1 Metrically Homogeneous Graphs

The Classification Problem

Γ connected, with graph metric \( d \).

Γ is metrically homogeneous if the metric space \((Γ, d)\) is (ultra)homogeneous.

(Cameron 1998) Classify the countable metrically homogeneous graphs.

Contexts: infinite distance transitive graphs, homogeneous graphs, homogeneous metric spaces

1.1 Finite Distance Transitive Graphs

Finite Distance Transitive Graphs

\( \text{distance transitivity} = \text{metric homogeneity for pairs} \)

Smith’s Theorem:

- Imprimitive case: Bipartite or Antipodal (or a cycle)
- Antipodal: maximal distance \( δ \)
- Reduction to the primitive case (halving, folding)

1.2 Homogeneous Graphs

Classification of Homogeneous Graphs

Metrically homogeneous diameter \( \leq 2 = \text{Homogeneous} \).

(The metric is the graph)

Fraïssé Constructions: Henson graphs \( H_n, H^c_n \)

Lachlan-Woodrow 1980 The homogeneous graphs are

- \( m \cdot K_n \) and its complement;
- The pentagon and the line graph of \( K_{3,3} \) (3 x 3 grid)
- The Henson graphs and their complements (including the Rado graph)

Method: Induction on Amalgamation Classes

Claim: If \( A \) is an amalgamation class of finite graphs containing all graphs of order 3, \( I_∞ \), and \( K_n \), then \( A \) contains every \( K_{n+1} \)-free graph.

Proof by induction on the order \(|A|\) where \( A \) is \( K_{n+1} \)-free

This doesn’t work directly, but a stronger statement can be proved by induction.

Induction via Amalgamation

\( A' \) is the set of finite graphs \( G \) such that any 1-point extension of \( G \) lies in \( A \).

Inductive claim: Every finite graph belongs to \( A' \).

Not making much progress yet, but . . .

1-complete: complete. 0-complete: co-complete.

\( A^p \) is the set of finite graphs \( G \) such that any finite \( p \)-complete graph extension of \( G \) belongs to \( A \).
\[ \mathcal{A}^p \subseteq \mathcal{A}' \]
\[ \mathcal{A}^p \text{ is an amalgamation class} \]

Target: The generators of \( \mathcal{A} \) all lie in one \( \mathcal{A}^p \), for some \( p \).

**Lachlan’s Ramsey Argument**

How to get into \( \mathcal{A}^p \):

1-point extensions of a large direct sum \( \oplus A_i \)

\[ \implies \]

\( p \)-extensions of one of the \( A_i \).

If \( A_i \) is itself a direct sum of generators, we get a fixed value of \( p \).


### 1.3 Homogeneous Metric Spaces

**Homogeneous Metric Spaces**

- Rational-valued Urysohn space.
- \( \mathbb{Z} \)-valued Urysohn space is a metrically homogeneous space.
- Or \( \mathbb{Z} \cap [0, \delta] \)-valued.
- \( S \)-valued: Van Thé AMS Memoir 2010

A metrically homogeneous graph of diameter \( \delta \) is:

A \( \mathbb{Z} \)-valued homogeneous metric space with bound \( \delta \), and all triangles \((1, i, i + 1)\) allowed (connectivity).

### 2 A Catalog

#### 2.1 Special Cases

**Special Cases**

- Diameter \( \leq 2 \) (Lachlan/Woodrow 1980)
- Locally finite (Cameron, Macpherson)
- \( \Gamma_1 \)-exceptional
- Imprimitive (Smith’s Theorem)

**The Locally Finite Case**

Finite of diameter at least 3 and vertex degree at least 3: Antipodal double covers of certain finite homogeneous graphs (Cameron 1980)

Infinite, Locally Finite: Tree-like \( T_{r,s} \) (Macpherson 1982)

Construction:
The graphs $T_{r,s}$

The trees $T(r,s)$: Alternately $r$-branching and $s$-branching.
Bipartite, metrically homogeneous if the two halves of the partition are kept fixed.

The graph obtained by “halving” on the $r$-branching side is $T_{r,s}$.
Each vertex lies at the center of a bouquet of $r$ $s$-cliques.

