Structure / Nonstructure in Finite Model Theory

Gregory Cherlin

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in fact, my friend, it’s not safe to make thin cuts; it’s safer to go along cutting through the middle of things, and that way one will be more likely to encounter real classes. ... whenever there is a class of anything, it is necessarily also a part of whatever it is called a class of, but it is not at all necessary that a part is a class.

Decision Problems with finite constraints

Tameness problems for classes of finite structures
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- $\mathcal{Q}$: a class of finite structures with an embedding relation $\leq$
- $\mathcal{P}$: a property of subsets of $\mathcal{Q}$
- $C \subseteq \mathcal{Q}$ a set of constraints
- $\mathcal{Q}_C : \{q \in \mathcal{Q} : c \nleq q \ (c \in C)\}$
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Problem $(\mathcal{Q}, \mathcal{P}, C)$: $\mathcal{P}(\mathcal{Q}_C)$?
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- $\mathcal{P} = WQO$: No infinite antichain
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*Two cases of interest:*
- \( \mathcal{P} = \text{WQO} \): No infinite antichain
- *Universality*: The class has a universal countable limit.
I  WQO
   • Worst case scenario
   • A Finiteness Theorem
   • Concrete Cases

II Universality
   • Beyond the pale
   • Within the pale
WQO

Universality
Theorem (Harvey Friedman)

There is a computable well-founded partial ordering $\leq^*$ of $\mathbb{N}$ for which the set

$$\{ n \in \mathbb{N} : (\mathbb{N}, \leq^*)_n \text{ is WQO} \}$$

is complete $\Pi^1_1$. 
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Construction: $(\mathbb{N}, \leq_1)$ computably linearly ordered so that

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$\leq^* = \leq \cap \leq_1$ — Then $(\mathbb{N}, \leq^*)_n$ is WQO iff $(\mathbb{N}, \leq_1)_n$ is WO
A Finiteness Theorem

Theorem (CL2000)

Let $Q$ be a well-founded quasiorder and $k$ fixed. Then there is a finite set $\Lambda_k$ of infinite antichains such that:

$\forall C \subseteq Q \text{ If } C \subseteq Q, |C| \leq k, \text{ and } Q_C \text{ is not WQO, then there is an antichain } l \in \Lambda_k \text{ with } l \subseteq^* Q_C.$
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then there is an antichain $I \in \Lambda_k$ with $I \subseteq^* Q_C$.

Idea (Nash-Williams): Use $I$ such that $Q \ll^{<I} = \{q : q \leq \text{almost all } a \in I\}$ is WQO.
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*Idea (Nash-Williams):* Use $I$ such that $Q << I = \{q : q \leq \text{ almost all } a \in I\}$ is WQO.

*Construction:*

$$\Lambda_{k+1} = \Lambda_k \cup \bigcup_{l_1, \ldots, l_{k+1} \in \Lambda_k} \{I_C : \text{critical } C \text{ in } \prod_i Q << l_i\}$$
Topology
Open sets $Q_C$ with $C$ finite.
Points $Q^<<l$. 
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Open sets $Q_C$ with $C$ finite.
Points $Q^{<<I}$.

Favorable Case:
- Isolated points are dense;
- Isolated points are effectively given ($Q^{<<I}$ decidable).
Open sets $\mathcal{Q}_C$ with $C$ finite.
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*Question: Does this happen in the cases of interest?*
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Points $Q^{<<I}$.

Favorable Case:
- Isolated points are dense;
- Isolated points are effectively given ($Q^{<<I}$ decidable).

*Question:* Does this happen in the cases of interest? (Yes in one simple case: vertex colored paths)
Cases of Interest: Graphs

- Graphs with forbidden subgraphs:
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  G. Ding 1992: $l_0 =$ cycles, $l_1 =$ bridges

  \[ \Lambda = \{ l_0 \} \]

  the unique isolated point.
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- Graphs with forbidden subgraphs:
  G. Ding 1992: $l_0 = \text{cycles}$, $l_1 = \text{bridges}$

$$\Lambda = \{l_0\}$$

the unique isolated point.

- Graphs with forbidden induced subgraphs:
  Unclear . . .
Cases of Interest: Tournaments

- Tournaments:
  \( \mathcal{A}_1 = \{ l_1, l_2 \} \): 
  \[ l_1 = \{ N_{1,n,D} : n \geq 7 \}, \quad l_2 = \{ N_{2,2n+1,H} : n \geq 4 \} \].

