Torsion in Groups of Finite Morley Rank

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I Connected Groups
  • Structure
II Permutation Groups
  • Bounds on Rank
III Torsion
  • Centralizers
  • Semisimplicity
  • Sylow Theorem
  • Weyl Group
1. Structure Theory

2. Permutation Groups

3. Torsion
Essential Notions—Generalities

- Morley rank ($\text{rk}(X)$)
- Connected group

$$[G : H] < \infty \implies G = H.$$  

$$X, Y \subseteq G \text{ generic} \implies X \cap Y \text{ generic}$$  

- $d(X)$: definable subgroup generated by $X$.
- **Fubini**: Zilber-Lascar-Borovik-Poizat
The Algebraicity Conjecture

Conjecture (Algebraicity)

\( G: \) finite Morley rank, connected.
\( H: \) maximal connected solvable normal, definable.

\[ 1 \rightarrow H \rightarrow G \rightarrow \bar{G} \rightarrow 1 \]

\( \bar{G}: \) a central product of algebraic groups.

Equivalently: The simple groups are algebraic.
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**Theorem (ABC, 2008)**

\[
1 \rightarrow U_2(G) \rightarrow G \rightarrow \bar{G} \rightarrow 1
\]

\( U_2(G) \): 1 \( \rightarrow \) \( O_2(G) \rightarrow \prod_i L_i \) (char 2, Altınel’s Jugendtraum - and his habilitation - and Wagner’s good tori)

\( \bar{G} \): Connected 2-Sylow divisible abelian. (“odd type”)
Theorem (Degenerate Type)

If there is no nontrivial connected abelian p-subgroup, then there is no p-torsion.

Theorem (Burdges-Altınel)

The centralizer of a divisible torsion subgroup is connected.
Odd Type: Torsion

**Theorem (Degenerate Type)**

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**Theorem (Burdges-Altınel)**

*The centralizer of a divisible torsion subgroup is connected.*

**Corollary**

*If there are no p-unipotent subgroups, then any p-element which centralizes a maximal divisible p-subgroup T lies in T.*
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Theorem (Degenerate Type)

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Theorem (Burdges-Altınel)

_The centralizer of a divisible torsion subgroup is connected._

Corollary

*If there are no p-unipotent subgroups, then any p-element which centralizes a maximal divisible p-subgroup T lies in T.*

Proof.

*T the definable hull of a maximal divisible p-subgroup. H = C(T)/T connected. H has no p-torsion.*
1. Structure Theory

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Definably primitive: no nontrivial $G$-invariant definable equivalence relation.

Theorem (BC)

$(G, X)$ definably primitive. Then $\text{rk}(G)$ is bounded by a function of $\text{rk}(X)$.

MPOSA = Macpherson-Pillay/O’Nan-Scott-Aschbacher
A description of the socle of a primitive permutation group, and the stabilizer of a point in that socle.

- **Affine:** The socle $A$ is abelian and can be identified with the set $X$ on which $G$ acts.
- **Non-affine:** The socle is a product of copies of one simple group.
Generic multiple transitivity

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Generic transitivity: one large orbit.

Generic \(t\)-transitivity: on \(X^t\).
**Generic multiple transitivity**

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Generic transitivity: one large orbit.

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**Proposition**

\((G, X)\) definably primitive. Then the degree of multiple transitivity of \(G\) is bounded by a function of \(rk (X)\).

(Special case of the theorem, but sufficient.)
Bounds on $t$

**Proposition**

$(G, X)$ definably primitive, generically $t$-transitive. Then $t$ is bounded by a function of $rk(X)$. 

Strategy: Let $T$ be a maximal 2-torus.

1. Derive an upper bound on the complexity of $T$ from $rk(X)$;
2. Derive a lower bound on the complexity of $T$ from $t$.

The upper bound: $rk(T/O_\infty(T)) \leq rk(X)$, this is because the stabilizer of a generic element of $X$ is torsion-free. But the lower bound requires attention.
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But the lower bound requires attention.
We want to show that a large degree of generic transitivity ($t$ large) blows up $rk\left(\frac{T}{T_\infty}\right)$ for $T$ the definable hull of a 2-torus.
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The group $G$ will induce the action of $Sym_t$ on any $t$ independent generic points.

Trading $T$ in for a smaller torus, and trading $t$ in for a smaller value as well (but not too small) we can set this up so that we have:

- a finite group $\Sigma$ operating on $T$, covering $Sym_t$,
- and sitting inside a connected group $H$ —
- such that $T$ is the definable hull of a maximal 2-torus in $H$.

Let us simplify considerably.
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Imagine the simplest case: $Sym_t$ sits inside $G$ and acts on $T$, the definable hull of a maximal 2-torus.
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It seems reasonable that this action can be exploited to blow up \(T\), and also \(T/T_\infty\).
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It seems reasonable that this action can be exploited to blow up $T$, and also $T / T_\infty$.

*There is a glaring hole in this argument.*
The Setup
$T$ inside $G$, $G$ connected, $\text{Sym}_t$ acts on $T$, $t$ large, and $T$ is the definable hull of a maximal 2-torus.

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But since this configuration is in a connected subgroup of $G$, and $T$ is a maximal 2-torus, the 2-elements of $Sym_t$ act nontrivially on $T$, and the action of $Alt_t$ is faithful.
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So we are done.
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More results on torsion

Assume no $p$-unipotents.
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- **Semisimplicity**
  If $G$ is connected, then every $p$-element is in a torus.

- **Sylow theory**
  For all primes $p$

- **Weyl groups** $N(T)/T$.
  If the Weyl group is nontrivial, it contains an involution.
  (Burdges-Deloro) If the group is minimal simple, the Weyl group is cyclic
Applications

1. Permutation Groups
2. Classification in odd type and low 2-rank
3. Bounds on 2-rank revisited?
### Other aspects

1. The Borovik Program: Signalizer functor theory, strong embedding, black box group theory . . .
2. Burdges unipotence theory and the Bender method
3. Generix strikes back [Nesin, Jaligot]
4. Conjugacy of Carter subgroups [Frécon]
5. Quasithin methods
   1. Amalgam method, representation theory (even type)
   2. Component analysis (odd type) [Borovik, Altseimer, Burdges]
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Desiderata

$L^*$-group theory in odd type (absolute bounds on 2-rank)
Control of actions of 2-tori on degenerate type groups.
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$L^*$-group theory in odd type (absolute bounds on 2-rank)
Control of actions of 2-tori on degenerate type groups.
and
Bad groups and non-commutative geometry . . . ?