .2 \\ .1 & .1 \end{bmatrix} = \text{nullspace } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{span } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]
and
\[
\text{nullspace } A - (3)I = \text{nullspace } \begin{bmatrix} -2 & 2 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \text{nullspace } \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \text{span } \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.
\]
So in this case there are two eigenvectors
\[
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}.
\]

Example 19.8. Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \) so the eigenvalues are 1, 1, 3. Then
\[
\text{nullspace } A - (1)I = \text{nullspace } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{nullspace } \begin{bmatrix} -1 & 2 \\ -1 & 1 \\ 1 & -2 \\ 0 & 0 \end{bmatrix} = \text{nullspace } \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \text{span } \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]
So “the” eigenvectors are
\[
v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\]
as we claimed above. Note these vectors are not unique: any multiples of \( v_1, v_2 \) are also eigenvectors.

Example 19.9. Every vector is an eigenvector for the identity matrix \( I \) with eigenvalue 1.

19.3. Properties of the eigenvalues. The characteristic polynomial \( \det(A - \lambda I) \) has degree \( n \), so there are at most \( n \) solutions to \( \det(A - \lambda I) = 0 \). If there are exactly \( n \) solutions \( \lambda_1, \ldots, \lambda_n \) to the equation \( \det(A - \lambda I) = 0 \) in the real numbers, so that
\[
\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda),
\]
we say that the eigenvalues of \( A \) are all real. This terminology will be explained later; for the moment we assume that all eigenvalues are real.

The number of times a factor \( (\lambda_i - \lambda) \) appears is the algebraic multiplicity of \( \lambda_i \).

Example 19.10. Suppose \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \). Then \( \det(A - \lambda I) = (1 - \lambda)^2(3 - \lambda) \) so \( \lambda = 1 \) is an eigenvalue with algebraic multiplicity 2 and \( \lambda = 3 \) is an eigenvalue with algebraic multiplicity 1.

Theorem 19.11. The number of real eigenvalues, counted with algebraic multiplicity, is at most \( n \). The determinant of \( A \) is the product of the eigenvalues taken with algebraic multiplicity.

Proof. We plug \( \lambda = 0 \) into the characteristic polynomial to get \( \det(A) = (\lambda_1)(\lambda_2) \cdots (\lambda_n) \).
\[\square\]
Corollary 19.12. A matrix is invertible only if 0 is not an eigenvalue.

Proof. A is invertible, iff det(A) ≠ 0, iff none of the λi’s is zero. □

There is one other coefficient of the characteristic polynomial which has a simple interpretation. The trace of a square matrix A is the sum of the diagonal entries:

\[ \text{Tr}(A) = a_{11} + \ldots + a_{nn}. \]

Theorem 19.13. The coefficient of \((-1)^{n+1} \lambda^{n-1}\) in the characteristic polynomial \(\det(A - \lambda I)\) is the trace \(\text{Tr}(A)\).

Proof. We expand to get

\[(\lambda_1 - \lambda)(\lambda_n - \lambda) = (-1)^n \lambda^n + (\lambda_1 + \ldots + \lambda_n)(-1)^{n-1} + O(\lambda^{n-2})\]

where \(O(\lambda^{n-2})\) means terms of order at most \(n - 2\) in \(\lambda\). On the other hand, the only term in \(\det(A - \lambda I)\) involving at least \(n - 1\) λ’s is

\[(a_{11} - \lambda) \ldots (a_{nn} - \lambda) = (-1)^n \lambda^n + (a_{11} + \ldots + a_{nn})(-1)^{n-1}\lambda^{n-1} + O(\lambda^{n-2}).\]

Equation the coefficients of \(\lambda^{n-1}\) finishes the proof. □

Theorem 19.14. The transpose \(A^T\) of a square matrix A has the same eigenvalues as A.

Proof. The characteristic polynomial

\[ \det(A^T - \lambda I) = \det(A^T - \lambda^T) = \det((A - \lambda I)^T) = \det(A - \lambda I). \]

So the eigenvalues, which are the roots of the characteristic polynomial, are also the same. □

Theorem 19.15. Suppose that the columns of A sum up to 1. Then \(\lambda = 1\) is an eigenvalue for A.

Proof. If the columns of A sum up to 1 then

\[
\begin{bmatrix}
1 & 1 & \ldots & 1
\end{bmatrix}
A =
\begin{bmatrix}
1 & 1 & \ldots & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]

which implies

\[
A^T
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]

which implies that \(\lambda = 1\) is an eigenvalue of \(A^T\). By the Theorem, this implies implies that \(\lambda = 1\) is an eigenvalue of A. □

Suppose A is a matrix which represents a physical system in which the total number is preserved, e.g. the matrix in the Coke/Pepsi example

\[
A = \begin{bmatrix}
.9 & .2 \\
.1 & .8
\end{bmatrix}
\]

Any such matrix has \(\lambda = 1\) as an eigenvalue. This means that there is a vector \(v\) that is an equilibrium for the system, that is \(Av = v\).

19.4. Properties of the eigenvectors. For any eigenvalue \(\lambda\) define

\[ E_\lambda = \text{nullspace}(A - \lambda I). \]

This is the \(\lambda\)-eigenspace for A. The dimension of \(E_\lambda\) is called the geometric multiplicity of \(\lambda\).

Theorem 19.16. If \(v_1, \ldots, v_r\) is a collection of vectors from different eigenspaces \(E_{\lambda_1}, \ldots, E_{\lambda_r}\) then \(v_1, \ldots, v_r\) are linearly independent.

Proof. Suppose one, say \(v_r\) is a combination of the others

\[ v_r = c_1 v_1 + \ldots + c_{r-1} v_{r-1}. \]

Applying A to both sides we get

\[ \lambda_r v_r = c_1 \lambda_1 v_1 + \ldots + c_{r-1} \lambda_{r-1} v_{r-1}. \]

Subtracting \(\lambda_r\) times the first equation we get

\[
0 = c_1(\lambda_1 - \lambda_r)v_1 + \ldots + c_{r-1}(\lambda_{r-1} - \lambda_r)v_{r-1}.
\]

By the inductive hypothesis, \(v_1, \ldots, v_{r-1}\) are independent so

\[
c_1(\lambda_1 - \lambda_r) = \ldots = c_{r-1}(\lambda_{r-1} - \lambda_r) = 0.
\]

Since all the eigenvalues \(\lambda_1, \ldots, \lambda_r\) are distinct, this implies that

\[ c_1 = \ldots = c_{r-1} = 0 \]

which shows that \(v_1, \ldots, v_r\) are independent. □
19.5. Problems.

(1) True or false? Explain.
   (a) A square matrix is not invertible, if and only if 0 is an eigenvalue.
(2) Show that \( \lambda \) is an eigenvalue of \( A \), if and only if \( \det(A - \lambda I) = 0 \).
(3) Show that if \( v_1, v_2 \) are eigenvectors of a matrix \( A \) with different eigenvalues \( \lambda_1, \lambda_2 \) then \( v_1, v_2 \) are linearly independent.

20. Diagonalization

If an \( n \times n \) matrix \( A \) has \( n \) independent eigenvectors \( v_1, \ldots, v_n \), \( A \) is called diagonalizable. In this case the eigenvectors \( v_1, \ldots, v_n \) form a basis for \( \mathbb{R}^n \) called an eigenbasis.

Theorem 20.1. If \( A \) is diagonalizable, then \( A = SDS^{-1} \) where \( S \) is the matrix whose columns are the eigenvectors of \( A \), and \( D \) is the diagonal matrix of eigenvalues.

Example 20.2. Suppose \( A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \). Then
\[ S = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} .7 & 0 \\ 0 & 1 \end{bmatrix} \]
then \( A = SDS^{-1} \).

Example 20.3. The matrix \( A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \) with eigenvalues 1, 1, 3 is not diagonalizable because there are only two independent eigenvectors
\[ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \].

Example 20.4. The identity matrix, or more generally any diagonal matrix is diagonalizable since it is already diagonal! In this case \( S = I \) and \( D = A \).

Finding the matrices \( S \) and \( D \) is called diagonalizing \( A \).

Theorem 20.5. If a matrix \( A \) has \( n \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \), then \( A \) is diagonalizable.

Proof. Choose an eigenvector \( v_i \) for each eigenvalue \( \lambda_i \). Since there are \( n \)-eigenvalues, there are \( n \) eigenvectors, independent by the theorem above.

20.1. Application to matrix powers. Suppose \( A \) is a square matrix, and we want to find a large power of \( A \), say \( A^t \). The best way to do this is using diagonalization:
\[ A^t = (SDS^{-1})^t = SDS^{-1}SDS^{-1} \cdots SDS^{-1} = SD^tS^{-1}. \]
Since \( D \) is diagonal, its matrix powers are easy to compute:
\[ D^t = \text{diag}(\lambda_1^t, \ldots, \lambda_n^t). \]

Example 20.6. Find \( A^t \), where \( A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \). Then
\[ A^t = SD^tS^{-1} = \begin{bmatrix} -1 & .2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} .7^t & 0 \\ 0 & 1^t \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} - 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 3 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} (.7)^t + 2 & -2(.7)^t + 2 \\ -.7)^t + 1 & 2(.7)^t + 1 \end{bmatrix}. \]
As \( t \) becomes very large this matrix approaches
\[ A^\infty := \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}. \]

20.2. Similarity. Two matrices \( A, B \) are said to be similar if there exists an invertible matrix \( S \) such that \( A = SBS^{-1} \).

