Permutation groups of Finite Morley rank

Gregory Cherlin

Rutgers The State University of New Jersey

June 29 – Edinburgh
Theorem

If $G$ is an algebraic group acting rationally, faithfully, and primitively on a variety $X$ then the dimension of $G$ can be bounded in terms of the dimension of $X$.

(Proved more broadly in the category of groups of finite Morley rank.)
Permutation groups of Finite Morley rank

Groups of Finite Morley rank

Methods

Permutation groups
Morley rank

\[ \text{rk} = \text{dim}: \mathcal{D}\mathcal{E}\mathcal{F}(G^n) \to \mathbb{N}. \]

Examples: Algebraic groups; differentially algebraic groups; compact complex groups.
Morley rank

\[ \text{rk} = \dim: \DEF(G^n) \to \mathbb{N}. \]

Examples: Algebraic groups; differentially algebraic groups; compact complex groups.
Diophantine applications: Abelian varieties, Manin kernel.
Typically: \textit{abelian} groups.
Morley rank
\[
\text{rk } = \dim: \mathcal{DEF}(G^n) \to \mathbb{N}.
\]
Examples: Algebraic groups; differentially algebraic groups; compact complex groups.
Diophantine applications: Abelian varieties, Manin kernel.
Typically: \textit{abelian} groups.

Conjecture

\textit{A simple group of finite Morley rank is algebraic.}
2-Tori

Theorem (Dichotomy)

Let $G$ be a simple group of finite Morley rank containing an involution. Then one of the following holds.

- $G$ is a simple algebraic group.
- $G$ contains a nontrivial 2-torus.
2-Tori

Theorem (Dichotomy)

Let $G$ be a simple group of finite Morley rank containing an involution. Then one of the following holds.

- $G$ is a simple algebraic group.
- $G$ contains a nontrivial 2-torus.

Theorem (ABC: Char 2 case)

Let $G$ be a simple group of finite Morley rank containing an infinite 2-group of bounded exponent. Then $G$ is algebraic, char. 2.
### Theorem (Dichotomy)

Let $G$ be a simple group of finite Morley rank containing an involution. Then one of the following holds.

- $G$ is a simple algebraic group.
- $G$ contains a nontrivial 2-torus.

### Theorem (BBC: Nondegeneracy)

Let $G$ be a connected group of finite Morley rank containing an involution. Then $G$ contains an infinite 2-subgroup.
1 Groups of Finite Morley rank

2 Methods

3 Permutation groups
Characteristic 2 type

Theorem (ABC: Char 2 case)

$G$ simple, finite Morley rank, with an infinite 2-group of bounded exponent. Then $G$ is algebraic.
Characteristic 2 type

Theorem (ABC: Char 2 case)

\(G\) simple, finite Morley rank, with an infinite 2-group of bounded exponent. Then \(G\) is algebraic.

Finite simple group theory:

- 1st generation: 15000 pages (condensed)
- 2nd generation: 5000 pages (in progress)
- Finite Morley rank: 300 pages.
Characteristic 2 type

**Theorem (ABC: Char 2 case)**

*G simple, finite Morley rank, with an infinite 2-group of bounded exponent. Then G is algebraic.*

Finite simple group theory:

- 1st generation: 15000 pages (condensed)
- 2nd generation: 5000 pages (in progress)
- Finite Morley rank: 300 pages.

Simplifying ingredient: fields are algebraically closed.
In finite group theory, the torus $\mathbb{F}_2^*$ reduces to a single element. This breaks “generic” arguments.
Sporadic groups are only one manifestation of this.
Characteristic 2 type

**Theorem (ABC: Char 2 case)**

*G simple, finite Morley rank, with an infinite 2-group of bounded exponent. Then G is algebraic.*

Finite simple group theory:

- 1st generation: 15000 pages (condensed)
- 2nd generation: 5000 pages (in progress)
- Finite Morley rank: 300 pages.

In finite group theory, the torus $\mathbb{F}_2^*$ reduces to a single element. This breaks “generic” arguments.

