Relational Complexity of a Finite Primitive Structure

Gregory Cherlin

Edinburgh, 19.9.2018
Contents

- Introduction
  - Structures and permutation groups
  - A little history
  - Questions, examples
- Small Complexity
- Natural examples
1 Structures and Permutation Groups

2 History

3 Questions, Examples

4 Very small $\rho$

5 Some natural examples
<table>
<thead>
<tr>
<th>Structure</th>
<th>Permutation Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>G</td>
</tr>
</tbody>
</table>

Remark

$A$ is homogeneous in the canonical language. (Orbits are isomorphism types.)
There and back again

<table>
<thead>
<tr>
<th>Structure</th>
<th>Permutation Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$G$</td>
</tr>
</tbody>
</table>

$A \xrightarrow{\text{Aut}} G$

Remark: $A$ is homogeneous in the canonical language. (Orbits are isomorphism types.)

Very small $\rho$

Some natural examples
There and back again

<table>
<thead>
<tr>
<th>Structure</th>
<th>Permutation Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$A^k / G$</td>
</tr>
<tr>
<td>$G$</td>
<td></td>
</tr>
</tbody>
</table>

Remark

$A$ is homogeneous in the canonical language. (Orbits are isomorphism types.)
Remark

A is **homogeneous** in the canonical language. (Orbits are isomorphism types.)
Example

\[
\begin{array}{c|c}
A & G \\
\hline
C_n & D_{2n} \\
\end{array}
\]

\[L_2: \text{path metric } d(x, y) = i\]
Example

\[
\begin{array}{c|c}
A & G \\
\hline
C_n & D_{2n}
\end{array}
\]

\[L_2: \text{ path metric } d(x, y) = i\]

- \textit{k-closed: } \( G = \text{Aut}(A|L_k) \)
- \textit{\( L_k \)-homogeneous: } \( L_k \)-isomorphism types determine \( G \)-orbits
$k$-closure and homogeneity

**Example (Petersen Graph)**

\[
\text{Aut}(P) = \text{Sym}(5) \text{ (2-closed).}
\]

$L_3$-homogeneous.
$k$-closure and homogeneity

**Example (Petersen Graph)**

$\text{Aut}(P) = \text{Sym}(5)$ (2-closed).
$L_3$-homogeneous.

Independent triples:
$\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ (triangle); $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ (star).
Relational Complexity

\[ \rho(G) = \min (r : A|L_r \text{ is } G\text{-homogeneous}) \]
Relational Complexity

\[ \rho(G) = \min\{ r : A|L_r \text{ is } G\text{-homogeneous} \} \]

rc-spectrum

\[ \{ r \mid \exists (a_1, \ldots, a_r), (a'_1, \ldots, a'_r) \] Not \( G \)-conjugate
all proper restrictions \( G \)-conjugate\} \n
\[ \rho(G) = \sup(\text{rc-spectrum}) \]
1. Structures and Permutation Groups

2. History

3. Questions, Examples

4. Very small \( \rho \)

5. Some natural examples
Lachlan Homogeneous for a finite relational language $ho$ bounded
$A^\rho/G$ bounded.
(Stability theory)
Model Theoretic Background

Lachlan Homogeneous for a finite relational language \( \rho \) bounded
\( A^\rho / G \) bounded.
(Stability theory)

Generalization: \( A^4 / G \) bounded.
Kantor-Liebeck-Macpherson Classified in the primitive case.
Classical or semi-classical geometries.
C-H Structure theory based on the primitive classification
(neostability theory)
1. Structures and Permutation Groups

2. History

3. Questions, Examples

4. Very small $\rho$

5. Some natural examples
Questions for the primitive case

- What can we say about $A$ if $\rho$ is bounded?
- What can we say about $\rho$ (and possibly the spectrum) when $A$ is “natural?”
- What is the meaning of gaps in the spectrum?
A few more examples

1. \( \text{SL}_n < G \leq \text{GL}_n: \ n + 1 \) (linear algebra)
   - \( \text{SL}_n: \ n \)
   - \( \text{ASL}_n < G \leq \text{AGL}_n: \ n + 2 \) unless \( n = 1, \ G = D_{2,q} \)

2. \( O^\pm(n, q), \ q \neq 2: \begin{cases} n \text{ isotropic} \\ 2 \text{ anisotropic} \end{cases} \)
   (linear algebra or inner products)

3. \( \mathcal{P}^1: \ 4 \) (cross ratio)

4. \( (\mathcal{P}([n]), \text{Sym}(n)): \lfloor \log_2 n \rfloor + 1 \)
   "\( |\alpha(\bar{S})| = i \)" \( \alpha \) a Boolean atom
Relational Complexity of a Finite Primitive Structure

Gregory Cherlin

Structures and Permutation Groups

2 History

3 Questions, Examples

4 Very small $\rho$

5 Some natural examples
Small $\rho$: $\rho = 2$

Conjecture (Binary Conjecture)

The (finite) primitive binary structures are

- $\bar{C}_p$ (regular action)
- $\text{Sym}(n)$ (theory of equality)
- $AO(n, q)$ anisotropic
Small $\rho$: $\rho = 2$

Conjecture (Binary Conjecture)

The (finite) primitive binary structures are

- $\tilde{C}_p$ (regular action)
- $\text{Sym}(n)$ (theory of equality)
- $AO(n, q)$ anisotropic

