Connected Groups of Finite Morley Rank

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MWMT
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Connected Groups of Finite Morley Rank

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I. Structure

Essential Notions

Algebraicity and Structure

II. Geometry

Good Tori

Carter subgroups

III. Application

Generic $t$-transitivity

Lower bounds for $T$

Desiderata
Essential Notions—Generalities

- Morley rank ($rk(X)$)
- **Generic set:** $rk(X) = rk(G)$
- Connected group
  
  $[G : H] < \infty \implies G = H.$
  
  $X, Y \subseteq G$ generic $\implies X \cap Y$ generic

- $d(X)$: definable subgroup generated by $X$.
- **Fubini:** Lascar-Borovik-Poizat
Essential Notions—\(p\)-groups and Types

- **\(p\)-torus**: divisible abelian \(p\)-group

- **Types**:
  - **Degenerate**: No infinite 2-subgroup
  - **Even**: Nondegenerate, no nontrivial 2-torus
    ("characteristic two type")

- **\(p\)-unipotent**: definable, connected, bounded exponent, nilpotent \(p\)-group
**The Algebraicity Conjecture**

**Conjecture (Algebraicity)**

\[ G: \text{finite Morley rank, connected.} \]
\[ H: \text{maximal connected solvable normal, definable.} \]

\[ 1 \rightarrow H \rightarrow G \rightarrow \tilde{G} \rightarrow 1 \]

\[ \tilde{G}: \text{a central product of algebraic groups.} \]

Equivalently: The simple groups are algebraic.
Borovik Programme

- FSG (15,000 pp., or 5,000 pp.)
- (No bad fields)
- **Minimal Counterexample**

... The perils of incomplete inductive arguments ...
Groups without 2-tori (I)

1 → \( O_2(G) \) → \( G \) → \( \bar{G} \) → 1

\( O_2(G) \): maximal normal unipotent 2-subgroup;

\[
\bar{G} = U_2(\bar{G}) \ast \hat{O}(\bar{G})
\]
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\( O_2(G) \): maximal normal unipotent 2-subgroup;

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- \( U_2(\bar{G}) \): product of algebraic groups;
- \( \hat{O}(G) \): no involutions
Groups without 2-tori (I)

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\[\bar{G} = U_2(\bar{G}) \ast \hat{O}(\bar{G})\]

- \(U_2(\bar{G})\): product of algebraic groups;
- \(\hat{O}(G)\): no involutions

Definition

\[U_2(G) = \langle U \leq G : 2\text{-unipotent} \rangle.\]
Groups without 2-tori (II)

\[ \bar{G} = U_2(\bar{G}) \ast \hat{O}(\bar{G}) \text{ (Algebraic } \ast \text{ degenerate.)} \]
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\[ \bar{G} = U_2(\bar{G}) \ast \hat{O}(\bar{G}) \]  (Algebraic * degenerate.)

**Ingredients**

**Theorem (E,M)**

A simple group of even type is algebraic.

There are no simple groups of finite Morley rank of mixed type.

**Theorem (D)**

A connected degenerate type group contains no elements of order two.
Groups without 2-tori (II)

\[ \bar{G} = U_2(\bar{G}) \ast \hat{O}(\bar{G}) \] (Algebraic \ast degenerate.)

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**Methods:** Finite group theory, good tori, Wagner on fields of finite Morley rank—**classification**

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Ingredients

**Theorem (E,M)**

A simple group of even type is algebraic.
There are no simple groups of finite Morley rank of mixed type.

Methods: Finite group theory, good tori, Wagner on fields of finite Morley rank—classification

**Theorem (D)**

A connected degenerate type group contains no elements of order two.

Methods: Black box group theory, genericity arguments—soft methods
The Three Waves

Theorem (E)

A simple group of even type is algebraic.
The Three Waves

**Theorem (E)**

*A simple group of even type is algebraic.*

1st  No bad fields, no degenerate type simple sections.
2nd  No degenerate type simple sections.
3rd  General case
The Three Waves

**Theorem (E)**

A simple group of even type is algebraic.

1st No bad fields, no degenerate type simple sections.

2nd No degenerate type simple sections.

3rd General case

The base case: Groups with strongly embedded subgroups.

1st Altınel’s Thesis

2nd Jaligot’s Thesis

3rd Altınel’s Habilitation . . . Limoncello
The Three Waves

**Theorem (E)**

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The base case: Groups with strongly embedded subgroups.

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From FSG to Geometry (good tori). (More below.)
1 \to U_2(G) \to G \to \tilde{G} \to 1

\tilde{G}: No nontrivial unipotent 2-subgroups.
Groups with 2-Tori

\[ 1 \to U_2(G) \to G \to \tilde{G} \to 1 \]

\( \tilde{G} \): No nontrivial unipotent 2-subgroups.
Back to the Borovik Programme: bounds on Prüfer 2-rank.