Wagner: in the multiplicative group of a field of finite Morley rank, in positive characteristic, torsion is dense (rigidity).
Nondegeneracy

Theorem (BBC: Nondegeneracy)

Let $G$ be a connected group of finite Morley rank containing an involution. Then $G$ contains an infinite 2-subgroup.
Theorem (BBC: Nondegeneracy)

Let $G$ be a connected group of finite Morley rank containing an involution. Then $G$ contains an infinite $2$-subgroup.

In a minimal counterexample, each involution $i$ lies outside $C^\circ(i)$. In particular $C(i)$ is disconnected.
Theorem (BBC: Nondegeneracy)

Let $G$ be a connected group of finite Morley rank containing an involution. Then $G$ contains an infinite $2$-subgroup.

In a minimal counterexample, each involution $i$ lies outside $C^o(i)$. In particular $C(i)$ is disconnected.

“Black Box group theory” . . .
Let $I$ be a conjugacy class of involutions and consider the structure of

$$d(ij) : i, j \in I \times I$$
Dihedral groups

Let $I$ be a conjugacy class of involutions and consider the structure of

$$d(ij) : i, j \in I \times I$$

Either $d(ij)$ contains a unique involution $k$ (commuting with $i$ and $j$) or $i, j$ are conjugate under the action of $d(ij)$. 
Dihedral groups

Let $I$ be a conjugacy class of involutions and consider the structure of

$$d(ij) : i, j \in I \times I$$

Either $d(ij)$ contains a unique involution $k$ (commuting with $i$ and $j$) or $i, j$ are conjugate under the action of $d(ij)$. Furthermore one of these two possibilities holds generically.
Let $I$ be a conjugacy class of involutions and consider the structure of

$$d(ij) : i, j \in I \times I$$

Either $d(ij)$ contains a unique involution $k$ (commuting with $i$ and $j$) or $i, j$ are conjugate under the action of $d(ij)$. Furthermore one of these two possibilities holds \textit{generically}. So the argument splits to two cases.
Case I: \( d(ij) \) contains no involution

\[ j = i^g = i^x \text{ with } x \in d(ij). \]
Case I: $d(ij)$ contains no involution

\[ j = i^g = i^x \text{ with } x \in d(ij). \]
\[ \zeta(g) = gx^{-1} \in gd(ij) \cap C(i). \]
Case I: $d(ij)$ contains no involution

\[ j = i^g = i^x \text{ with } x \in d(ij). \]
\[ \zeta(g) = gx^{-1} \in gd(ij) \cap C(i). \]

Covariant generically defined map $\zeta : G \to C(i)$

$(\zeta(cg) = c\zeta(g))$. 
Case I: $d(ij)$ contains no involution

$j = i^g = i^x$ with $x \in d(ij)$.

$\zeta(g) = gx^{-1} \in gd(ij) \cap C(i)$.

Covariant generically defined map $\zeta : G \to C(i)$

($\zeta(cg) = c\zeta(g)$).

Conclusion:

- Fiber ranks of $\zeta$ over points of $C(i)$ are constant;
- $\deg(G) \geq \deg(C(i))$

Hence $G$ connected implies $C(i)$ connected.
We use a generically defined covariant map to transfer coarse structure between $G$ and $C(i)$. 
Black Box Group Theory

$G$ large finite. Choose elements independently with a uniform distribution and perform various operations or measurements on them (compute orders, multiply, ...). Allowable operations vary.
Black Box Group Theory

$G$ large finite. Choose elements independently with a uniform distribution and perform various operations or measurements on them (compute orders, multiply, ...). Allowable operations vary.

Problem: analyze $G$—is it simple? and if so, what group is it?
**Black Box Group Theory**

$G$ large finite. Choose elements independently with a uniform distribution and perform various operations or measurements on them (compute orders, multiply, ...). Allowable operations vary.

Problem: analyze $G$—is it simple? and if so, what group is it?

Determine $O_p(G)$ in characteristic $p$. 
Black Box Group Theory

$G$ large finite. Choose elements independently with a uniform distribution and perform various operations or measurements on them (compute orders, multiply, ...). Allowable operations vary.

