Simple Groups of Finite Morley Rank: 2004

Luminy

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Algebraicity Conjecture

An infinite simple group of finite Morley rank is an algebraic group

(Chevalley group, algebraically closed base field)

rk(S) = dim(S) Subgroup, conjugacy class, ...

Borovik's Program

Determine the 2-Sylow structure of a minimal counterexample.

Morley rank

Macintyre: An infinite field of finite Morley rank is algebraically closed.

Zilber: An $\aleph_1$-categorical structure which is not almost strongly minimal involves an infinite group of finite Morley rank.

Hrushovski: Non-algebraic Abelian groups of finite Morley rank are associated with abelian varieties (Mordell-Lang, Manin, Buium) and their model theory has number theoretic consequences.

Analogies with algebraic groups

Connectedness (indecomposability).

G°: generic subsets, irreducibility

Generation: $\langle X_i : i \in I \rangle = \prod X_{i_k}^{\pm 1}$ definable, connected

$[G, X]$ definable, connected.

Borel subgroups?

- Conjugacy?
- Nilpotent?

Bad group: Minimal connected simple, all Borels nilpotent.

- No involutions (geometry of involutions)
- No involutory automorphisms
- Borels conjugate, and disjoint.

Finite Group Theory

Sylow theory

Schur-Zassenhaus

Carter subgroups

Strong embedding

Signalizer functor theory

Amalgam method

Structure of K-groups

Generation results

Etc., etc. . . .

2-**Sylow**$^\circ$ **structure in algebraic groups**

Characteristic 2:

unipotent—[bounded exponent, definable]

Other characteristics:

semisimple—[divisible abelian]

2-**Sylow**$^\circ$ **structure in groups of FMR**

$$S^\circ = U * T:$$

2-Unipotent $\times$ 2-torus
with finite intersection

"Prüfer rank" (dimension of the 2-torus, Lie rank)

Types

U / T	$\neq 1$	$= 1$
$\neq 1$	Mixed	Odd
$= 1$	Even	Degenerate

Even	Odd	Mixed	Degenerate
U (bdd exp)	T (bdd width)	$U * T$	1

“Theorem” I

- Mixed type does not exist.
- Even type is algebraic.

“Theorem” II

A minimal counterexample to the algebraicity conjecture has Prüfer rank at most two.

Mixed and Even Type

Mixed type: reduces to even type.

Even type:

- Strong embedding, weak embedding
- Strongly closed abelian subgroups
- Standard components of type SL_2.
- Pushing-up
- $C(G,T)$
- Parabolic subgroups
- Amalgams

Conjugacy Theorems

2-Sylow subgroups

Hall and Carter subgroups

Borel subgroups

Maximal tori

Good tori

Odd Type

Generic identification: Berkman

The B-conjecture (unipotence): Burdges

Minimal connected simple groups: Jaligot

Soluble groups: Frécon

Tameness

Bad field $(K; T)$; $K \rtimes T$

Tame: no bad field.

Proposition A tame connected solvable group without involutions is nilpotent.

Corollary A tame minimal connected simple group of degenerate type is a bad group, hence contains no involutions.

Recent Theme: removal of "tameness" hypothesis.

Recognition (Generic Case)

Berkman

Model: SL_n

$S = T_2 \rtimes W_2$: $T =$ diagonal, $W = \mathrm{Sym}_n$, Coxeter group

Dynkin diagram A_{n-1}: structure of W, elementary transpositions $(i, i+1)$

Root SL_2's:

$$\begin{pmatrix} * & * & 0 & 0 & \dots \\ * & * & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & * & * & 0 & \dots \\ 0 & * & * & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Recognition

$\mathcal{E}$: copies of SL_2 normalized by S°.

$W_0 = \langle r_L : L \in \mathcal{E} \rangle$

Identify W_0 geometrically (Complex Reflection Groups) derive the Dynkin diagram, and verify "Curtis-Tits-Phan" relations.

Reductivity: $OC(i) = 1$ (*Strong B-conjecture*)

Generation: $\langle \mathcal{E} \rangle = G$.

Generic Identification:

High Prüfer rank + Reductivity + Generation $\Rightarrow$ Algebraicity

The B-Conjecture

Burdges

Killing $OC(i)$ (or limiting its effect): *Signalizer Functor Method*

Origins: Feit-Thompson paper, Gorenstein, Walter.

Solvable Signalizer Functor Theorem (finite group theory)

Nilpotent Signalizer Functor Theorem (our context)

Reduction: The Nilpotent Subfunctor Theorem

The Subfunctor Theorem

Idea: $\theta(i) = U(O(C(i)))$ where U is the "unipotent radical"

What does *unipotent* mean? At least:

- The unipotent radical of a solvable group is definably characteristic and nilpotent.

- If the unipotent radical of a solvable group is trivial, that group is a "torus".

Dichotomy for minimal nonalgebraic groups:

The B-conjecture, or minimal connected simple

Generation principle: If a 4-group V acts on a connected K-group H of odd type, then H is generated by $C^\circ(i)$ $(i \in V)$.

Proof: in a counterexample, the proper subgroup of H that arises is a "*subsystem subgroup*", of full Lie rank, and one eliminates possibilities by inspection.

Minimal Simple Groups

Theorem: The Prüfer rank is at most two.

Tame Case

Borel subgroups.

First application of tameness:

Jaligot's Lemma The intersection of Borel subgroups is disjoint from their Fitting subgroups.

Leads to: Standard Borel subgroups are nilpotent.

Second application of tameness:

The torus T enveloping a Sylow 2-subgroup involves almost all primes.

$W = N(T)/C(T)$: W acts semi-regularly in each prime, and regularly on the involutions, leading by number theory (Dirichlet) to $d = 2$.

Elimination of Tameness.

Number theory replaced by elementary "generic subsets" arguments.

Jaligot's Lemma replaced by close analysis of *maximal intersections* of Borel subgroups.

Carter Subgroups

Frécon

Carter subgroup: self-normalizing, nilpotent

Useful Lemma: A Carter subgroup of a standard Borel subgroup contains the Sylow 2-subgroup

Corollary: The intersection of two standard Borel subgroups is not a Carter subgroup.

Philosophy: in a minimal simple group, this intersection wants to be a Carter subgroup.

Absolute Carter subgroups, via the unipotence theory.

Decomposition theorem for nilpotent groups via the unipotence theory.

Geometrical themes

Geometry of involutions (bad groups)

Groups generated by pseudo-reflection subgroups (even type, SCA)

Complex reflection groups (generic identification, recognition of Coxeter group)

Good tori and bad fields (Wagner), Linearization (Poizat), Conjugacy theorems

Intersections of Borel subgroups in minimal connected simple groups and the unipotence theory.

The amalgam method of finite group theory

An open problem

Degenerate type: bound on 2-rank.