I Homogeneous Structures

- Distance Homogeneous Graphs

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- Trees
I. Homogeneous Structures

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3. Universality

4. Applications

5. Questions
Homogeneity

\[ A \simeq B \implies A \sim B \text{ under } \text{Aut}(\Gamma) \]

E.g. \((\mathbb{Q}, <)\)

Around Homogeneous Universal Graphs

Gregory Cherlin

I. Homogeneous Structures

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Homogeneity

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E.g. \((\mathbb{Q},<)\)

Urysohn 1927 (Ph.D. 1921; d. 1924, aged 26): \( \mathbb{U} \)

Rado 1964: \( G \)

Fraïssé 1954: \( \Gamma \leftrightarrow \text{Sub}(\Gamma) \)
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Amalgamation

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Amalgamation of Metric Spaces

1-point extensions: \( A_i = A_0 \cup \{u_i\} \).

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  d^+(u_1, u_2) &= \min(d(u_1, a) + d(u_2, a)) \\
  d^-(u_1, u_2) &= \max|d(u_1, a) - d(u_2, a)|
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Any positive \( d \) in \([d^+, d^-]\) will do.
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\( \mathbb{U}_0 \): The universal homogeneous countable rational-valued metric space.

\( \mathbb{U} \): The completion of \( \mathbb{U}_0 \).
Homogeneous Graphs and Digraphs

Henson 1971: $G_n$ ($K_n$-free graph), its automorphisms and structure
Henson 1972: $D_{\neg\mathcal{T}}$ ($\mathcal{T}$-free digraph)
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**Lachlan-Woodrow 1980**: Homogeneous graphs classified. Imprimitive or Degenerate: $(mK_n)^\pm$; Primitive finite: $P$, $E(K_{3,3})$
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Lachlan 1984: Homogeneous tournaments classified
$I_1$, $C_3$, $Q$, $S$, $T_\infty$
Cherlin 1993 (Banff): Homogeneous directed graphs
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Tools: Fraïssé, Finite Ramsey theorem
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Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

*Color classes* $c_1 \leq c_2 \leq c_1$, densely colored; *connections between pairs of color class components; triples*. Fraïssé for existence.
Some recent developments

Torrezão de Souza/Truss 2008: Colored PO

Kechris-Pestov-Todorcevic 2005: Fraïssé+Ramsey+Top. Dynamics

Glasner: “This remarkable paper is a tour de force where three experts in disparate areas—model theory, structural Ramsey theory and topological dynamics—collaborate in creating a unified and beautiful theory.”
Some recent developments

Kechris-Pestov-Todorcevic 2005: Fraïssé+Ramsey+Top. Dynamics

Minimal flows: compact actions with every orbit dense. Extremely amenable: no nontrivial minimal flow
Some recent developments

Kechris-Pestov-Todorcevic 2005: Fraïssé+Ramsey+Top. Dynamics

- The extremely amenable closed subgroups of $\text{Sym}_\infty$ are exactly the groups of the form $\text{Aut}(\mathbb{A})$ with $\mathbb{A}$ the Fraïssé limit of a Fraïssé order class with the Ramsey property.
- If $\mathbb{A}$ is one of the following structures, then the universal minimal flow $M(G)$ of the group $G = \text{Aut}(\mathbb{A})$ is its action on the space of linear orderings of the universe of $\mathbb{A}_0$:
  - $G_n \ (n \leq \infty)$;
  - $(\mathbb{N}, =)$;
  - $\mathbb{U}_0$
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A catalog?
A catalog

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A catalog

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3. Fraïssé type
   - $\delta \leq d$;
   - Omit $(1, d)$-subspaces ($d \geq 3$);
   - Omit odd cycles up to order $2K + 1$;
   - Omit triangles of perimeter $\geq C$.

Some interactions in these constraints.
Exceptional $\Gamma_1 \rightarrow$ Exceptional $\Gamma$. 
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Difficulty: $\Gamma_k$
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Homogeneous metric space; not necessarily with the graph metric, because of the parity condition.

But $(\Gamma_{k-1}, \Gamma_k)$ should be.

