Between model theory and combinatorics: Homogeneity, WQO, Universality

Gregory Cherlin

Colloque en l’honneur de Chantal Berline
June 4
(9:30–10:25)
Between Model Theory and Combinatorics

I Homogeneous Structures
  • Distance Homogeneous Graphs
II Universal Graphs
  • Trees
III Well quasi-orders
  • Finiteness Theorem
Homogeneity

\[ A \cong B \implies A \sim B \text{ under } \text{Aut}(\Gamma) \]

E.g. \((\mathbb{Q}, <)\)

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Homogeneity
Recent Developments
Universality
Applications
Well quasi-orders
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Homogeneity

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Michael Aschbacher, The theory of finite groups (1986), p. 82

Urysohn 1927 (Ph.D. 1921; d. 1924, aged 26): \(\mathbb{U}\)

Rado 1964: \(G_\infty\)

Berline-Cherlin 1980-1983: QE rings
(cf. Boffa/Macintyre/Point, Baldwin/Rose, Saracino/Wood)
Fraïssé 1954: $\Gamma \leftrightarrow \text{Sub}(\Gamma)$
Amalgamation

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Amalgamation of Metric Spaces

1-point extensions: $A_i = A_0 \cup \{u_i\}$.

\[
d^+(u_1, u_2) = \min(d(u_1, a) + d(u_2, a))
\]

\[
d^-(u_1, u_2) = \max|d(u_1, a) - d(u_2, a)|
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Any positive $d$ in $[d^-, d^+]$ will do.
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Any positive \( d \) in \([d^-, d^+]\) will do.

\( \mathbb{U}_0 \): The universal homogeneous countable rational-valued metric space.

\( \mathbb{U} \): The completion of \( \mathbb{U}_0 \).
Homogeneous Graphs and Digraphs

Henson 1971: $G_n$ ($K_n$-free graph), its automorphisms and structure
Henson 1972: $D_{\sim T}$ ($T$-free digraph)
Homogeneous Graphs and Digraphs

Henson 1971: $G_n$ ($K_n$-free graph), its automorphisms and structure
Henson 1972: $D_{-T}$ ($T$-free digraph)

Imprimitive or Degenerate: $(mK_n)^\pm$; Primitive finite: $P$, $E(K_{3,3})$
Primitive infinite: $(G_n)^\pm$
Homogeneous Graphs and Digraphs

Henson 1971: $G_n$ ($K_n$-free graph), its automorphisms and structure
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Lachlan-Woodrow 1980: Homogeneous graphs classified. Imprimitive or Degenerate: $(mK_n)^\pm$; Primitive finite: $P$, $E(K_3,3)$
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Lachlan 1984: Homogeneous tournaments classified $I_1, C_3, \mathbb{Q}, \mathbb{S}, \mathcal{T}_\infty$
Cherlin 1993 (Banff proceedings): Homogeneous directed graphs
Homogeneous Graphs and Digraphs

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$I_1$, $C_3$, $\mathbb{Q}$, $\mathbb{S}$, $T_\infty$
Cherlin 1993 (Banff proceedings): Homogeneous directed graphs
Tools: Fraïssé, Finite Ramsey theorem
Between model theory and combinatorics: Homogeneity, WQO, Universality

1. Homogeneity
2. Recent Developments
3. Universality
4. Applications
5. Well quasi-orders
Torrezão de Souza/Truss 2008: Colored PO

\[ c_1 \leq c_2 \leq c_1 \], densely colored; connections between pairs of color class components; triples. Fraïssé for existence.
Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

Kechris-Pestov-Todorcevic 2005:
Fraïssé+Ramsey+Top. Dynamics

Glasner: “This remarkable paper is a tour de force where three experts in disparate areas—model theory, structural Ramsey theory and topological dynamics—collaborate in creating a unified and beautiful theory.”
Some recent developments

Kechris-Pestov-Todorcevic 2005: Fraïssé+Ramsey+Top. Dynamics

Minimal flows: compact actions with every orbit dense. Extremely amenable: no nontrivial minimal flow
Some recent developments

Kečršis-Pestov-Todorkevici 2005: Fraïssé+Ramsey+Top. Dynamics

• The extremely amenable closed subgroups of $\text{Sym}_\infty$ are exactly the groups of the form $\text{Aut}(\mathcal{C})$ with $\mathcal{C}$ the Fraïssé limit of a Fraïssé order class with the Ramsey property.
• If $\mathcal{C}$ is one of the following structures, then the universal minimal flow $M(G)$ of the group $G = \text{Aut}(\mathcal{C})$ is its action on the space of linear orderings of the universe of $\mathcal{C}_0$:
  - $\mathcal{G}_n \ (n \leq \infty)$;
  - $(\mathbb{N}, =)$;
  - $\mathcal{U}_0$
Distance homogeneous graphs?