**Theorem (Borovik, Burdges, Cherlin, Jaligot)**

*In a minimal connected nonalgebraic simple group of finite Morley rank, the Prüfer 2-rank is at most 2.*

—Burdges unipotence theory for elimination of hypotheses on bad fields.
—Analysis of minimal simple groups: Deloro (with technology of Burdges, Frécon).
Groups without 2-unipotent subgroups

In a more geometrical vein . . .

**Theorem**

- 2-elements are toral.
- Maximal 2-tori are conjugate.
- Any 2-element in the centralizer of a maximal 2-torus belongs to that 2-torus.
- The generic element of $G$ belongs to $C^\circ(T)$ for a unique maximal 2-torus $T$.

But this is a shift in emphasis . . .
I. Structure
   • Essential Notions
   • Algebraicity and Structure

II. Geometry
   • Good Tori
   • Carter subgroups

III. Application
   • Generic $t$-transitivity
   • Lower bounds for $T$

Desiderata
Definition

A definable divisible abelian subgroup $T$ of $G$ is a **good torus** if every definable subgroup of $T$ is the definable hull of its torsion subgroup.

The multiplicative group of a field of finite Morley rank is a good torus [Wagner]. Maximal good tori are conjugate [Cherlin].

Limoncello (Even type with strongly embedded subgroups IV): finiteness of the number of conjugacy classes of 1-dimensional algebraic tori contained in a fixed definable subgroup.
Good tori

Definition

A definable divisible abelian subgroup $T$ of $G$ is a **good torus** if every definable subgroup of $T$ is the definable hull of its torsion subgroup.

Rigidity properties:

**R-I** $N^o(T) = C^o(T)$

**R-II** Any uniformly definable family of subgroups of $T$ is finite.
Definition

A definable divisible abelian subgroup $T$ of $G$ is a **good torus** if every definable subgroup of $T$ is the definable hull of its torsion subgroup.

Theorem

- *The multiplicative group of a field of finite Morley rank is a good torus* [Wagner].
- *Maximal good tori are conjugate* [Cherlin].
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Limoncello (Even type with strongly embedded subgroups IV):

finiteness of the number of conjugacy classes of 1-dimensional algebraic tori contained in a fixed definable subgroup.
Theorem \((T_p)\)

If \(T\) is a \(p\)-torus and \(H = C^\circ(T)\), then the union of the conjugates of \(H\) is generic in \(G\).
Generic Covering and Conjugacy

**Theorem** $(T_p)$

*If $T$ is a $p$-torus and $H = C^o(T)$, then the union of the conjugates of $H$ is generic in $G$.*

**Properties of $H = C^o(T)$:**
- Almost self-normalizing (Rigidity-I)
- Generically disjoint from its conjugates: $H \setminus (\bigcup H^{[G \setminus N(H)]})$ generic in $H$. 

**Lemma (Genericity Lemma)**

If a definable subgroup $H$ of $G$ is almost self-normalizing and generically disjoint from its conjugates then:

- For $X \subseteq H$, we have in $H$. 

**Definition**

$X$ is generous in $G$ if the union of its conjugates is generic in $G$. 

**Desiderata**

I. Structure
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Desiderata
Theorem (T\(_p\))

If \( T \) is a \( p \)-torus and \( H = C^\circ(T) \), then the union of the conjugates of \( H \) is generic in \( G \).

Lemma (Genericity Lemma)

If a definable subgroup \( H \) of \( G \) is almost self-normalizing and generically disjoint from its conjugates then:

- \( \bigcup H^G \) is generic in \( G \);
- For \( X \subseteq H \), we have \( \bigcup X^G \) generic in \( G \) if and only if \( \bigcup X^H \) is generic in \( H \).
Generic Covering and Conjugacy

**Theorem (Tₚ)**

*If T is a p-torus and \( H = C^\circ(T) \), then \( H \) is generous in \( G \).*

**Lemma (Genericity Lemma)**

*If a definable subgroup \( H \) of \( G \) is almost self-normalizing and generically disjoint from its conjugates then:*

- \( \bigcup H^G \) is generic in \( G \);
- For \( X \subseteq H \), we have \( \bigcup X^G \) generic in \( G \) if and only if \( \bigcup X^H \) is generic in \( H \).*

**Definition**

\( X \) is generous in \( G \) if the union of its conjugates is generic in \( G \).
Generic Covering and Conjugacy

**Theorem (T_p)**

If $T$ is a $p$-torus and $H = C^o(T)$, then in $G$.

**Lemma (Genericity Lemma)**

If a definable subgroup $H$ of $G$ is almost self-normalizing and generically disjoint from its conjugates then:

- $H$ is generous in $G$;
- For $X \subseteq H$, we have $X$ is generous in $G$ if and only if $X$ is generous in $H$.

**Definition**

$X$ is generous in $G$ if the union of its conjugates is generic in $G$. 
Definition

A **Carter subgroup** of $G$ is a connected definable nilpotent subgroup which is almost self-normalizing.
Carter Subgroups

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Theorem (Frécon-Jaligot)

They exist.
Carter Subgroups

**Definition**

A **Carter subgroup** of $G$ is a connected definable nilpotent subgroup which is almost self-normalizing.