Problem: analyze $G$—is it simple? and if so, what group is it?

Determine $O_p(G)$ in characteristic $p$.

Using centralizers of involutions ($p$ odd).

Question: If $G$ is a black box group, is $C(i)$ a black box group?
Is $C(i)$ a black box group?

In the favorable case $\zeta : G \rightarrow C(i)$ generically, $\zeta(g)$ picks elements of $C(i)$ randomly.

The uniform measure on $G$ is carried to the uniform measure on $C(i)$. 
1 Groups of Finite Morley rank

2 Methods

3 Permutation groups
Main Theorem

Theorem (Borovik/Cherlin)

If $G$ is a group of finite Morley rank acting definably, faithfully, and primitively on a set $X$ then the rank of $G$ can be bounded in terms of the rank of $X$. 
Imprimitive Example

\[ T = K^\times \text{ acting on } V \text{ via } (t^i v_i), \ W \leq V \text{ hyperplane in general position.} \]

\[ \hat{G} = V \rtimes T \text{ acting on the coset space } X = W \backslash \hat{G} \text{ (faithfully).} \]

\[ rk \ (X) = 2, \ rk \ (\hat{G}) = \dim(V) + 1. \]
**Generic Multiple Transitivity**

$X^{(t)}$: $t$-tuples of *distinct* elements.

$t$-transitive: transitive on $X^{(t)}$
Generic Multiple Transitivity

\( X^{(t)} \): \( t \)-tuples of distinct elements.

\( t \)-transitive: transitive on \( X^{(t)} \)

Generically transitive: one large orbit \( \Omega \) on \( X \), i.e.

\[ rk(X \setminus \Omega) < rk(X) \]
Generic Multiple Transitivity

$X^{(t)}$: $t$-tuples of distinct elements.

t-transitive: transitive on $X^{(t)}$

Generically transitive: one large orbit $\Omega$ on $X$, i.e.

$$rk (X \setminus \Omega) < rk (X)$$

Generically $t$-transitive: generically transitive on $X^t$. 

Generic Multiple Transitivity

$X^{(t)}$: $t$-tuples of distinct elements.

t-transitive: transitive on $X^{(t)}$

Generically transitive: one large orbit $\Omega$ on $X$, i.e.

$$rk (X \setminus \Omega) < rk (X)$$

Generically $t$-transitive: generically transitive on $X^t$.

Not fully classified, even for actions of algebraic groups (Popov in char. 0).
Generic Multiple Transitivity

$X^{(t)}$: $t$-tuples of distinct elements.

t-transitive: transitive on $X^{(t)}$

Generically transitive: one large orbit $\Omega$ on $X$, i.e.

$$rk (X \setminus \Omega) < rk (X)$$

Generically $t$-transitive: generically transitive on $X^t$.

$$rk (G) \geq t \cdot rk (X)$$
Generic Multiple Transitivity

$X^{(t)}$: $t$-tuples of distinct elements.

t-transitive: transitive on $X^{(t)}$

Generically transitive: one large orbit $\Omega$ on $X$, i.e.

$$rk\left(X \setminus \Omega\right) < rk\left(X\right)$$

Generically $t$-transitive: generically transitive on $X^t$.

$$rk\left(G\right) \geq t \cdot rk\left(X\right)$$

If we wish to bound $rk\left(G\right)$ we must bound $t$. Conversely, in the primitive case, this is enough.
Bounds on $t$

$(G, X)$ finite Morley rank, $r = rk(X)$ (fixed)

t$(G, X) = \sup(t: G \text{ generically } t\text{-transitive})$
Bounds on $t$

$(G, X)$ finite Morley rank, $r = rk (X)$ (fixed)

$t(G, X) = \sup(t: G$ generically $t$-transitive$)$

$\tau(r): \sup t(G, X)$ i.e. the degree of generic multiple transitivity, with $(G, X)$ definably primitive.
(G, X) finite Morley rank, \( r = \text{rk}(X) \) (fixed)

\( t(G, X) = \sup(t: G \text{ generically } t\text{-transitive}) \)

\( \tau(r): \sup t(G, X) \) i.e. the degree of generic multiple transitivity, with \( (G, X) \) definably primitive.