Extend the classification project?
Komjáth-Mekler-Pach 1988: Universal graphs omitting paths; or omitting cycles of odd length
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Data: Finitely many constraints $C$ (finite, connected “forbidden” graphs).
Universal countable $C$-free graph? ? Decidable ?
Existentially complete $\mathcal{C}$-free graphs. (Generalizes Fraïssé.)
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If the existentially complete countable graph is unique, then it is universal.

And there is an exact criterion for this in terms of the algebraic closure.
Forbid $\mathcal{C}$. What is $acl_\mathcal{C}(A)$?
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- Forbid $C_4$. Then for points $u, v$ at distance 2, the “midpoint” is a definable function $f(u, v)$. Such points are in the “definable closure” of $u, v$. 

Algebraic Closure
Forbid $\mathcal{C}$. What is $acl_{\mathcal{C}}(A)$?

- Forbid $C_4$. Then for points $u, v$ at distance 2, the “midpoint” is a definable function $f(u, v)$. Such points are in the “definable closure” of $u, v$.

- Forbid a star $S_k$. Then for any $u$, the neighbors of $u$ are “algebraic” over $u$: they lie in a $u$-definable finite set.
Theorem (CSS 1999)

Let \( C \) be a finite set of forbidden graphs, \( T \) the theory of the existentially complete \( C \)-free graphs. Then the following are equivalent.

1. \( T \) has a unique countable model
2. The algebraic closure operator is locally finite.
Theorem (CSS 1999)

Let $\mathcal{C}$ be a finite set of forbidden graphs, $T$ the theory of the existentially complete $\mathcal{C}$-free graphs. Then the following are equivalent.

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Proof.

$\Rightarrow$: General nonsense (Ryll-Nardzewski, Engeler, Svenonius)

$\Leftarrow$: Close analysis: over any finite algebraically closed set, the set of types is finite.
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Conjectured by Menachem Kojman:

**Theorem**

*If $\mathcal{C}$ is closed under homomorphism (i.e., the image of a constraint in $\mathcal{C}$ under graph homomorphism is $\mathcal{C}$-forbidden) then acl is degenerate and there is a universal $\mathcal{C}$-free graph.*

Example. Odd cycles.
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Example. Odd cycles.

**Theorem (Cherlin-Shi 1996)**

For $C$ a finite set of cycles the following are equivalent.

1. There is a universal $C$-free graph.
2. $C$ consists of all odd cycles up to a fixed length.
Theorem (Cherlin-Shelah 2007)

For $\mathcal{C} = \{ T \}$ a single tree, the following are equivalent.

1. There is a universal $\mathcal{C}$-free graph.
2. The tree $T$ is an extension of a path by at most one additional edge.
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Shelah’s idea: Pruning

To prune a tree $T$: $T'$ is obtained by removing all leaves.
Pruning Trees

Lemma

If there is a $T$-free universal graph $G$ then there is a $T'$-universal graph $G^*$, consisting of the vertices of $G$ of infinite degree.
Pruning Trees

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Minimal trees: those which prune to a path or near-path. (15 cases).
In general: Remove a minimal block-leaf. (Or a downward-closed family.)

Conjectures
General Pruning

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*If there is C-free universal graph, then C has complete blocks and a path-like structure, with very few exceptions.*
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**Conjectures**

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*If there is C-free universal graph, then C has complete blocks and a path-like structure, with very few exceptions.*

**Conjecture**

*For a single connected constraint C, the problem of determining whether there is a universal C-free graph is algorithmically decidable.*
Two Questions

- Is the generic triangle-free graph $G_3$ pseudofinite (i.e., are its properties shared by finite graphs)?

  “Alice’s Restaurant” extension properties
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Vershik: there is a random construction of $G_3$. Namely, build a graph on $\mathbb{R}$ for which the extension properties are satisfied on open sets, and take a countable subgraph at random, with respect to a probability measure on $\mathbb{R}$.
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- The Hairy Ball Problem Let $K$ be a finite graph consisting of a complete graph together with a single finite path attached to each vertex. Is there a universal $K$-free graph?
A Concrete Example

The Bouquet $K_5 \land K_5$

$(K_5 + K_5)$ - free

(Algebraic closure running along the mid-line)