**Theorem (Frécon-Jaligot)**

*They exist.*

**Theorem (Frécon)**

*In a $K^*$-group, Carter subgroups are conjugate.*

A tour de force. This is a case where a minimal counterexample eventually dies completely. Along the way, Burdges’ Bender method is used, and many other things.
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Desiderata
Theorem (BC)

\((G, X)\) definably primitive. Then \(\text{rk} (G)\) is bounded by a function of \(\text{rk} (X)\).

**Definably primitive:** no nontrivial \(G\)-invariant definable equivalence relation.

MPOSA = Macpherson-Pillay/O’Nan-Scott-Aschbacher
A description of the **socle** of a primitive permutation group, and the stabilizer of a point in that socle.

- **Affine:** The socle \(A\) is abelian and can be identified with the set \(X\) on which \(G\) acts.
- **Non-affine:** The socle is a product of copies of one simple group.
Theorem

\((G, X)\) definably primitive. Then \(\text{rk}(G)\) is bounded by a function of \(\text{rk}(X)\).
Generic multiple transitivity

Theorem

\((G, X)\) definably primitive. Then \(\text{rk}(G)\) is bounded by a function of \(\text{rk}(X)\).

Generic transitivity: one large orbit.

Generic \(t\)-transitivity: on \(X^t\).
Generic multiple transitivity

**Theorem**

\((G, X)\) definably primitive. Then \(\text{rk}(G)\) is bounded by a function of \(\text{rk}(X)\).

Generic transitivity: one large orbit.

**Proposition**

\((G, X)\) definably primitive. Then the degree of multiple transitivity of \(G\) is bounded by a function of \(\text{rk}(X)\).

(Special case of the theorem, but sufficient.)
Bounds on $t$

**Proposition**

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $\text{rk} (X)$. 
Bounds on $t$

**Proposition**

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $rk(X)$.

**Strategy:** Let $T$ be the definable hull of a maximal 2-torus. Derive an upper bound on the complexity of $T$ from $rk(X)$, and a lower bound on the complexity of $T$ from $t$. 
Bounds on $t$

**Proposition**

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The upper bound: $\text{rk}(T/O_{\infty}(T)) \leq \text{rk}(X)$. This is because the stabilizer of a generic element of $X$ is torsion-free.
Bounds on $t$

**Proposition**

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $rk(X)$.

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The upper bound: $rk(T/O_{\infty}(T)) \leq rk(X)$. This is because the stabilizer of a generic element of $X$ is torsion-free. But the lower bound requires attention.
We want to show that a large degree of generic transitivity ($t$ large) blows up $rk\left(T / T_{\infty}\right)$ for $T$ the definable hull of a 2-torus.

Let us simplify considerably.
We want to show that a large degree of generic transitivity ($t$ large) blows up $rk\left(\frac{T}{T_{\infty}}\right)$ for $T$ the definable hull of a 2-torus.

Let us simplify considerably.

The group $G$ will induce the action of $Sym_t$ on any $t$ independent generic points.

Trading $T$ in for a smaller torus, and trading $t$ in for a smaller value as well (but not too small) we can set this up so that we have:

- a finite group $\Sigma$ operating on $T$, and
- covering $Sym_t$, and
- sitting inside a connected group $H$ such that
- $T$ is the definable hull of a maximal 2-torus in $H$. 

Imagine the simplest case: $Sym_t$ sits inside $G$ and acts on $T$, the definable hull of a maximal 2-torus. It then seems reasonable that this action can be exploited to blow up $T$, and also $T/\widetilde{T}$. 

There is a glaring hole in this argument.
We want to show that a large degree of generic transitivity (\(t\) large) blows up \(rk(\, T / T_\infty\,)\) for \(T\) the definable hull of a 2-torus.

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Plugging a hole

The Setup

$T$ inside $G$, $G$ connected, $Sym_t$ acts on $T$, $t$ large, and $T$ is the definable hull of a maximal 2-torus.

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However: at this point $G$ can again be taken to be simple (via MPOSA) and therefore a dichotomy applies:

- Either $G$ is algebraic or
- $G$ contains no unipotent 2-subgroup.
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In the former case, we can trade 2 off for a prime different from the characteristic and use the bound on $rk\left(\frac{T}{T_\infty}\right)$ to control the rank of $G$—structure theory.
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In the latter case, recall that 2-elements in the centralizer of $T$ belong to $T$. It follows easily that the action of $\text{Sym}_t$ on $T$ is faithful.
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And so, we are done!
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Desiderata
The original goal: a list of “intractable” minimal configurations.

I would like to see that in final form!

Better bounds on primitive permutation groups, particularly in the algebraic case.

Popov in characteristic 0, with algebraic actions.

L-group theory for odd type groups.

Frécon perhaps, in his work on conjugacy of Carter subgroups.
Desiderata

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- **Absolute bounds** on Prüfer rank of groups of odd type

- Construction of bad field towers.

- Construction of bad groups
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And a pony!