Another point of view: the graph on the neighbors of a fixed vertex:
$\Gamma_1 : r \cdot K_{s-1}$.

From this point of view, we may also take $r$ or $s$ to be infinite!

$\Gamma_i$

$\Gamma_i = \Gamma_i(v)$: Distance $i$, with the induced metric.

Remark 1. If distance 1 occurs, then the connected components of $\Gamma_i$ are metrically homogeneous.

In particular $\Gamma_1$ is a homogeneous graph.

Exceptional Cases: finite, imprimitive, or $H_n^\infty$.
The finite case is Cameron+Macpherson, the imprimitive case leads back to $T_{r,s}$
with $r$ or $s$ infinite, and $H_n^\infty$ does not occur for $n > 2$ (Cherlin 2011)

In other words, the nonexceptional cases are

- $I_\infty$
- Henson graphs $H_n$ including Rado’s graph.

Imprimitive Graphs

“Smith’s Theorem” (Amato/Macpherson, Cherlin):

Part I: Bipartite or antipodal, and in the antipodal case with classes of order 2 and
the metric antipodal law for the pairing:

$$d(x,y') = \delta - d(x,y)$$
Hence no triangles of diameter greater than $2\delta$:
\[ d(x, z) \leq d(x, y') + d(y', z) = 2\delta - d(x, y) - d(x, z) \]

Part II: The bipartite case reduces by halving to a case in which $\Gamma_1$ is the Rado graph.
On the other hand, the antipodal case does not reduce: while distance transitivity is inherited after “folding,” metric homogeneity is not.
There is also a bipartite antipodal case.

2.2 Generic Cases

Some Amalgamation Classes
Within $\mathcal{A}^\delta$: finite integral metric spaces with bound $\delta$:
- $\mathcal{A}^\delta_{K,\text{even}}$: No odd cycles below $2K + 1$.
- $\mathcal{A}^\delta_{C,\text{bounded}}$: Perimeter at most $C$.
- $(1, \delta)$-constraints.

The first two classes are given (implicitly) in Komjath/Mekler/Pach 1988 as examples of constraints admitting a universal graph, which is constructed by amalgamation.
The last is a generalization of Henson’s construction. A $(1, \delta)$-space is a space in which only the distances 1 and $\delta$ occur (a vacuous condition if $\delta = 2$).
Any set $S$ of $(1, \delta)$-constraints may be imposed.
Mixing: $\mathcal{A}^\delta_{K,C:S}$

Expectations ca. 2008
- The generic case is $\mathcal{A}^\delta_{\Delta:S}$ with $\Delta$ some set of forbidden triangles …
- and $\Delta$ is a mix of parity constraints $K$ and size constraints $C$.
Not quite …

Variations on a theme
More examples
- $C = (C_0, C_1)$: $C_0$ controls large even parity, $C_1$ controls large odd parity
- $K = (K_1, K_2)$: $K_1$ controls odd cycles at the bottom, $K_2$ controls odd cycles midrange.
  - $(i, j, k)$: $P = i + j + k$
  - For $P$ odd, forbid
    \[ P < 2K_1 + 1 \]  \[ P > 2K_2 + i \]
Triangle Constraints

**Theorem 1.** If $A$ is a geodesic amalgamation class of finite integral metric spaces with diameter $\delta$, determined by triangles, then $A$ is one of the classes $A^\delta_{K,C,S}$

with $K = (K_1, K_2)$ and $C = (C_0, C_1)$.

But not all such classes work . . . .

**Definability in Presburger Arithmetic**

The classes $A^\delta_{K,C}$ are uniformly definable in Presburger arithmetic from the parameters $K_1, K_2, C_0, C_1, \delta$.

The $k$-amalgamation property is amalgamation for diagrams of order at most $k$.

With constraints of order 3, one expects $k$-amalgamation for some low $k$ to imply amalgamation. (In the event, $k = 5$.)

**Observation 1.** $k$-amalgamation is a definable property in Presburger arithmetic for the classes $A^\delta_{K,C}$.

Therefore it should be expressible using inequalities and congruence conditions on linear combinations of the parameters.