Cameron: classify connected graphs which are homogeneous as metric spaces in the graph metric.
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$\delta \leq 2$: Lachlan-Woodrow
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\[ \Gamma_1 = \Gamma(v_*): \text{Homogeneous graph} \]
Distance homogeneous graphs?

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$\delta \leq 2$: Lachlan-Woodrow

$\Gamma_1 = \Gamma(v^*)$: Homogeneous graph

A catalog?
A catalog

1. $\delta \leq 2 \text{ (L-W);}$
A catalog

1. $\delta \leq 2$ (L-W);
2. Locally finite and limits of such
A catalog

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   2. “Doubles” (more generally: antipodal graphs)
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Homogeneity
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   - $C_n$ ($n \leq \infty$)
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   - Tree-like ($r$-tree of $s$-cliques: $r, s \leq \infty$)
A catalog

1. \( \delta \leq 2 \) (L-W);
2. Locally finite and limits of such
   1. \( C_n \) (\( n \leq \infty \))
   2. “Doubles” (more generally: antipodal graphs)
   3. Tree-like (\( r \)-tree of \( s \)-cliques: \( r, s \leq \infty \))
3. Fraïssé type
   1. \( \delta \leq d \);
   2. Omit \((1, d)\)-subspaces (\( d \geq 3 \));
   3. Omit odd cycles up to order \( 2K + 1 \);
   4. Omit triangles of perimeter \( \geq C \).
   Some interactions in these constraints.
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Recent Developments

Universality

Applications

Well quasi-orders

Exceptional $\Gamma_1 \rightarrow$ Exceptional $\Gamma$. 
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Homogeneity
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\[ \Gamma_1 \]

Exceptional \( \Gamma_1 \) \( \rightarrow \) Exceptional \( \Gamma \).

Difficulty: \( \Gamma_k \)
Exceptional $\Gamma_1 \rightarrow$ Exceptional $\Gamma$.

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**Difficulty:** $\Gamma_k$

Homogeneous metric space; not necessarily with the graph metric, because of the parity condition.

But $(\Gamma_{k-1}, \Gamma_k)$ should be.

Extend the classification project?
Between model theory and combinatorics: Homogeneity, WQO, Universality

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Universal Graphs

Komjáth-Mekler-Pach 1988: Universal graphs omitting paths; or omitting cycles of odd length
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Data: Finitely many constraints $C$ (finite, connected “forbidden” graphs).
Universal countable $C$-free graph?

? Decidable ?
Existentially complete $C$-free graphs.
(Generalizes Fraïssé.)
Existentially complete $C$-free graphs. (Generalizes Fraïssé.)

If the existentially complete countable graph is unique, then it is universal.

And there is an exact criterion for this in terms of the algebraic closure.
Our forbidden structures are forbidden in the graph theorist’s sense, not the model theorist’s (“induced”) sense. 

N.B.: if one takes induced substructures then one gets domino problems if the language is rich enough (maybe not in graphs??)
Algebraic Closure

Forbid $C$. What is $acl_C(A)$?
Algebraic Closure

Forbid $\mathcal{C}$. What is $acl_{\mathcal{C}}(A)$?

• Forbid $C_4$. Then for points $u, v$ at distance 2, the “midpoint” is a definable function $f(u, v)$. Such points are in the “definable closure” of $u, v$. 
Forbid $\mathcal{C}$. What is $acl_{\mathcal{C}}(A)$?

- Forbid $C_4$. Then for points $u, v$ at distance 2, the “midpoint” is a definable function $f(u, v)$. Such points are in the “definable closure” of $u, v$.

- Forbid a star $S_k$. Then for any $u$, the neighbors of $u$ are “algebraic” over $u$: they lie in a $u$-definable finite set. (So the algebraic closure of a point is its connected component.)
Theorem (CSS 1999)

Let $C$ be a finite set of forbidden graphs, $T$ the theory of the existentially complete $C$-free graphs. Then the following are equivalent.

1. $T$ has a unique countable model
2. The algebraic closure operator is locally finite.
Theorem (CSS 1999)

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1. $T$ has a unique countable model
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Proof.

$\implies$: General nonsense (Ryll-Nardzewski, Engeler, Svenonius)

$\impliedby$: Close analysis: over any finite algebraically closed set, the set of types is finite.
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1 Homogeneity
2 Recent Developments
3 Universality
4 Applications
5 Well quasi-orders
Conjectured by Menachem Kojman:

**Theorem**

*If $C$ is closed under homomorphism (i.e., the image of a constraint in $C$ under graph homomorphism is $C$-forbidden) then acl is degenerate and there is a universal $C$-free graph.*

Example. Odd cycles.
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**Theorem**

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Example. Odd cycles.