Proposition 20.7. (1) \( A \) is similar to itself. (2) If \( A \) is similar to \( B \), then \( B \) is similar to \( A \). (3) If \( A \) is similar to \( B \) and \( B \) is similar to \( C \) then \( A \) is similar to \( C \).

Theorem 20.8. If \( A \) and \( B \) are similar, then they have the same characteristic polynomial. As a result, they have the same eigenvalues, with the same algebraic multiplicities.

A matrix is diagonalizable if and only if it is similar to a diagonal matrix. Let’s apply this to prove the following statement:
**Proposition 20.9.** The geometric multiplicity of an eigenvalue $\lambda_i$ is between 1 and the algebraic multiplicity of $\lambda_i$.

**Proof.** First

$$\text{geommult } \lambda_i = \dim(\text{nullspace}(A - \lambda_i I)) \geq 1$$

since $\det(A - \lambda_i I) = 0$.

Second, let $v_1, \ldots, v_r$ be a basis for $E_{\lambda_i}$ and extend it to a basis $v_1, \ldots, v_n$ for $\mathbb{R}^n$. Let $S$ be the matrix whose columns are $v_1, \ldots, v_n$. Note that

$$SAS^{-1}e_j = SA v_j = S \lambda_i v_j = \lambda_i e_j$$

for $j \leq r$, so $SAS^{-1}$ is of the form

$$SAS^{-1} = \begin{bmatrix} \lambda_i I_r & * \\ 0 & * \end{bmatrix}$$

that is, block upper-triangular with $\lambda_i I_r$ in the upper-left corner. Hence

$$\det(A - \lambda I) = \det(SAS^{-1} - \lambda I)$$

has at least $r$ copies of $\lambda - \lambda_i$. So the algebraic multiplicity is at least $r$. \qed

20.3. Problems.

(1) True or false? Explain.

(a) The identity matrix is diagonalizable.
(b) An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues is $n$.
(c) The sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ matrix is $n$, if complex eigenvalues are included.
(d) Any projection matrix $P$ is diagonalizable.
(e) If $A$ is diagonalizable, then so is $A^2$.
(f) Any diagonalizable matrix has $n$ distinct eigenvalues.
(g) If an invertible square matrix $A$ is diagonalizable, then so is $A^{-1}$.
(h) If $A$ is diagonalizable, then so is $A^T$.

21. Complex eigenvalues

To define complex numbers, we suppose that $-1$ has a square root, called $i$, the imaginary unit, so

$$i^2 = -1.$$ 

This is similar to how negative numbers are introduced: $-x$ is the number which satisfies the equation $-x + x = 0$.

An imaginary number is any real multiple $bi$ of the imaginary unit $i$. A complex number is the sum of a real number plus an imaginary number. The sum of complex numbers is defined by summing the real and imaginary parts

$$(5 + 2i) + (3 - 4i) = 8 - 2i.$$ 

Differences are similar:

$$(5 + 2i) - (3 - 4i) = 2 + 6i.$$ 

The product of complex numbers is again a complex number, using that $i^2 = -1$:

$$(5 + 2i)(3 - 4i) = 15 - 20i + 6i - 8i^2 = 15 - 14i - 8(-1) = 23 - 14i.$$ 

Geometrically, complex numbers are represented as points in the complex plane, which has horizontal axis the real axis and vertical axis the imaginary axis.

Sum and subtraction of complex numbers is the same as addition and subtraction of two-vectors.
The complex conjugate of a complex number \( z = a + bi \) is the reflection of that number over the real axis,

\[ \bar{z} = a - bi. \]

The norm \( |z| \) of a complex number \( z = a + bi \) is the length of the corresponding 2-vector,

\[ |z| = \sqrt{a^2 + b^2}. \]

The norm can also be defined using the conjugate:

\[ \overline{z\bar{z}} = (a + ib)(a - ib) = a^2 + b^2 \quad \text{so} \quad |z| = \sqrt{\overline{z\bar{z}}}. \]

This gives us a way to define inverses of complex numbers: We have

\[ \frac{z}{\overline{z\bar{z}}} = 1 \quad \text{so} \quad z^{-1} = \frac{\overline{z}}{|z|^2}. \]

**Example 21.1.** Find the inverse of \( 5 + 2i \).

\[
(5 + 2i)^{-1} = \frac{5 - 2i}{(5 + 2i)(5 - 2i)} = \frac{5 - 2i}{25 + 4} = \frac{5}{29} - \frac{2}{29}i.
\]

**21.1. Polar form.** The geometric meaning of multiplication is best explained by the polar form of a complex number. Define the argument \( \arg z \) to be the angle \( \theta \) between \( z \) and the positive real axis. Define \( r = |z| \) the modulus of \( z \).

Then the adjacent (resp.) side in the picture is

\[ a = r \cos(\theta), \quad b = r \sin(\theta). \]

The polar form of \( z \) is

\[ z = r \cos(\theta) + ri \sin(\theta). \]

The old way \( z = a + bi \) is called Cartesian form.

**Example 21.2.** Find the polar form of \( 1, i, 1 + i, -1, -1 - i \).

From the picture: For \( z = 1 \) we have \( r = 1, \theta = 1 \). For \( z = i \) we have \( r = 1, \theta = \pi/2 \). For \( z = 1 + i \) we have \( r = \sqrt{2}, \theta = \pi/4 \).

**Proposition 21.3.** Complex conjugation in polar form reverses the sign of the angle \( \theta \): If \( z = r \cos(\theta) + ri \sin(\theta) \) then \( \bar{z} = r \cos(-\theta) + r \sin(-\theta)i \).

**Proof.** \( \bar{z} = r \cos(\theta) - r \sin(\theta)i = r \cos(-\theta) + r \sin(-\theta)i. \)

Multiplication in polar form is simpler than in Cartesian form. Recall the Taylor series expansions

\[
e^x = 1 + x + x^2/2! + x^3/3! + \ldots.
\]

\[
\cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots.
\]

\[
\sin(x) = x - x^3/3! + x^5/5! - x^7/7! + \ldots.
\]

From the Taylor series for \( e^x \) we get

\[
e^{i\theta} = 1 + i \theta + (i \theta)^2/2! + (i \theta)^3/3! + \ldots
\]

\[
= (1 - \theta^2/2! + \theta^4/4! - \theta^6/6! + \ldots) + (\theta - \theta^3/3! + \theta^5/5! + \ldots)i
\]

\[
= \cos(\theta) + i \sin(\theta).
\]

This shows

**Theorem 21.4 (Euler).** \( e^{i\theta} = \cos(\theta) + i \sin(\theta). \)
Now we can re-write the polar form
\[ z = r \cos(\theta) + r \sin(\theta)i = re^{i\theta}. \]
This implies the following geometric interpretation of multiplication of complex numbers.

**Proposition 21.5.** Multiplication of complex numbers is given by multiplying the lengths and adding the angles. That is, if \( z_1 = r_1e^{i\theta_1} \) and \( z_2 = r_2e^{i\theta_2} \) then
\[ z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}. \]

21.2. **Application: Angle-sum formulas.** It’s easy to derive from Euler’s formula the formulas for the cosine or sine of the sum of two, three, or more angles. For instance,
\[ e^{i(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2). \]
On the other hand,
\[
\begin{align*}
e^{i(\theta_1+\theta_2)} &= e^{i\theta_1}e^{i\theta_2} \\
&= \cos(\theta_1) + i\sin(\theta_1)(\cos(\theta_2) + i\sin(\theta_2)) \\
&= (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)).
\end{align*}
\]
Matching up the real and imaginary parts, we get

**Proposition 21.6.** \( \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) \) and \( \sin(\theta_1 + \theta_2) = \cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2) \).

21.3. **Application: Powers of complex numbers.** The best way to find a large power of a complex number is to first write it in polar form.

**Example 21.7.** Find \( z^{20} \) where \( z = 1 + i \). Since the length \( r = \sqrt{2} \) and the angle \( \theta = \pi/4 \), the polar form is \( z = \sqrt{2}e^{i\pi/4} \). So
\[
z^{20} = \sqrt{2}^{20}(e^{i\pi/4})^{20} = 2^{10}e^{i\pi} = 1024(\cos(5\pi) + i\sin(5\pi)) = 1024(-1) = -1024.
\]

21.4. **Complex vectors and matrices.** A complex vector or matrix is a vector or matrix with complex entries. For example,
\[
v = \begin{bmatrix} 1 + i \\ 1 - i \end{bmatrix}
\]
is a complex column vector. Similarly
\[
A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}
\]
is a complex matrix. We can multiply matrices and vectors just as before:
\[
Av = \begin{bmatrix} 1(1 + i) + i(1 - i) \\ i(1 + i) + (1 - i) \end{bmatrix} = \begin{bmatrix} 2 + 2i \\ 2 \end{bmatrix}.
\]

For complex vectors \( v \), the norm \( \|v\| \) is defined by
\[
\|v\| = \sqrt{v \cdot \overline{v}}.
\]
If \( v = [a_1 + b_1i \ a_2 + b_2i \ \ldots \ a_n + b_ni] \) then
\[
\|v\| = \sqrt{a_1^2 + b_1^2 + \ldots + a_n^2 + b_n^2}.
\]
For any complex number \( z \) and complex vector \( v \), \( \|zv\| = |z|\|v\| \). The only vector with norm 0 is the zero vector.

21.5. **The fundamental theorem of algebra.** If we allow complex numbers, then any quadratic polynomial now has exactly two roots (counted with multiplicity) since \( \sqrt{b^2 - 4ac} \) always makes sense as a complex number.