\( \tau_S(r): \sup t(G, X) \) i.e. the degree of generic multiple transitivity, with \( (G, X) \) definably primitive and simple.
Bounds on \( t \)

\((G, X)\) finite Morley rank, \( r = rk(X) \) (fixed)

\[ t(G, X) = \sup(t: G \text{ generically } t\text{-transitive}) \]

\( \tau(r) \): \( \sup t(G, X) \) i.e. the degree of generic multiple transitivity, with \((G, X)\) definably primitive.

\( \tau_S(r) \): \( \sup t(G, X) \) i.e. the degree of generic multiple multiple transitivity, with \((G, X)\) definably primitive and simple.

Reduction to the simple case via Macpherson-Pillay form of O’Nan-Scott-Aschbacher.
Key Lemma

Lemma

$T$ the definable closure of p-torus, acting faithfully on $X$. Then $\text{rk} \left( T/O_0(T) \right) \leq \text{rk} \left( X \right)$. 
Bounds on $t(G)$

$G$ simple, generically highly transitive on $X$. 
Bounds on $t(G)$

$G$ simple, generically highly transitive on $X$.

$G$ contains involutions.
Bounds on $t(G)$

$G$ simple, generically highly transitive on $X$.

$G$ contains involutions.

$G$ is algebraic or contains a nontrivial 2-torus $T$. 
Bounds on $t(G)$

$G$ simple, generically highly transitive on $X$.

$G$ contains involutions.

$G$ is algebraic or contains a nontrivial 2-torus $T$.

$G$ algebraic: $T$ maximal $p$-torus, Lie rank of $G$ bounded by $rk(X)$, done.
**Bounds on $t(G)$**

$G$ simple, generically highly transitive on $X$.

$G$ contains involutions.

$G$ is algebraic or contains a nontrivial 2-torus $T$.

$G$ algebraic: $T$ maximal $p$-torus, Lie rank of $G$ bounded by $rk\ (X)$, done.

$G$ contains a nontrivial 2-torus $T$: persuade $Sym_t$ to act on $T$ nontrivially and show that $rk\ (T)$ controls $t$. 

**Ingredient:** Every involution belongs to some 2-torus (cf. torality theorem, Burdges-Cherlin).
Bounds on $t(G)$

$G$ simple, generically highly transitive on $X$.

$G$ contains involutions.

$G$ is algebraic or contains a nontrivial 2-torus $T$.

$G$ algebraic: $T$ maximal $p$-torus, Lie rank of $G$ bounded by $rk(X)$, done.

$G$ contains a nontrivial 2-torus $T$: persuade $Sym_t$ to act on $T$ nontrivially and show that $rk(T)$ controls $t$.

Ingredient: Every involution belongs to some 2-torus (cf. torality theorem, Burdges-Cherlin).
Better bounds?

\[
r = 1; \ t \leq 3; \\
r = 2; \ t \leq 27.
\]

Conjecture

\[t \leq r + 2\]
Better bounds?

\[ r = 1; \ t \leq 3; \]
\[ r = 2; \ t \leq 27. \]

**Conjecture**

\[ t \leq r + 2 \]

Known in characteristic 0 with rational actions.
Not known in characteristic \( p \) with rational actions or in characteristic 0 with definable actions.
Conclusion

Though we do not have an explicit classification of the simple groups of finite Morley rank, we can apply the theory as it stands in much the way that we would apply a full classification.
Conclusion

Though we do not have an explicit classification of the simple groups of finite Morley rank, we can apply the theory as it stands in much the way that we would apply a full classification.

Another “old chestnut”:

**Theorem**

*If $G$ is connected and satisfies the following equation generically:*

\[ x^{2^n} = 1 \]

*then indeed $G$ is a 2-group of exponent at most $2^n$.*

Again, this seems to need the classification theory, in spite of its apparently elementary character.