**Acceptable Parameters**

- $\delta \geq 3$.
- $1 \leq K_1 \leq K_2 \leq \delta$ or $K_1 = \infty$ and $K_2 = 0$;
- $2\delta + 1 \leq C_{\min} < C_{\max} \leq 3\delta + 2$, with one even and one odd.

**Conditions for amalgamation (or 5-amalgamation):**

**Conditions on $K, C$**

- If $K_1 = \infty$:
  
  $$K_2 = 0, C_1 = 2\delta + 1,$$

- If $K_1 < \infty$ and $C \leq 2\delta + K_1$:
  
  $$C = 2K_1 + 2K_2 + 1, K_1 + K_2 \geq \delta, \text{ and } K_1 + 2K_2 \leq 2\delta - 1$$
  
  If $C' > C + 1$ then $K_1 = K_2$ and $3K_2 = 2\delta - 1$. 

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• If $K_1 < \infty$, and $C > 2\delta + K_1$:

$K_1 + 2K_2 \geq 2\delta - 1$ and $3K_2 \geq 2\delta$.
If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$.
If $C' > C + 1$ then $C \geq 2\delta + K_2$.

Notes:
$C = \min(C_0, C_1), C' = \max(C_0, C_1)$
$C' > C + 1$ means we need both $C_0$ and $C_1$.

Conditions on $\mathcal{S}$

• If $K_1 = \infty$:

$\mathcal{S}$ is \begin{align*}
\text{empty} & \quad \text{if } \delta \text{ is odd, or } C_0 \leq 3\delta \\
\text{a set of } \delta\text{-cliques} & \quad \text{if } \delta \text{ is even, } C_0 = 3\delta + 2
\end{align*}

• If $K_1 < \infty$ and $C \leq 2\delta + K_1$:

If $K_1 = 1$ then $\mathcal{S}$ is empty.

• If $K_1 < \infty$, and $C > 2\delta + K_1$:

If $K_2 = \delta$ then $\mathcal{S}$ cannot contain a triangle of type $(1, \delta, \delta)$.
If $K_1 = \delta$ then $\mathcal{S}$ is empty.
If $C = 2\delta + 2$, then $\mathcal{S}$ is empty.

2.3 Proofs

Antipodal Variations

• $\mathcal{A}_n^\delta = \mathcal{A}_{1,\delta-1;2\delta+1;2\delta+1;0}$ is the set of finite integral metric spaces in which no triangle has perimeter greater than $2\delta$.

• $\mathcal{A}_{a,n}^\delta$ is the subset of $\mathcal{A}_n^\delta$ containing no subspace of the form $I_2^{\delta-1}[K_k, K_{\ell}]$ with $k + \ell = n$; here $I_2^{\delta-1}$ denotes a pair of vertices at distance $\delta - 1$ and $I_2^{\delta-1}[K_k, K_{\ell}]$ stands for the corresponding composition, namely a graph of the form $K_k \cup K_{\ell}$ with $K_k, K_{\ell}$ cliques (at distance 1), and $d(x, y) = \delta - 1$ for $x \in K_k, y \in K_{\ell}$.
In particular, with $k = n, \ell = 0$, this means $K_n$ does not occur.
Necessity: Amalgamation diagrams

**Lemma 2.** Let $A$ be an amalgamation class of diameter $\delta$ determined by triangle constraints with associated parameters $K_1, K_2, C, C'$. Then

$$C > \min(2\delta + K_1, 2K_1 + 2K_2)$$

We suppose

$$C \leq 2\delta + K_1$$

and we show that

$$C > 2K_1 + 2K_2$$

Set $j = \lfloor \frac{C - K_1}{2} \rfloor$, and $i = (C - K_1) - j$. Then $1 < j \leq i \leq \delta$.