**Example 21.8.** Find the roots of \( f(x) = x^2 + x + 3 \). Ans \( x = (-1 \pm \sqrt{-8})/2 \).

In fact, we have the following theorem.

**Theorem 21.9 (Fundamental Theorem of Algebra).** If \( f(x) = c_0 + c_1x + \ldots + c_nx^n \) is a polynomial of degree \( n \), then \( f \) can be factored \( f(x) = c_n(x - z_1)(x - z_2)\ldots(x - z_n) \) where \( z_1, \ldots, z_n \) are complex numbers. That is, any degree \( n \) polynomial has exactly \( n \) roots, counted with multiplicity.

**Corollary 21.10.** Any \( n \times n \) matrix has exactly \( n \) eigenvalues, counted with algebraic multiplicity, if we allow complex eigenvalues.

**Proof.** Apply the fundamental theorem of algebra to the characteristic polynomial \( \det(A - \lambda I) \). \( \square \)

If we have a complex eigenvalues, we can define complex eigenvectors just as before.
Example 21.11. Find the complex eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Ans. The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 + 1$ which has roots $\lambda_\pm = \pm i$. The eigenspaces are: $\lambda = i$.

$$E_i = \text{nullspace}(A - iI) = \text{nullspace} \begin{bmatrix} -i1 \\ -1 & i \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ i \end{bmatrix}.$$ Similarly, the eigenspace $E_{-i}$ is the span of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

The geometric multiplicity of a complex eigenvector $\lambda$ is the dimension of the (complex) subspace $E_\lambda$. It is at least 1 and at most the algebraic multiplicity.

Corollary 21.12. A matrix is diagonalizable over the complex numbers if and only if the geometric multiplicity equals the algebraic multiplicity of $\lambda$, for each complex eigenvalue $\lambda$.

Example 21.13. Find the eigenvalues and eigenvectors for the “shift matrix” $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

Ans. $\det(A - \lambda I) = \lambda^4 - 1 = 0$ implies that $\lambda^4 = 1$, so writing $\lambda = re^{i\theta}$ we get $r^4e^{4i\theta} = 1 \implies r = 1, 4\theta = 2\pi k$ so $\lambda = \exp(2k\pi i/4)$ for $k = 0, 1, 2, 3$. Using $e^{i\pi/2} = i$ we get $\lambda = 1, i, -1, -i$.

The eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ -i \\ 1 \\ -1 \end{bmatrix}$. So the diagonalization is $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{-1}$.


(1) True/False:
(a) An $n \times n$ matrix with $n$ even has at least one real
(b) An $n \times n$ matrix with $n$ odd has at least one complex eigenvalue.
(c) An $n \times n$ matrix with $n$ odd has at least one real eigenvector.

(2) Let $z = 1 + i$.
(a) Find the polar form of $z$.
(b) Find the complex conjugate $\bar{z}$ of $z$.
(c) Find the inverse of $z$.
(d) Find $z^{20}$. (Hint: use your answer to (a).)

(3) Let $z = -1 + i$. Find (a) The polar form of $z$.
(b) The conjugate of $z$.
(c) The inverse of $z$.
(d) The power $z^{10}$ of $z$.

(4) (a) Find the (real and complex) eigenvalues for the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (b) Find the diagonalization of $A$. (c) Find $A^{15}$, using the diagonalization from (b).

22. Symmetric matrices and quadratic forms

One class of matrices for which complex eigenvectors and eigenvalues never appear is symmetric matrices.

Proposition 22.1. If $A$ is symmetric and real, $A = A^T = A$, then all the eigenvalues are real.
Proof. If $v$ is a (possibly complex) eigenvector with (possibly complex) eigenvalue $\lambda$ then either $\lambda = 0$, which is real, or
\[
\|v\| = v^T \overline{\sigma} = (\frac{Av}{\lambda})^T \overline{\sigma}
\]
\[
= \frac{1}{\lambda} v^T A^T \overline{\sigma} = \frac{1}{\lambda} v^T A \overline{\sigma}
\]
\[
= \frac{1}{\lambda} v^T A v = \frac{1}{\lambda} v^T \lambda v
\]
\[
= \frac{1}{\lambda} \|v\|^2
\]
Since $\|v\| \neq 0$, we must have $\lambda = \overline{\lambda}$, which means that $\lambda$ is on the real axis. \qed

Proposition 22.2. If $A$ is a symmetric real matrix, and $Av$ and $w$ are perpendicular vectors, then so are $v$ and $Aw$.

Proof. $0 = (Av) \cdot w = (Av)^T w = v^T A^T w = v^T Aw = v \cdot (Aw)$. \qed

Theorem 22.3. If $A$ is a symmetric $n \times n$ matrix then $A$ is diagonalizable, and there exists an orthonormal basis of eigenvectors $v_1, \ldots, v_n$.

Proof. By induction on the size $n$. Let $v_1$ be an eigenvector normalized to have length one, and $v_2, \ldots, v_n$ vectors so that $v_1, \ldots, v_n$ is an orthonormal basis. Then $\lambda_1 v_1 = Av_1$ is perpendicular to $v_j$, $j > 1$, which implies that $Av_j$ is perpendicular to $v_1$. Hence $Av_j$ is a combination of the vectors $v_2, \ldots, v_n$.

Let $S$ be the matrix with columns $v_1, \ldots, v_n$. Since these form an orthonormal basis, $S^{-1} = S^T$. Then
\[
S^T A S e_1 = S^T A v_1 = \lambda_1 e_1
\]
and
\[
S^T A S e_j = S^T A v_j
\]
is a combination of the vectors $e_2, \ldots, e_n$. Therefore, $S^T A S$ has block diagonal form $\text{diag}(\lambda_1, A_1)$ for some $n - 1 \times n - 1$-matrix $A_1$. Since $A$ is symmetric and $S$ is orthogonal, $S^T A S$ is symmetric. So $A_1$ is symmetric as well. By the inductive hypothesis, $R^{n-1}$ has an orthonormal basis of eigenvectors $w_2, \ldots, w_n$ for $A_1$. So $e_1, w_2, \ldots, w_n$ is an orthonormal eigenbasis for $S^T A S$. But then the vectors $S^T e_1, \ldots, S^T w_n$ are an orthonormal eigenbasis for $A$. \qed

Example 22.4. ATT, MCI, Sprint are competing for customers. Each loses 20 percent of its customers to each of its competitors, each month. If $a(t), m(t), s(t)$ denote the number of customers in month $t$, then
\[
\begin{bmatrix}
  a(t+1) \\
  m(t+1) \\
  s(t+1)
\end{bmatrix} = 
\begin{bmatrix}
  .6 & .2 & .2 \\
  .2 & .6 & .2 \\
  .2 & .2 & .6 
\end{bmatrix}
\]
Because the columns of this matrix sum up to one, we know that $\lambda = 1$ is an eigenvalue. To find the others, we long divide $(\lambda - 1)$ into the characteristic polynomial ....

22.1. Quadratic Forms. Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. A quadratic form if a function $q(x_1, \ldots, x_n)$ that is degree two in the variables $x_1, \ldots, x_n$.

Example 22.5. Let $q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2$ is a quadratic form in two variables $x_1$ and $x_2$. We can try to graph the function $q$ in three dimensions. First we can graph the function when $x_1 = 0$, and then when $x_2 = 0$. You might think that the function is always “going up”. But it’s not. It actually looks like a saddle.

To prove this, let’s write $q$ in matrix form. Let $Q$ be the matrix whose diagonal entries are the coefficients of $x_1^2$ and $x_2^2$, and whose off-diagonal entries are $\frac{1}{2}$ the coefficient of $x_1x_2$:
\[
Q = \begin{bmatrix}
  1 & -2 \\
  -2 & 1
\end{bmatrix}
\]
Let’s find the eigenvectors and eigenvalues for $Q$. The characteristic polynomial is
\[
\det(Q - \lambda I) = (1 - \lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3)
\]
so the eigenvalues are
\[
\lambda = -1, 3.
\]
We can find the eigenvalues: For $\lambda = 1$ the eigenspace is
\[
E_{-1} = \text{nullspace} \begin{bmatrix}
  2 & -2 \\
  -2 & 2
\end{bmatrix} = \text{span} \begin{bmatrix}
  1 \\
  1
\end{bmatrix}/\sqrt{5}.
\]
Figure 2. Saddle

\[ E_3 = \text{nullspace} \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{5}. \]

So \( Q \) can be diagonalized

\[ Q = SDS^T, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \]

This means that if we define new coordinates

\[ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = S^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Then

\[ q(y_1, y_2) = y^T D y = -y_1^2 + 3y_2^2. \]

This quadratic form is easy to understand; it is a “saddle”. See Figure ??

Another way of graphing the quadratic form is to draw it’s level sets

\[ q(y_1, y_2) = c. \]

For the example above, these are hyperbolas.

**Example 22.6.** Graph the quadratic form \( q(x_1, x_2) = 2x^2 + 2xy + 2y^2. \) Describe the level set \( q(x_1, x_2) = 4. \)

There is a similar story with three or more variables.