**Theorem (Cherlin-Shi 1996)**

For $C$ a finite set of cycles the following are equivalent.

1. There is a universal $C$-free graph.
2. $C$ consists of all odd cycles up to a fixed length.
Theorem (Cherlin-Shelah 2007)

For $\mathcal{C} = \{T\}$ a single tree, the following are equivalent.

1. There is a universal $\mathcal{C}$-free graph.
2. The tree $T$ is an extension of a path by at most one additional edge.
Theorem (Cherlin-Shelah 2007)

For $\mathcal{C} = \{T\}$ a single tree, the following are equivalent.

1. There is a universal $\mathcal{C}$-free graph.
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(←→: Cherlin-Tallgren 2007, based on KMP)
... and trees

Theorem (Cherlin-Shelah 2007)

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(\( \Longleftrightarrow \): Cherlin-Tallgren 2007, based on KMP)

Shelah’s idea: Pruning

To prune a tree \( T \): \( T' \) is obtained by removing all leaves.
Lemma

If there is a $T$-free universal graph $G$ then there is a $T'$-universal graph $G^*$, consisting of the vertices of $G$ of infinite degree.
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Minimal trees: those which prune to a path or near-path. (15 cases).
In general: Remove a minimal block-leaf. (Or a downward-closed family.)
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Conjectures

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If there is $C$-free universal graph, then $C$ has complete blocks and a path-like structure, with very few exceptions.
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If there is $C$-free universal graph, then $C$ has complete blocks and a path-like structure, with very few exceptions.

Conjecture

For a single connected constraint $C$, the problem of determining whether there is a universal $C$-free graph is algorithmically decidable.
A Concrete Example

The Bouquet $K_5 \land K_5$

$(K_5 + K_5)$ - free

(Algebraic closure running along the mid-line)
The Hairy Ball Graph

- **The Hairy Ball Problem** Let $K$ be a finite graph consisting of a complete graph together with a single finite path attached to each vertex. Is there a universal $K$-free graph?

Equivalently: if one strings together an infinite series of “canonical obstructions” ($K$ minus part of one path) along a 2-way infinite path, does the graph $K$ necessarily appear?
1. Homogeneity

2. Recent Developments

3. Universality

4. Applications

5. Well quasi-orders
WQO

Well-founded: no descending chains.
WQO: no descending chains or infinite antichains.

Classes of finite structures ordered by embedding (in either of the two common senses) are well-founded, but not in general WQO.

Robertson-Seymour: Finite graphs under “graph minor” are WQO.

Friedman: this is (formally speaking) not easy to prove—that is, it requires impredicative methods.
A dichotomy?

\[ Q: \text{our favorite quasi-order (e.g., all finite tournaments)} \]
\[ C \subseteq Q \text{ finite (constraints, “forbidden” points)} \]
\[ Q_C : C\text{-free elements} - \neg \exists c \in C( x \geq c) \]
A dichotomy?

\[ \mathcal{Q} : \text{our favorite quasi-order (e.g., all finite tournaments)} \]
\[ C \subseteq \mathcal{Q} \text{ finite (constraints, “forbidden” points)} \]
\[ \mathcal{Q}_C : C\text{-free elements} - \nexists c \in C \ (x \geq c) \]

Problem: Is \( \mathcal{Q}_C \) WQO?
A dichotomy?

\[ Q: \text{our favorite quasi-order (e.g., all finite tournaments)} \]
\[ C \subseteq Q \text{ finite (constraints, “forbidden” points)} \]
\[ Q_C: C\text{-free elements—} \neg \exists c \in C(x \geq c) \]

Problem: Is \( Q_C \) WQO?

Meta-Problem: Can you tell?

*Thesis*: This is a *dichotomy* only if one can decide algorithmically which case one is in.
An Example (Friedman)

For $L$ a linear order, let $L_{WO}$ be the largest initial segment of $L$ which is well ordered.

Let $<_1$ be a recursive ordering of $\mathbb{N}$ so that $(\mathbb{N}, <)_{WO}$ is complete $\Pi^1_1$.

Let $Q^*$ be the quasiorder of $\mathbb{N}$ defined by

$$m \leq^* n \iff (m \leq n \& m \leq_1 n)$$
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Let $Q^*$ be the quasiorder of $\mathbb{N}$ defined by

$$m \leq^* n \iff (m \leq n \& m \leq_1 n)$$

Then:
- $Q^*$ is well-founded;
- for $m \in Q^*$, $Q^*_m$ is wqo iff the initial segment determined by $m$ in $(\mathbb{N}, <_1)$ is well ordered.
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Then:
- $Q^*$ is well-founded;
- for $m \in Q^*$, $Q_m^*$ is wqo iff the initial segment determined by $m$ in $(\mathbb{N}, <_1)$ is well ordered.