$C > \min(2\delta + K_1, 2K_1 + 2K_2)$

In the following amalgamation, vertices $u_1, u_2$ force $d(a_1, a_2) = K_1$ and $|a_1a_2c| = C$:

![Diagram](image)

\[d(c, u_1) = d(c, u_2) = i - 1\]

So omit $ca_2u_1$ or $ca_2u_2$, with $P \geq 2K_1 + 1, \ldots$

**Proofs of amalgamation**

Three amalgamation strategies:

- $d^-(a, b) = \max(d(a, x) - d(a, b))$
- $d^+(a, b) = \inf d(a, x) + d(x, b)$
- $\tilde{d}(a, b) = \inf[C - (d(a, x) + d(a, b))]$
Amalgamation for $A_{K,C}^\delta$

- If $C \leq 2\delta + K_1$:
  - If $d^-(a_1, a_2) \geq K_1$ then take $d(a_1, a_2) = d^-(a_1, a_2)$.
  - Otherwise:
    - If $C' = C + 1$ then:
      * If $d^+(a_1, a_2) \leq K_2$ then take $d(a_1, a_2) = \min(d^+(a_1, a_2), \bar{d}(a_1, a_2))$
      * If $d^-(a_1, a_2) < K_1$ and $K_2 < d^+(a_1, a_2)$ then take $d(a_1, a_2) = \bar{d}(a_1, a_2)$ if $\bar{d}(a_1, a_2) \leq K_2$ and $d(a_1, a_2) = K_1$ otherwise.
    - If $C' > C + 1$ then:
      * If $d^+(a_1, a_2) < K_2$ then take $d(a_1, a_2) = d^+(a_1, a_2)$;
      * If $d^-(a_1, a_2) < K_2 \leq d^+(a_1, a_2)$ then take
        \[
        d(a_1, a_2) = \begin{cases} 
        K_2 - 1 & \text{if there is } v \in A_0 \text{ with } d(a_1, v) = d(a_2, v) = \delta \\
        K_2 & \text{otherwise}
        \end{cases}
        \]

- If $C > 2\delta + K_1$:
  - If $C' = C + 1$ then:
    * If $d^+(a_1, a_2) \leq K_1$ then take $d(a_1, a_2) = \min(d^+(a_1, a_2), \bar{d}(a_1, a_2))$;
    * If $d^+(a_1, a_2) > K_1$ then take
      \[
      d(a_1, a_2) = \begin{cases} 
        K_1 + 1 & \text{if there is } v \in A_0 \text{ with } \\
        d(a_1, v) = d(a_2, v) = \delta, \\
        \text{and } K_1 + 2K_2 = 2\delta - 1 \\
        K_1 & \text{otherwise}
        \end{cases}
        \]
  - If $C' > C + 1$ then:
    * If $d^+(a_1, a_2) < K_2$ then take $d(a_1, a_2) = d^+(a_1, a_2)$;
    * If $d^+(a_1, a_2) \geq K_2$ then take $d(a_1, a_2) = \min(K_2, C - 2\delta - 1)$.

3 Conclusion

Completeness?

Good points:

- All cases with exceptional $\Gamma_1$
• $\delta \leq 3$, probably (Amato/Cherlin/Macpherson)
• Exact as far as triangle constraints are concerned
• Smith’s Theorem

Weak points
• Smith’s Theorem
  – Bipartite to be completed inductively
  – Antipodal description may be incomplete
• Induction to $\Gamma_i$ is not always available

In fact, for antipodal graphs omitting $K_n$, triangles and $(1, \delta)$-constraints do not suffice.
That class was found on an ad hoc basis. (And is invisible in diameter 3.)

**Toward a classification theorem**

**Strategy?**

• (Step 0) Prepare diameter 4 and $\Gamma_2$ generally? (Prudent)
• (Step 1) Characterize triangles occurring in amalgamation classes
• (Step 2) Show that if the triangle constraints are as expected, then $\Gamma_i$ has the expected constraints.
• (Step 3) Assuming the first two conditions, characterize $\Gamma$.

(Works in diameter 3)
... With Lachlan’s Ramsey method in reserve.

**Furthermore**

No need to wait for a classification:
• Ramsey theory for these homogeneous metric spaces
• Topological dynamics
• Other aspects of the automorphism group (normal subgroups, subgroups of small index)