**Example 22.7.** Find the diagonalization of the quadratic form \( q(x_1, x_2, x_3) = 6x_1x_2 + 8x_2x_3. \)

The matrix \( Q \) is

\[ Q = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}. \]

The eigenvalues of \( Q \) are found by setting

\[ 0 = \det(Q - \lambda I) \]

\[ = \det \begin{bmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 4 \\ 0 & 4 & -\lambda \end{bmatrix} \]

\[ = -\lambda^3 + 4\lambda \]

\[ = -\lambda(\lambda^2 - 25) \]

\[ = -\lambda(\lambda - 5)(\lambda + 5). \]

Therefore, the eigenvalues are

\[ \lambda = 0, 5, -5. \]

The eigenvectors are

\[ E_0 = \text{nullspace} Q \]

\[ = \text{nullspace} \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \]

\[ = \text{nullspace} \begin{bmatrix} 2 \leftrightarrow 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \]

\[ = \text{nullspace} \begin{bmatrix} 3 \end{bmatrix} - (4/3) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ = \text{span} \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}. \]
The other eigenvectors are

\[ E_{\pm 2} = \text{nullspace } Q \mp 5I \]

\[ = \text{nullspace } \begin{bmatrix} \mp 5 & 3 & 0 \\ 3 & \mp 5 & 4 \\ 0 & 4 & \mp 5 \end{bmatrix} \]

\[ = \text{nullspace } \begin{bmatrix} 1/5 \\ -3/5 \end{bmatrix} \]

\[ = \text{nullspace } (5/16) \begin{bmatrix} \mp 3/5 & 0 \\ 0 & \mp 1/5 & 4 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ = \text{span } \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}. \]

An orthonormal eigenbasis is obtained by dividing by the lengths:

\[ \frac{1}{5} \begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{25}} \begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}, \frac{1}{\sqrt{25}} \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}. \]

Therefore,

\[ Q = S^T DS = \begin{bmatrix} -4/5 & 0 & 3/5 \\ 3/\sqrt{25} & 1/\sqrt{25} & 4/\sqrt{25} \\ 3/\sqrt{25} & -1/\sqrt{25} & 4/\sqrt{25} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -4/5 & 3/\sqrt{25} & 3/\sqrt{25} \\ 3/\sqrt{25} & 4/\sqrt{25} & 4/\sqrt{25} \end{bmatrix} \]

Setting

\[ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_1/5 + 3x_2/\sqrt{25} + 3x_3/\sqrt{25} \\ x_2/\sqrt{2} - x_3/\sqrt{2} \\ 3x_1/5 + 4x_2/\sqrt{25} + 4x_3/\sqrt{25} \end{bmatrix} \]

we get

\[ q(y_1, y_2, y_3) = 5y_2^2 - 5y_3^2. \]

22.2. Problems.

(1) Let \( q(x_1, x_2) = 2x_1^2 - 4x_1x_2 - x_2^2. \) Find coordinates \( y_1, y_2 \) and numbers \( a, b \) such that \( q(y_1, y_2) = ay_1^2 + by_2^2. \)

(2) (a) Find the eigenvalues for the matrix \( A = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \) (b) Find the diagonalization of \( A. \) (c) Using diagonalization, find \( A^5. \) (d) Using diagonalization, find the eigenvalues of \( I - A. \)

23. Linear Dynamical Systems

By dynamical system we will mean a mathematical model for the time evolution of a system.

Example 23.1. A typical example is the Coke/Pepsi example discussed earlier. Recall: Suppose \( c(t) \) (resp. \( p(t) \)) is the number of Coke (resp. Pepsi) drinkers at time \( t \) months. Suppose each month, 10 percent of the Coke drinkers switch to Pepsi drinkers, and 20 percent of the Pepsi drinkers switch to Coke. If we start with 100 Pepsi drinkers and no Coke drinkers, what happens as \( t \) goes to infinity?

We set up this as a linear algebra problem we write

\[ c(t + 1) = .9c(t) + .2p(t) \]

\[ p(t + 1) = .1c(t) + .8p(t) \]

or in matrix form

\[ \mathbf{x}(t + 1) = A\mathbf{x}(t) \text{ where } A = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \text{ and } \mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}. \]

This implies that

\[ \mathbf{x}(t) = A\mathbf{x}(t - 1) = A^2\mathbf{x}(t - 2) = \ldots = A^t\mathbf{x}(0) \]
The state of the system in any month $t$ is described by the vector of Coke/Pepsi drinkers

$$\mathbf{x}(t) = \begin{bmatrix} c(t) \\ p(t) \end{bmatrix}.$$

The time evolution of the system is an equation for $\mathbf{x}(t+1)$ in terms of $\mathbf{x}(t)$:

$$\mathbf{x}(t+1) = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \mathbf{x}(t).$$

The matrix in this equation is the time evolution matrix.

**Example 23.2.** Two companies are competing for customers. Each year, company A loses 60 percent of its customers to company B, while company B each year loses 70 percent of its customers to company A. (Clearly neither of the company’s produce a very high quality product!)

(a) Write down the state vector and time evolution matrix for this system. That is, represent the system in the form $\mathbf{x}(t+1) = A\mathbf{x}(t)$, for some matrix $A$. (b) Find the diagonalization of the matrix $A$. (c) Suppose that initially, 100 customers are with company A and none with company B. Find a formula for the number of customers with company A at time $t$. (d) How many customers does A have, for $t$ very large. (e) Show the evolution of the system on the graph with axes A, B.

### 23.2. A biological example.

**Example 23.3.** The model for the flu epidemic we discussed earlier. Suppose that in a population of 80 students, at any point in time there are $w$ well students, $s$ sick students, and $i$ students who have already been sick and developed immunity. Suppose each week 20 percent of the well students get sick, 50 percent of the sick students get better and develop immunity, but after one week the immunity wears off.

The state vector is the number of well, sick, and immune students

$$\mathbf{x}(t) = \begin{bmatrix} w(t) \\ s(t) \\ i(t) \end{bmatrix}$$

and the time evolution matrix is

$$A = \begin{bmatrix} .8 & 0 & 1 \\ 2 & .5 & 0 \\ 0 & +.5 & 0 \end{bmatrix}.$$
Proof. Since $\mathbf{u}A = \mathbf{u}$, we have $A^T \mathbf{u}^T = \mathbf{u}^T$ so $\mathbf{u}^T$ is an eigenvector of $A^T$ with eigenvalue 1. Since $A$ and $A^T$ have the same eigenvalues, 1 is also an eigenvalue for $A$.

Knowing that 1 is an eigenvalue of $A$ helps to find its other eigenvalues. For instance, to find the eigenvalues of a three-by-three Markov matrix you could compute its characteristic polynomial $\det(A - \lambda I)$, factor out $(\lambda - 1)$ using long division, and then factor the remaining degree two polynomial using the quadratic formula.

Some other properties of Markov matrices that we will not show are

1. All the other eigenvalues have norm $\|\lambda\|$ at most 1
2. If all the entries of $A$ are positive, then $A$ has a unique eigenvector with eigenvalue 1.

23.5. Problems.

1. GM and Ford are competing for customers. Suppose that each year, thirty percent of GM’s customers leave for Ford, while ten percent of Ford’s customers leave for GM. The remaining customers remain loyal. Let $g[t], f[t]$ be the number of customers with GM and Ford in year $T$.
   (a) Find a matrix $A$ such that
   $$
   \begin{bmatrix}
   g[t + 1] \\
   f[t + 1]
   \end{bmatrix}
   = A
   \begin{bmatrix}
   g[t] \\
   f[t]
   \end{bmatrix}.
   $$

   (b) Find the eigenvalues and eigenvectors for the matrix $A$ you found in part (a).

   (c) Let $g[0] = 100, f[0] = 0$. Compute $[g[t], f[t]]$ for $t = 1, 2$.

   (d) Find the number of customers with GM and with Ford, in the limit $t \to \infty$.

   (e) Draw a sketch showing the sequence of points $[g[t], f[t]]$.

24. Recursive sequences

A recursive sequence is a sequence of numbers where the $n$-th number is defined by a formula involving previous numbers. The most famous example, the Fibonacci sequence, is

$$
1, 1, 2, 3, 5, 8, 13, \ldots
$$

Each number is the sum of the two previous numbers. We will find a closed formula for the $n$-th Fibonacci number, using eigenvalues and eigenvectors.

To express this as a linear system, we do the following trick which is very important not only for linear algebra but also differential equations.

$$
f(n + 1) = f(n) + f(n - 1)
$$

$$
f(n) = f(n)
$$

What’s the point of writing the second equation, which is obvious? The point is that the vector $\begin{bmatrix} f(n + 1) \\ f(n) \end{bmatrix}$ can now be written as a matrix times $\begin{bmatrix} f(n) \\ f(n - 1) \end{bmatrix}$:

$$
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
f(n + 1) \\
f(n)
\end{bmatrix}
= A^n
\begin{bmatrix}
f(1) \\
f(0)
\end{bmatrix}.
$$

This means that

$$
\begin{bmatrix}
f(n + 1) \\
f(n)
\end{bmatrix}
= A^n
\begin{bmatrix}
1 \\
0
\end{bmatrix}.
$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. To find $A^n$, we diagonalize $A$.