Corollary

In the effectively given quasiorder $Q^*$, recognizing those constraints $c$ which correspond to wqo ideals is as difficult as it could possibly be.
Failures of WQO: Examples

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Paths with colored vertices:
Tournaments:
Permutations:
These are minimal antichains:
\( Q \prec I \) is wqo

Lemma
Below any antichain there is a minimal antichain.
(Minimal bad sequence argument)
These antichains are also isolated: there is a finite set of constraints \( C \) such these are the only antichains in \( Q^C \), up to equivalence.
(I.e., up to \( Q^C \prec I = Q^C \prec J \).)
Failures of WQO: Examples

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Failures of WQO: Examples

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Well quasi-orders
Failures of WQO: Examples

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These are **minimal antichains**: $Q^<I$ is wqo

**Lemma**

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These antichains are also **isolated**: there is a finite set of constraints $C$ such these are the only antichains in $Q_C$, up to equivalence. (I.e., up to $Q^<I = Q^<J$.)
Isolated antichains

**Density Hypothesis:** The isolated minimal antichains are dense (any non-wqo $Q_C$ contains an isolated antichain).
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Examples:
- Graphs
- Colored Paths
Isolated antichains

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**Examples:**
- **Graphs**  Just 2 minimal antichains
  - \( I_0 \): Cycles (degree at most 2—unique isolated)
  - \( I_1 \): Bridges (not isolated)
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- **Colored Paths**

**Proposition**

*Among vertex-colored paths, the minimal antichains are quasi-periodic, that is they consist of a periodic part augmented by a first and last vertex which break the periodicity.*
Density Hypothesis: The isolated minimal antichains are dense (any non-wqo $Q_C$ contains an isolated antichain). Examples:
- Graphs
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Corollary

In the cases of graphs and colored paths, the isolated minimal antichains are dense, the associated ideals are effectively recognizable, and the recognition of wqo classes given by finitely many constraints is effective, in polynomial time.
A finiteness Theorem

Theorem (Cherlin-Latka 2000)

Let $Q$ be a wellfounded quasiorder. Then for each $k$, there is a finite set $\Lambda_k$ of minimal antichains, such that any non-wqo $Q_C$ with $|C| \leq k$ allows one of the antichains in $\Lambda_k$ (up to a finite set).
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**Proof.**

Induction. Start with $\Lambda_{k+1} = \Lambda_k$ and consider constraints $C = \{c_1, \ldots, c_{k+1}\}$ for which this is inadequate. $C_i = C \setminus \{c_i\}$. If $Q_{C_i}$ is wqo, no worries.
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Suppose some $I \in \Lambda_k$ is compatible with $Q_{C_i}$. If $I$ is compatible with $c_i$, no worries.

Remaining case: $c_i \in Q^{<I_i}$, $I_i \in \Lambda_k$. $C \in \prod_i Q^{<I_i}$ a wqo.
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Remaining case: $c_i \in Q^{<l_i}$, $l_i \in \Lambda_k$.

$C \in \prod_i Q^{<l_i}$ a wqo. So there are finitely many minimal cases; expand $\Lambda_k$ by witnesses for the minimal cases.
A finiteness Theorem

Theorem (Cherlin-Latka 2000)

Let $Q$ be a wellfounded quasiorder. Then for each $k$, there is a finite set $\Lambda_k$ of minimal antichains, such that any non-wqo $Q_C$ with $|C| \leq k$ allows one of the antichains in $\Lambda_k$ (up to a finite set).

Corollary

If the ideals $Q^<_I$ are computable for $I \in \Lambda_k$, then the decision problem for wqo with respect to $k + 1$ constraints is decidable.
Friedman again

Claims:

1. The finiteness theorem for $k = 1$ is provably equivalent to $\Pi^1_1 - CA_0$ over $RCA_0$, even for locally finite quasiorders.

2. There is a finite signature with just constant and function symbols, such that model theoretic embeddability of finite structures gives a quasiorder for which the set of forbidden points defining a wqo ideal is complete $\Pi^1_1$. 

Claims:

1. The finiteness theorem for $k = 1$ is provably equivalent to $\Pi^1_1 \land \text{CA}_0$ over $\text{RCA}_0$, even for locally finite quasiorders.

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At what the other extreme we may conjecture:

**Conjecture**

The isolated minimal antichains are dense for $Q$ the quasiorder of tournaments, and the corresponding ideals are uniformly recursive.
A final question

Old chestnut:

• Is the generic triangle-free graph $G_{3p}$ pseudofinite (i.e., are its properties shared by finite graphs)?

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