- \( N_{k,n} \): Linear of order \( n \), but with successors and edges \( (i, j) \) with \( i \equiv j \mod k \) reversed.
- \( N_{k,kn+1,D} \) or \( N_{k,kn,H} \): “mark” the ends (1 or 2 marker vertices)
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Theorem (Latka)

A set of finite tournaments determined by one “forbidden tournament” is wqo iff the infinite antichains \( l_1, l_2 \) are incompatible with the constraint.

Proof: tree decompositions and Kruskal’s Lemma.
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A set of finite tournaments determined by one "forbidden tournament" is wqo iff the infinite antichains $l_1, l_2$ are incompatible with the constraint.

Proof: tree decompositions and Kruskal’s Lemma.

**Corollary**

The WQO problem for classes of tournaments determined by at most two forbidden tournaments is (p-time) decidable.

*Remark*. No actual bound on the degree of the polynomial...
Cases of Interest: Pattern Classes

- Pattern Classes of Permutations

**Theorem (Knuth, 1969)**

The permutations which can be sorted using a stack are those omitting the pattern \((231)\) and their number is given by the Catalan numbers (cf. Macmahon 1915).
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**Theorem (Knuth, 1969)**

*The permutations which can be sorted using a stack are those omitting the pattern (231) and their number is given by the Catalan numbers (cf. Macmahon 1915).*

Themes:
- Characterize permutations sortable by variations on stacks
- Algorithmic problems for such classes of permutations
- Rates of growth for the the numbers of such permutations
- WQO (Antichains)
Tournaments: Infinite Antichains

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$|\Lambda_1| = 3$ [Atkinson/Murphy/Ruškuc 2002]
### Tournaments: Infinite Antichains

**An antichain:**

\[
\Lambda_1 = 3 \quad \text{[Atkinson/Murphy/Ruškuc 2002]}
\]

**References:**

N. Ruškuc, “Decidability questions for pattern avoidance classes of permutations,” in *Third International Conference on Permutation Patterns, Gainesville, Fla.*, 2005

The Main Question

Once more:

\(Q\): Finite relational structures with signature \(\sigma\) (with symmetry conditions).

- Isolated points are dense?
- Isolated points are effectively given (\(Q^{<<l}\) decidable)?
1. WQO

2. Universality
Universality

$(\mathcal{Q}, \leq)$: \textit{weak} substructure.

Property $\mathbb{P}$: existence of a universal countable \textit{limit}
Universality

\((\mathcal{Q}, \leq)\): weak substructure.

Property \(P\): existence of a universal countable limit

(?) When does a finite set of forbidden structures allow a universal structure?
Universality

$(Q, \leq)$: weak substructure. Property $\mathbb{P}$: existence of a universal countable limit

(?) When does a finite set of forbidden structures allow a universal structure?

**Examples (Graphs)**

- Forbid $K_n$ (Henson via Fraïssé)
- $C$ a set of cycles: *forbid odd cycles up to some fixed size* (CSS, 1999)
Universality

\((Q, \leq)\): weak substructure.
Property \(\mathbb{P}\): existence of a universal countable limit

(?) When does a finite set of forbidden structures allow a universal structure?

Theorem

*With forbidden induced subgraphs this question is undecidable*

(which is to be expected)
(Q, ≤): weak substructure.
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When does a finite set of forbidden structures allow a universal structure?

**Theorem**

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(which is to be expected) ;

**Theorem**

*with forbidden weak substructures, there is a good theory.*
Universality

$(Q, \leq)$: weak substructure.

Property $P$: existence of a universal countable limit $(?)$ When does a finite set of forbidden structures allow a universal structure?

**Theorem**

*With forbidden induced subgraphs this question is undecidable;*

**Theorem**

*with forbidden weak substructures, there is a good theory.*

... why the difference? ...
Undecidability

Tiling Problems

0-1 tilings
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When there is a tiling then some further decoration of \(\mathbb{Z}^2\) gives \(2^{\aleph_0}\) countable variations, and no universal structure.
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When there is no tiling then there is a bound on the sizes of connected components, and there is a universal homogeneous structure.
Undecidability

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To forbid a pattern places a condition on induced subgraphs.
Algebraic Closure

$\text{acl}_C(A)$: e.g., if you bound the vertex degree by forbidding a star, then the algebraic closure of a point is its connected component.
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Theorem

If the algebraic closure operator is locally finite then the model completion of the theory of $C$-free graphs is $\aleph_0$-categorical (and its model is universal).
Other tameness conditions
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- Number of models
Other tameness conditions

- Number of models
- **Stability and its kin (for the associated theory).**

**Question**

*Is stability (and so on) a decidable property, as a function of the constraint set $C$? Is this combinatorially interesting (or robust) at the finite level?*