To find the eigenvalues, we set

$$
0 = \det(A - \lambda I)
= (1 - \lambda)(-\lambda) - 1
= \lambda^2 - \lambda - 1
= (\lambda - (1 + \sqrt{5})/2)(\lambda - (1 - \sqrt{5})/2).
$$

The eigenvectors are

$$
\mathbf{v}_+ = \begin{bmatrix}
1 + \sqrt{5} \\
2
\end{bmatrix}, \quad
\mathbf{v}_- = \begin{bmatrix}
1 - \sqrt{5} \\
2
\end{bmatrix}.
$$
This means that the diagonalization of $A$ is
\[ A = SDS^{-1} \]
\[ = \begin{bmatrix}
\frac{1 + \sqrt{5}}{2} & 1 \\
\frac{1 - \sqrt{5}}{2} & 1
\end{bmatrix}
\begin{bmatrix}
\frac{1 + \sqrt{5}}{2} & 0 \\
0 & \frac{1 - \sqrt{5}}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{1 + \sqrt{5}}{2} & 1 \\
\frac{1 - \sqrt{5}}{2} & 1
\end{bmatrix}^{-1}.\]
The inverse of $S$ is
\[ S^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix}
1 & -\frac{1 - \sqrt{5}}{2} \\
-1 & \frac{1 + \sqrt{5}}{2}
\end{bmatrix} \]
so that the Fibonacci numbers are
\[ \begin{bmatrix}
f(t + 2) \\
f(t + 1)
\end{bmatrix} = A^t \begin{bmatrix}
1 \\
1
\end{bmatrix} \]
\[ = S^{-1} D^t S \begin{bmatrix}
1 \\
1
\end{bmatrix} \]
\[ = \frac{1}{\sqrt{5}} \begin{bmatrix}
\frac{1 + \sqrt{5}}{2} & 1 \\
\frac{1 - \sqrt{5}}{2} & 1
\end{bmatrix}
\begin{bmatrix}
(\frac{1 + \sqrt{5}}{2})^t & 0 \\
0 & (\frac{1 - \sqrt{5}}{2})^t
\end{bmatrix}
\frac{1}{\sqrt{5}} \begin{bmatrix}
1 & -\frac{1 - \sqrt{5}}{2} \\
-1 & \frac{1 + \sqrt{5}}{2}
\end{bmatrix} \]
\[ = \frac{1}{\sqrt{5}} \begin{bmatrix}
\frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\
\frac{1 - \sqrt{5}}{2} & \frac{1 + \sqrt{5}}{2}
\end{bmatrix}
\begin{bmatrix}
\frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\
\frac{1 - \sqrt{5}}{2} & \frac{1 + \sqrt{5}}{2}
\end{bmatrix} \]
\[ = \frac{1}{\sqrt{5}} \begin{bmatrix}
\frac{(1 + \sqrt{5})^t}{2} & \frac{1 - \sqrt{5}}{2} \\
\frac{1 - \sqrt{5}}{2} & \frac{(1 + \sqrt{5})^t}{2}
\end{bmatrix} \]

So the Fibonacci number
\[ f(t) = \frac{1}{\sqrt{5}} \left( (\frac{1 + \sqrt{5}}{2})^t - (\frac{1 - \sqrt{5}}{2})^t \right). \]

**Example 24.1.** Let's look now at some other recursive formula, for instance
\[ f(n + 1) = f(n) - f(n - 1) \]
which gives the sequence
\[ 1, 1, 0, -1, -1, 0, 1, 1, \ldots \]
Find a closed formula for $f(n)$.

To solve the equation we introduce a second equation
\[ f(n) = f(n) \]
so that we get a system of linear equations
\[ \begin{bmatrix}
f(n + 1) \\
f(n)
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
f(n) \\
f(n - 1)
\end{bmatrix}.\]
The characteristic polynomial is
\[ \det(A - \lambda I) = (1 - \lambda)(-\lambda) + 1 = \lambda^2 - \lambda + 1 \]
which has roots
\[ \lambda_\pm = \frac{1 \pm \sqrt{3}}{2} = e^{\pm \frac{2\pi i}{3}}. \]
This means that the matrix $A$ is diagonalizable
\[ A = SDS^{-1}, \quad D = \begin{bmatrix}
e^{2\pi i/3} & 0 \\
0 & e^{-2\pi i/3}
\end{bmatrix}.\]
If $n$ is a multiple of 3 then
\[ A^n = SD^n S^{-1} = SS^{-1} = I. \]
This explains why the sequence is periodic with period 3.

**Example 24.2.** Define a sequence $f(n)$ by $f(n + 1) = 2f(n) + f(n - 1)$. Find a closed formula for $f(n)$ using linear algebra.

### 24.1. Problems.

1. Fibonacci considered a model for a population of rabbits where each pair of adult rabbits produces a pair of juvenile rabbits each month, and each pair of juvenile rabbits becomes adult after one month. A more realistic model takes into account death of the adult rabbits. Suppose 1/2 of the adult rabbits die each month. Then the equations for the model are
\[ j(t + 1) = a(t), \quad a(t + 1) = \frac{1}{2}a(t) + j(t), \]
where $j(t), a(t)$ are the number of pairs of juvenile and adult rabbits at time $t$.

(a) Find the time evolution matrix $A$ for this system.
(b) Find the eigenvalues and eigenvectors for $A$.
(c) Find the ratio of adult rabbits to juvenile rabbits, as $t$ approaches infinity. (Hint: express the initial state as a linear combination $c_1 v_1 + c_2 v_2$ of the eigenvectors. You do not need to find $c_1$ and $c_2$.)
(d) If you did (b) correctly, the eigenvectors of $A$ are perpendicular. What property does $A$ have which guarantees this is so?

(e) Write a recursive formula for $a(t)$ in terms of $a(t-1)$ and $a(t-2)$, similar to the Fibonacci formula $F(t) = F(t-1) + F(t-2)$.

(f) Draw a graph showing the trajectory (time evolution) of the system. Indicate the eigenspaces on your graph by dotted lines.

(2) Consider the recursive sequence defined by $f(n) = f(n-1) - f(n-2)$ with $f(1) = f(2) = 1$. Call these the Ibonacci numbers. The first few are $1, 1, 0, -1, -1, 0, \ldots$.

(a) Find a matrix $A$ so that
\[
\begin{bmatrix}
  f(n+1) \\
  f(n)
\end{bmatrix} = A \begin{bmatrix}
  f(n) \\
  f(n-1)
\end{bmatrix}.
\]

(b) Find the (complex) eigenvalues and eigenvectors of $A$.

(c) Find a formula for the $n$-th Ibonacci number. It should be clear from your formula that the sequence of Ibonacci numbers is repeating.

25. SINGULAR VALUE DECOMPOSITION

Let $A$ be a matrix.

**Proposition 25.1.** The matrix $A^T A$ has $n$ non-negative real eigenvalues.

*Proof.* $A^T A$ is symmetric: $(A^T A)^T = A^T (A^T)^T = A^T A$. Therefore the eigenvalues are real. Let $v$ be an eigenvector of $A^T A$ with eigenvalue $\lambda$. Then
\[
\lambda(v \cdot v) = v \cdot A^T A v = (Av) \cdot (Av) \geq 0.
\]
Since $v \cdot v > 0$, this implies $\lambda \geq 0$.

**Definition 25.2.** The singular values of a matrix $A$ are the square roots of the eigenvalues of $A^T A$.

**Proposition 25.3.** If $A$ is a matrix and $Q$ is an orthogonal matrix, then the singular values of $A$ are the same as those of $QA$ or $AQ$, if these products are defined.

*Proof.* The singular values of $QA$ are the square roots of eigenvalues of $(QA)^T QA = A^T Q^T QA = A^T A$ since $Q$ is orthogonal. The proof for $AQ$ is similar.

**Theorem 25.4.** Let $A$ be any matrix. There exist orthogonal matrices $Q_1, Q_2$ and a diagonal matrix $D$ such that $A = Q_1 D Q_2$. The diagonal elements of $D$ are the singular values of $A$, and $D$ is unique up to ordering of the diagonal elements. If $A$ is invertible, then given a choice of $D$ the matrices $Q_1$ and $Q_2$ are also unique.

*Proof.* By diagonalization of symmetric matrices, the diagonalization of $A^T A$ is $Q_1^T D_1 Q_1$ for some orthogonal $Q_1$ and diagonal $D_1$. Since the eigenvalues are non-negative, $D_1 = D^T D$ where $D$ is the diagonal matrix of square roots of $D_1$. Hence
\[
A^T = Q_1^T D^T D Q_1.
\]
Choose a basis $v_1, \ldots, v_k$ for nullspace($A$)$^\perp$. Then the vectors
\[
w_1 = A v_1, \ldots, w_p = A v_k
\]
are an orthonormal basis for $	ext{col}(A)$. Let $w_{p+1}, \ldots, w_n$ be an orthonormal basis for $	ext{col}(A)^\perp$. Let
\[
u_1 = D Q_1 v_1, \ldots, \nu_p = D Q_1 v_p
\]
and $\nu_{p+1}, \ldots, \nu_n$ an orthonormal basis for $	ext{col}(D Q_1)^\perp$. By (8)
\[
w_j \cdot w_k = (A v_j) \cdot (A v_k) = (D Q_1 v_j) \cdot (D Q_1 v_k) = \nu_j \cdot \nu_k, j, k = 1, \ldots, p.
\]
Since $w_j$ and $\nu_j$, $j \geq p + 1$ are orthonormal,
\[
w_j \cdot w_k = \nu_j \cdot \nu_k
\]
for any $j, k = 1, \ldots, n$. Let $S_1$ be the matrix whose columns are $w_1, \ldots, w_n$ and let $S_2$ be the matrix whose columns are $\nu_1, \ldots, \nu_n$. Let $Q_2 = S_1 S_2^{-1}$. Then
\[
Q_2 D Q_1 v_j = S_1 e_1 = A v_j, \quad j = 1, \ldots, p.
\]
so $Q_2 D Q_1 = A$. By (9),
\[
(Q_2 w_j) \cdot (Q_2 w_k) = \tilde{\nu}_j \cdot \tilde{\nu}_k = w_j \cdot w_k, \quad \forall j, k = 1, \ldots, n
\]
so $Q_2$ is orthogonal.
The decomposition

\[ A = Q_1 D Q_2 \]

is called a singular value decomposition of \( A \).

Here is the geometric meaning of singular values. Assume \( A \) is invertible, and let \( S \) be the unit sphere in \( \mathbb{R}^n \):

\[ S = \{ v_1^2 + \ldots + v_n^2 = 1 \}. \]

Multiplication by \( Q_2 \) maps \( S \) to itself:

\[ \mathbf{v} \in S \implies Q_2 \mathbf{v} = 1. \]

Multiplication by \( D \) maps \( S \) to an ellipsoid

\[ E = DS = \{ \lambda_1^{-2} v_1^2 + \ldots + \lambda_n^{-2} v_n^2 = 1 \}. \]

The numbers \( \lambda_1, \ldots, \lambda_n \) are the axis lengths of \( E \). Multiplication by \( Q_1 \) rotations \( E \) into another ellipsoid. See Figure 3.

Let’s summarize the discussion in a theorem:

**Theorem 25.5.** Let \( S \) be the unit sphere in \( \mathbb{R}^n \), and \( A \) a square matrix. The set of vectors \( A \mathbf{v} \) for \( \mathbf{v} \in S \) form an ellipsoid in \( \mathbb{R}^n \), whose semiaxis lengths are the singular values of \( A \).

### 26. Jordan normal form

Let \( \lambda \) be a number. A Jordan block with eigenvalue \( \lambda \) is a matrix

\[
\begin{bmatrix}
\lambda & 1 & 0 & \ldots & 0 \\
0 & \lambda & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda
\end{bmatrix}
\]

with \( \lambda \)'s on the diagonal and 1’s above the \( \lambda \)'s.

A matrix is in Jordan normal form if \( J \) is a block diagonal matrix and each block is a Jordan block.

**Theorem 26.1.** Any square matrix \( A \) is similar to a matrix \( J \) in Jordan normal form, which is unique up to ordering of the blocks.

**Proof.** Let \( m_1, \ldots, m_k \) be the algebraic multiplicities of the eigenvalues so that

\[
\det(A - \lambda I) = (\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_k)^{m_k}.
\]

For each \( i, j \) let

\[ E_{\lambda_j}^i = \text{nullspace}((A - \lambda_j I)^i). \]

If \( \mathbf{v} \) lies in \( E_{\lambda_j}^i \) then \( (A - \lambda_j I)\mathbf{v} \) lies in \( E_{\lambda_j}^{i-1} \). Let

\[ V_j^i = E_{\lambda_j}^i \cap (E_{\lambda_j}^{i-1})^\perp. \]

Starting with \( i = m_i \) and continuing downwards, choose a basis

\[ B_j^i = \{ v_{j,k}^i, k = 1, \ldots, \dim(V_j^i) \} \]

for each \( V_j^i \). Applying \( \det(A - \lambda_j I) \) to each vector gives a subset

\[ \det(A - \lambda_j I)B_j^i \subset E_{\lambda_j}^{i-1}. \]

Since \( \det(A - \lambda_j I)\mathbf{v} \) is non-zero for \( \mathbf{v} \in V_j^i \), \( \det(A - \lambda_j I)B_j^i \) is linearly independent. Extend it to a basis \( B_j^{i-1} \) for \( V_j^{i-1} \) and so on. Repeat this for each eigenvalue \( \lambda_j \).

We claim that the union

\[ \mathcal{B} := \bigcup_{j,i} B_j^i \]

is a basis for \( \mathbb{R}^n \).
Linearly independent: ............... Span: ....................

Let $S$ be the matrix whose columns are the vectors $v^i_{j,k}, (A - \lambda_j I)v^i_{j,k}, (A - \lambda_j I)^2v^i_{j,k}, \ldots$. Then

$$SAS^{-1}v^i_{j,k} = (A - \lambda_j I)v^i_{j,k} + \lambda_j v^i_{j,k}$$

which implies that $A$ is in Jordan normal form. □

**Example 26.2.** Find the Jordan normal form for $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The eigenvalue is 1 with algebraic multiplicity 3. The nullspaces are

$E_1^1 = \text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.

$E_1^2 = \text{nullspace}(A - I)^2 = \text{nullspace}(0) = \mathbb{R}^3$.

The spaces

$V_1^1 = E_1^1$, $V_1^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

We choose

$B_1^2 = \{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$.

Applying $A - I$ gives

$(A - I)B_1^2 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$.

We extend this to a basis of $E_1^1$

$B_1^1 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$.

The matrices $S, J$ are

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

27. More geometry of determinants

There is also a formula for the volumes of simplices in terms of the distances between the vertices. Let $S$ be an $n - 1$-simplex and let $d_{ij}$ denote the distance between the $i$-th and $j$-th vertex.

The Cayley-Menger matrix for $S$ is the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & d_{12}^2 & \ldots & d_{1n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{2n}^2 & d_{12}^2 & \ldots & 0 \end{bmatrix}.$$  

**Theorem 27.1.** The volume of the simplex $S$ is

$$(S) = \frac{2^{-(n-1)}}{n!} \sqrt{\det(A)}.$$  

**Proof.** Let $v_1, \ldots, v_n$ denote the vectors with tails at 0 and heads at the vertices of $S$. Form new vectors $w_1, \ldots, w_n$ by putting a 1 in front of each vector. Let $S'$ denote the simplex with edge vectors $w_1, \ldots, w_n$. Then $S$ is the base of $S'$ and $S'$ has height 1 so

$$(S') = \frac{1}{n} (S)$$

and if $W$ is the matrix with columns $w_1, \ldots, w_n$ then

$$(S') = \frac{1}{n!} |\det(W)|.$$  

The matrix $W^TW$ has determinant

$\det(W^TW) = \det(W^T)\det(W) = \det(W)^2$

and entries

$$w_i^T w_j = 1 + v_i^T v_j.$$
If we rescale the vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) by a scalar \( c \), then \( (S) \) and also \( (S') \) scale by \( c^{n-1} \). It follows that the determinant

\[
W^TW = \begin{bmatrix}
1 + \mathbf{v}_1^T \mathbf{v}_1 & 1 + \mathbf{v}_1^T \mathbf{v}_2 & \cdots & 1 + \mathbf{v}_1^T \mathbf{v}_n \\
1 + \mathbf{v}_2^T \mathbf{v}_1 & 1 + \mathbf{v}_2^T \mathbf{v}_2 & \cdots & 1 + \mathbf{v}_2^T \mathbf{v}_n \\
\vdots & \vdots & \ddots & \vdots \\
1 + \mathbf{v}_n^T \mathbf{v}_1 & 1 + \mathbf{v}_n^T \mathbf{v}_2 & \cdots & 1 + \mathbf{v}_n^T \mathbf{v}_n
\end{bmatrix}
\]

is the sum of terms containing a single 1. This is the same as the determinant of

\[
\det \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & \mathbf{v}_1^T \mathbf{v}_1 & \mathbf{v}_1^T \mathbf{v}_2 & \cdots & \mathbf{v}_1^T \mathbf{v}_n \\
1 & \mathbf{v}_2^T \mathbf{v}_1 & 1 & \cdots & \mathbf{v}_2^T \mathbf{v}_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \mathbf{v}_n^T \mathbf{v}_1 & \mathbf{v}_n^T \mathbf{v}_2 & \cdots & 1 + \mathbf{v}_n^T \mathbf{v}_n
\end{bmatrix}
\]

Multiply the first row by \( \frac{1}{2} \mathbf{v}_j^T \mathbf{v}_j \), and subtract it from the \( j+1 \)-st column, for each \( j = 1, \ldots, n \). Multiply the first column by \( \frac{1}{2} \mathbf{v}_j^T \mathbf{v}_j \), and subtract it from the \( j+1 \)-st column, for each \( j = 1, \ldots, n \). Since

\[
-\frac{1}{2} \mathbf{v}_i^T \mathbf{v}_i + \mathbf{v}_i^T \mathbf{v}_j - \frac{1}{2} \mathbf{v}_j^T \mathbf{v}_j = -\frac{1}{2} (\mathbf{v}_i - \mathbf{v}_j)^T (\mathbf{v}_i - \mathbf{v}_j) = -\frac{1}{2} d_{ij}^2
\]

this gives the matrix

\[
\begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
1 & 0 & -\frac{1}{2} d_{12}^2 & \cdots & -\frac{1}{2} d_{1n}^2 \\
1 & -\frac{1}{2} d_{12}^2 & 0 & \cdots & -\frac{1}{2} d_{2n}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & -\frac{1}{2} d_{1n}^2 & -\frac{1}{2} d_{1(n-1)}^2 & \cdots & 0
\end{bmatrix}
\]

Multiplying each row by \(-2\), then dividing the first column by \(-2\), gives the matrix \( A \). The total change in the determinant under these row operations is by a factor of \((-2)^{n-1}\); the proposition follows.

**Corollary 27.2.** The area of a triangle with edge lengths \( a, b, c \) is

\[
(T) = \frac{1}{4} \sqrt{s(s-a)(s-b)(s-c)}
\]

where \( s = (a + b + c)/2 \).

**Proof.** The determinant of

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & a^2 & b^2 \\
1 & a^2 & 0 & c^2 \\
b^2 & c^2 & 0 & 0
\end{bmatrix}
\]

is \( c^4 + 2a^2c^2 + 2b^2c^2 + b^4 + a^4 + 2a^2b^2 = (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \).

\[\square\]

**Example 27.3.** Find the area of the triangle with edge lengths 3, 4, 5. The semiperimeter \( s = 12/2 = 6 \). The area is

\[
\frac{1}{2} \sqrt{(6-3)(6-4)(6-5)} = \sqrt{6} = 6.
\]

Since this is a right triangle with base 3 and height 4, you could also answer \( \frac{1}{2}(3)(4) = 6 \).

The point of Heron’s formula is that it does not require the triangle to be a right triangle.

**Proposition 27.4 (Cayley).** Four points in \( \mathbb{R}^3 \) lie in a plane if and only if the determinant of the Cayley-Menger matrix is zero.

**Example 27.5.** The distances between Chicago, New York, New Orleans, and Miami (except for the Miami-New Orleans distance) are given in the following table (in thousands of kilometers). Find the distance between Miami and New Orleans.

<table>
<thead>
<tr>
<th>Distance</th>
<th>Chicago</th>
<th>NY</th>
</tr>
</thead>
<tbody>
<tr>
<td>NY</td>
<td>1.149</td>
<td></td>
</tr>
<tr>
<td>New Orleans</td>
<td>1.342</td>
<td>1.887</td>
</tr>
<tr>
<td>Miami</td>
<td>1.916</td>
<td>1.766</td>
</tr>
</tbody>
</table>

The determinant of \( A \) is

\[
\det(A) = -32.291 + 30.543y^2 - 2.6404 = -2.6404(y^2 - 3.223^2)(y^2 - 1.084^2).
\]

Since \( y = 3.223 \) is much to large, the correct solution to \( \det(A) = 0 \) is \( y = 1.084 \). The actual answer is 1.073; the difference is due to the curvature of the earth’s surface.

**Proposition 27.6.** Four points in \( \mathbb{R}^3 \) lie on the sphere of radius \( R \) if and only if \( \det(A) = 0 \), where \( A \) is the matrix whose \( ij \)-th entry is \( \cos(d_{ij}/R) \), and \( d_{ij} \) is the distance between the \( i \)-th and \( j \)-th point along the surface of the sphere.
Example 27.7. Find the distance between Miami and New Orleans using the table above, assuming the earth is round with radius 6378 kilometers.

The determinant of the matrix $A$ has roots .9873 and .8876. Taking the arccosine gives 1.075 and 3.225. Since 3.225 is much too big, the answer is 1.075 (which is now only 2 kilometers off the actual distance.) Part of this is round-off error, but also the earth is not exactly spherical.

28. Eigenvalues of Sums and Singular Values of Products

Let $A$ and $B$ be square matrices of the same size. $A$ and $B$ commute if $AB = BA$.

Example 28.1. A square matrix $A$ always commutes with itself.

$A$ and $B$ are simultaneously diagonalizable if there exists a matrix $S$ such that $SAS^{-1}$ and $SBS^{-1}$ are both diagonal.

Proposition 28.2. If $A$ and $B$ are diagonalizable, then they commute if and only if they are simultaneously diagonalizable.

Proof. Suppose that $A = SDS^{-1}$, where $D$ is the diagonal matrix whose entries are the eigenvalues of $A$, and $S$ is the matrix of eigenvectors. Let $E_{\lambda}$ be the $\lambda$-eigenspace for $A$. If $v$ is in $E_{\lambda}$, then so is $Bv$ since

$$Av = \lambda v \implies A(Bv) = B(Av) = B\lambda v = \lambda Bv.$$ 

It follows that $S^{-1}BS$ is block diagonal. Let $B_1, \ldots, B_k$ denote the blocks. Since $B$ is diagonalizable, so are each of the blocks $B_1, \ldots, B_k$. For each block $B_j$, there exists a matrix $S_j$ such that $S_j^{-1}B_jS_j$ is diagonal. Let $S'$ denote the block diagonal matrix with blocks $S_1, \ldots, S_k$. Then $S'^{-1}S^{-1}BSS' = S'^{-1}S^{-1}DSS' = D$ since $D$ is block diagonal with each block a multiple of the identity. 

Proposition 28.3. If $A$ and $B$ are diagonalizable and commute, then each eigenvalue of $A+B$ is a sum of an eigenvalue of $A$ and an eigenvalue of $B$, and each eigenvalue of $AB$ is a product of an eigenvalue of $A$ and an eigenvalue of $B$.

Proof. Suppose that $A = SDS^{-1}$ and $B = SDS^{-1}$ where $D_A$ is the diagonal matrix of eigenvalues of $A$ and $D_B$ is the diagonal matrix of eigenvalues of $D_B$. Then

$$A + B = S(D_A + D_B)S^{-1}, \quad AB = S(D_AD_B)S^{-1}.$$ 

so the eigenvalues of $A+B$ are the entries of $D_A+D_B$, and the eigenvalues of $AB$ are the entries of $D_AD_B$. 

If $A$ and $B$ do not commute, then the eigenvalues of $A+B$ are related in a much more complicated way to those of $A$ and $B$.

Suppose that $A$ and $B$ are symmetric, so that the eigenvalues of $A$, $B$, and $A+B$ are real. The possible eigenvalues of $A+B$ are related to those of $A$ and $B$ by the trace equality

$$\text{Tr}(A) + \text{Tr}(B) = \sum \lambda_j(A) + \sum \lambda_j(B) = \sum \lambda_j(A+B) = \text{Tr}(A + B)$$

and a set of linear inequalities. There is a “puzzles” game which determines which inequalities hold. The puzzle board looks like

To find the inequalities for $n \times n$ matrices, there should be $n$ little triangles along each big edge in the board. The puzzle pieces are together

with their rotations.

A puzzle is a way of filling in the puzzle board with puzzle pieces so that all of the edges match.
Example 28.4. An example of a puzzle is

For each puzzle, let $I$ denote the positions of the 1’s on the northwest boundary, $J$ the positions of the 1’s on the northeast boundary, and $K$ the positions of the edge along the southern boundary, reading left to right.

Example 28.5. For the puzzle in the previous example,

$I = \{2, 4\}, \; J = \{2, 4\}, \; K = \{2, 3\}$.

Theorem 28.6. If there is a puzzle whose 1’s on the boundary are in positions $I, J, K$ then

1. the inequality
   $$\sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B) \leq \sum_{k \in K} \lambda_k(A + B)$$
   holds for any real symmetric (or complex Hermitian) matrices $A, B$, where $\lambda_i$ denotes the $i$-th eigenvalues.

2. the inequality
   $$\prod_{i \in I} \sigma_i(A) \prod_{j \in J} \sigma_j(B) \leq \prod_{k \in K} \sigma_k(AB)$$
   holds for any invertible real (or complex) matrices $A, B$, where $\sigma_i$ denotes the $i$-th singular value.

Example 28.7. The puzzle above gives the inequalities

$$\lambda_2(A) + \lambda_4(A) + \lambda_2(B) + \lambda_4(B) \leq \lambda_2A + B) + \lambda_3(A + B).$$

$$\sigma_2(A)\sigma_4(A)\sigma_2(B)\sigma_4(B) \leq \sigma_2AB\sigma_3(AB).$$

If there is a unique puzzle with boundary 1’s in places $I, J, K$, the puzzle is called rigid.

Theorem 28.8. The inequalities corresponding to rigid puzzles together with the trace equality give necessary and sufficient conditions for the eigenvalue sum and singular value product problems, and are a minimal set of inequalities with this property.

29. Introduction to Proofs

A proof is a justification of a mathematical statement, using definitions and axioms. Here are some useful logical symbols

$$\implies$$ implies
$$\iff$$ if and only if
$$\exists$$ there exists
$$\forall$$ for all

Example 29.1. The definition of invertibility is

$A$ is invertible $\iff \exists B, \; BA = AB = I$.

Example 29.2. Vectors $v_1, \ldots, v_n$ are linearly independent if

$$c_1v_1 + \ldots + c_nv_n = 0 \implies c_1 = c_2 = \ldots = c_n = 0.$$
You might realize why the property (1) holds: “It doesn’t matter what order numbers are added, therefore it doesn’t matter what order vectors are added either.” This is right, but confusing. For instance, what does “adding the number together” mean? Does it mean adding them all to get a single number, or adding each pair to get a vector?

To be precise you have to use equations. Let’s start with the “A vector is a list of numbers”. Equations corresponding to this statement might be

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \]

Many students might start out by writing

\[ u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}. \]

This assumes that the vectors are three-vectors, which isn’t an assumption in the statement of property (1). To prove a statement, you can only make the assumptions that are made in the statement, in this case, that \( u \) and \( v \) are vectors of any size.

Now the next statement, “Vector addition means adding the components.” The corresponding equations are

\[ u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}, \quad v + u = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}. \]

The statement “It doesn’t matter what order numbers are added” becomes the equations

\[ u_1 + v_1 = v_1 + u_1, \quad u_2 + v_2 = v_2 + u_2, \ldots, \quad u_3 + v_3 = v_3 + u_3. \]

Finally, because vectors are equal if and only if their components are equal, this gives

\[ u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix} = v + u. \]

To get the formal proof we eliminate a lot of the discussion. For instance, see Section 2. Many proofs about vectors begin with the statement like “Let \( v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) be an \( n \)-vector.” However, it isn’t always necessary to have variable names for the components. A good example is the proof of the property (8). A person trying to prove (8)

\[ \|cv\| = |c|\|v\|. \]

might have the initial thoughts: \( \|v\| \) means the length of \( v \). Also \( c \) is a scalar, so \( cv \) must mean scalar multiplication, that is, multiplying each component by the scalar. So the left hand side is the length of \( v \), after we have multiplied by the scalar \( c \). The right hand side is the length of \( v \), multiplied by the absolute value of \( c \).

He might think of an example: Suppose that \( c = -2 \) and \( v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \). Then the left hand side is

\[ \| -2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \| = \| \begin{bmatrix} -6 \\ -8 \end{bmatrix} \| = \sqrt{36 + 64} = \sqrt{100} = 10 \]

while the right hand side is

\[ | -2| \| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \| = 2\sqrt{9 + 16} = 2\sqrt{25} = 2(5) = 10. \]

After these initial observations, it still might not be clear to the person whether or not the statement is true for all vectors, of any size, as the statement claims. So the person might try to write down equations corresponding to the statements above.
“Multiplying each component by the scalar” might become: Let \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) be an \( n \)-vector and \( c \) be a scalar. Then
\[
\mathbf{c} \mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.
\]

“The left-hand side is the length” might become:
\[
\|\mathbf{c}\mathbf{v}\| = \sqrt{(cv_1)^2 + (cv_2)^2 + \ldots + (cv_n)^2}.
\]

On the other hand, the right hand side is
\[
|c|\|\mathbf{v}\| = |c|\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.
\]

It still might be a mystery at this point how to make these expressions equal. So he stops to make some more observations. He looks at the two expressions, and sees that the problem is how to get the \( c \)'s in the first expression, out in front as it is in the second. Now he notices, there is a \( c \) in every term! At this point he is very hopeful that it will all work out. He writes
\[
\begin{align*}
&= \sqrt{c^2(v_1)^2 + c^2(v_2)^2 + \ldots + c^2(v_n)^2} \\
&= \sqrt{c^2(v_1)^2 + c^2(v_2)^2 + \ldots + (v_n)^2} \\
&= \sqrt{c^2(v_1)^2 + c^2(v_2)^2 + \ldots + (v_n)^2} \\
&= |c|\sqrt{(v_1)^2 + (v_2)^2 + \ldots + (v_n)^2}
\end{align*}
\]

Hurray! He is done. End of proof.

This type of sequence is very typical in linear algebra proofs. The first step is often to have some vague thoughts in words about what the equation means. It usually isn’t clear at this point how to prove the statement, or whether it is true. So he tries to make the thoughts into equations, in order to understand better what the statement means. After writing the equations, he stops again to have some more vague thoughts. Then he messes around a little bit with the equations. Finally by doing some algebra he arrives at the statement he wants to prove.

There is absolutely nothing wrong with the proof given above. But in fact, there is a different proof which is much shorter and doesn’t begin with “Let \( \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \) be an \( n \)-vector.” Instead, it uses properties (6) and (7).
\[
\begin{align*}
\|\mathbf{c}\mathbf{v}\| &= \sqrt{(\mathbf{c}) \cdot (\mathbf{c})} \quad \text{By Prop. (7)} \\
&= \sqrt{c(\mathbf{v} \cdot \mathbf{c})} \quad \text{By Prop. (6)} \\
&= \sqrt{c^2(\mathbf{v} \cdot \mathbf{v})} \quad \text{By Prop. (6) again} \\
&= \sqrt{c^2\mathbf{v} \cdot \mathbf{v}} \quad \text{From algebra} \\
&= |c|\|\mathbf{v}\| \quad \text{By Prop. (7) again}
\end{align*}
\]

This proof is nice and short.

Many students are tempted to say, “I don’t care about the proofs. I just want to be able to solve problems.” You can learn a lot of linear algebra without being able to do proofs like this. But you won’t be able to solve problems well in linear algebra, unless you try to do and understand proofs. Linear algebra simply has so many interesting rules that it is impossible to memorize all of them.

29.2. Proof by deduction. Proof by deduction is used to prove “If ... then ...” statements. Assume the If statement, and deduce the Then statement.

Example 29.3. Prove that if \( AB = BA \), then \( AB^2 = B^2A \).

Proof: Assume \( AB = BA \). Then
\[
AB^2 = A(AB) \quad \text{definition of matrix powers}
\]
\[
= (AB)B \quad \text{matrix associativity}
\]
\[
= (BA)B \quad \text{assumption}
\]
\[
= B(AB) \quad \text{matrix associativity}
\]
\[
= B(BA) \quad \text{assumption}
\]
\[
= (BB)A \quad \text{matrix associativity}
\]
\[
= B^2A \quad \text{definition of matrix powers}
\]
Example 29.4. Prove that if $A$ is invertible, then so is $A^2$.

$A$ is invertible \hspace{1cm} \text{assumption}
\exists B, \ AB = BA = I \hspace{1cm} \text{definition}
A(AB)B = AIB = AB = I \hspace{1cm} \text{using previous line}
(AA)(BB) = I \hspace{1cm} \text{matrix associativity}
A^2B^2 = I \hspace{1cm} \text{definition of matrix powers}
B^2A^2 = B(BA)A = BIA = I \hspace{1cm} \text{similar reasoning}
\exists C, A^2C = CA^2 = I \hspace{1cm} \text{take } C = B^2
A^2 \text{ is invertible} \hspace{1cm} \text{definition}

Example 29.5. Prove that if $A$ is invertible, then so is $A^2$. (Different answer.)

$A$ is invertible \hspace{1cm} \text{assumption}
det(A) \neq 0 \hspace{1cm} \text{theorem}
det(A^2) = det(A) \det(A^2) \hspace{1cm} \text{theorem}
det(A^2) \neq 0 \hspace{1cm} \text{previous two lines}
A^2 \text{ is invertible} \hspace{1cm} \text{theorem}

Sometimes you have to use a “proof within a proof”. For example, to prove that vectors are independent, you should assume $c_1v_1 + \ldots + c_nv_n$ and prove that $c_1 = c_2 = \ldots = c_n = 0$, as in Example 29.2.

Example 29.6. Prove that if $v_1$ and $v_2$ are non-zero, orthogonal vectors, then they are linearly independent.

(1) $v_1, v_2$ are orthogonal \hspace{1cm} \text{assumption}
(2) $v_1 \cdot v_2 = 0$ \hspace{1cm} \text{from line 1}
(3) $v_1, v_2 \neq 0$ \hspace{1cm} \text{assumption}
(4) $0 = c_1v_1 + c_2v_2$ \hspace{1cm} \text{assumption}
(5) $(c_1v_1 + c_2v_2) \cdot v_1 = c_1v_1 \cdot v_1 + c_2v_2 \cdot v_1$ \hspace{1cm} \text{distributive property}
(6) $0 = c_1v_1 \cdot v_1$ \hspace{1cm} \text{from lines 4, 5}
(7) either $c_1 = 0$ or $v_1 \cdot v_1 = 0$ \hspace{1cm} \text{from line 6}
(8) $v_1 \cdot v_1 \neq 0$ \hspace{1cm} \text{from line 3}
(9) $c_1 = 0$ \hspace{1cm} \text{from lines 7, 8}
(10) $(c_1v_1 + c_2v_2) \cdot v_2 = c_2v_2 \cdot v_2 = 0$ \hspace{1cm} \text{line 2, distributivity}
(11) $c_2 = 0$ \hspace{1cm} \text{line 3, 10}
(12) $v_1, v_2$ are linearly independent \hspace{1cm} \text{lines 9, 11 and definition}

29.3. Proof by contradiction. This method is also used to prove “If ... then .....” statements. Assume the if part and the negative of the then statement, and show a contradiction. Prove that if $Av = 0$ for some non-zero vector $v$, then $A$ is not invertible.

$\exists v, \ Av = 0$ and $v \neq 0$ \hspace{1cm} \text{assumption}
$A$ is invertible \hspace{1cm} \text{assumption}
$\exists B, \ BA = AB = I$ \hspace{1cm} \text{definition}
$B(Av) = B0 = 0$ \hspace{1cm} \text{by first line}
$B(0v) = (BA)v = Iv = v$ \hspace{1cm} \text{by third line}
$v = 0$ \hspace{1cm} \text{by last two lines}
$0 \neq 0$ \hspace{1cm} \text{by last and first line}

29.4. Proof by induction. This type of proof is used to show statements that begin “For any positive integer $n$, .....”. To use proof by induction, show the statement for $n = 1$. Then show that if the statement holds for $n$, then it holds for $n + 1$. The assumption the statement holds for $n$ is called the inductive hypothesis.

Example 29.9. Prove that if $det(A^n) = det(A)^n$. For $n = 1$,

$det(A^n) = det(A^1) = det(A) = det(A)^1$ so the statement holds for $n = 1$.

Suppose that the statement holds for $n$, that is,

$det(A^n) = det(A)^n$.

We have to show that it holds for $n + 1$, that is, we want to show that

$det(A^{n+1}) = det(A)^{n+1}$.

We write

$det(A^n) = det(A)^n$ \hspace{1cm} \text{inductive hypothesis}
$det(A^{n+1}) = det(A^nA)$ \hspace{1cm} \text{definition of matrix powers}
$det(A^nA) = det(A^n) \det(A)$ \hspace{1cm} \text{theorem } det(AB) = det(A)det(B)
$det(A^n) \det(A) = det(A)^n \det(A)$ \hspace{1cm} \text{from line 1}
$det(A)^n \det(A) = det(A)^{n+1}$ \hspace{1cm} \text{algebra}
$det(A^{n+1}) = det(A)^{n+1}$ \hspace{1cm} \text{using lines 2-5}