Cherlin, Wiscons: reduced to almost simple case (Very dependent on the value $\rho = 2$)
Almost Simple Case

Gill, Spiga, Dalla Volta, Hunt, Liebeck
Almost Simple Case

Gill, Spiga, Dalla Volta, Hunt, Liebeck

Theorem (Gill, Spiga)

The Binary Conjecture holds for alternating socle.
Almost Simple Case

Gill, Spiga, Dalla Volta, Hunt, Liebeck

Theorem (Gill, Spiga)

*The Binary Conjecture holds for alternating socle.*

The easy cases:

- $\text{Sym}(n)$ on $k$-sets: $\lceil \log_2 k \rceil + 2$
  (bounded family, but not usually 2)
- $\text{Sym}(n = n_1 n_2)$ on partitions of shape $n_1 \times n_2$: At least
  $$\max(n_1, \lceil \log_2 2(n_2 - 1) \rceil)$$
Alternating Socle: Primitive Point Stabilizer

The hard case
Primitive point stabilizer $M = G_*$

**Key device:** Elements of $M$ have few fixed points on $[n]$
If $M$ has an element of order 4 with a fixed point $\alpha = (0)(1243)\cdots \in M$, $\beta = (01234)$ not in $M$. $H = \langle \alpha, \beta \rangle \simeq \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, acting naturally on $\{0, 1, 2, 3, 4\}$. 

$$
\begin{array}{c|c}
M \backslash G & [n] \\
\hline 
\alpha & ? \quad (0)(1243)\cdots \in M \\
\beta & ? \quad (01234) \notin M \\
\end{array}
$$
If $M$ has an element of order 4 with a fixed point

\[
\begin{align*}
\alpha &= (0)(1243) \cdots \in M, \\
\beta &= (01234) \notin M
\end{align*}
\]

$H = \langle \alpha, \beta \rangle \simeq F_5 \rtimes F_5^\times$, acting naturally on $\{0, 1, 2, 3, 4\}$. Let $\tilde{0}$ be $M$ in $M \setminus G$ and let $\tilde{O} = \tilde{0} \cdot H = (\tilde{0}, \tilde{1}, \ldots, \tilde{4})$. 
If $M$ has an element of order 4 with a fixed point

\[
\begin{array}{c|c}
M \setminus G & [n] \\
\alpha & (\bar{0})(\bar{1}, \bar{2}, \bar{4}, \bar{3}) \cdots (0)(1243) \cdots \in M \\
\beta & (\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}) \cdots (01234) \notin M
\end{array}
\]

$\alpha = (0)(1243) \cdots \in M$. $\beta = (01234)$ not in $M$

$H = \langle \alpha, \beta \rangle \simeq \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, acting naturally on $\{0, 1, 2, 3, 4\}$.

Let $\bar{0}$ be $M$ in $M \setminus G$ and let $\bar{O} = \bar{0} \cdot H = (\bar{0}, \bar{1}, \ldots, \bar{4})$.

Then $H_\bar{0} = H_0 = \langle \alpha \rangle$ and $H$ acts \textit{doubly transitively} on $\bar{O}$. 
If $M$ has an element of order 4 with a fixed point

$$\begin{array}{c|c}
M \backslash G & [n] \\
\alpha & (\tilde{0})(\tilde{1}, \tilde{2}, \tilde{4}, \tilde{3}) \cdots (0)(1243) \cdots \in M \\
\beta & (\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}) \cdots (01234) \notin M \\
\end{array}$$

$\alpha = (0)(1243) \cdots \in M$. $\beta = (01234)$ not in $M$

$H = \langle \alpha, \beta \rangle \cong \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, acting naturally on $\{0, 1, 2, 3, 4\}$.

Let $\tilde{0}$ be $M$ in $M \backslash G$ and let $\tilde{O} = \tilde{0} \cdot H = (\tilde{0}, \tilde{1}, \ldots, \tilde{4})$.

Then $H\tilde{0} = H_0 = \langle \alpha \rangle$ and $H$ acts doubly transitively on $\tilde{O}$.

Binarity: $G$ induces $\text{Sym}(\tilde{O})$ on $\tilde{O}$. 
If $M$ has an element of order 4 with a fixed point

\[ M \backslash G \quad [n] \]

\[ \alpha = (\tilde{0})(\tilde{1}, \tilde{2}, \tilde{4}, \tilde{3}) \cdots \quad (0)(1243) \cdots \quad \in M \]

\[ \beta = (\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}) \cdots \quad (01234) \quad \notin M \]

$\alpha = (0)(1243) \cdots \in M$. $\beta = (01234)$ not in $M$

$H = \langle \alpha, \beta \rangle \simeq F_5 \rtimes F_5^\times$, acting naturally on $\{0, 1, 2, 3, 4\}$.

Let $\tilde{0}$ be $M$ in $M \backslash G$ and let $\tilde{O} = \tilde{0} \cdot H = (\tilde{0}, \tilde{1}, \ldots, \tilde{4})$.

Then $H_{\tilde{0}} = H_0 = \langle \alpha \rangle$ and $H$ acts doubly transitively on $\tilde{O}$.

**Binarity:** $G$ induces $\text{Sym}(\tilde{O})$ on $\tilde{O}$.

In particular $\beta$ has a conjugate $\beta'$ such that $\beta \beta'$ is nontrivial and fixes $\tilde{0}$.

**Return to $[n]$:** Many fixed points, in $M$: contradiction! (or $n$ is not very large).
If $M$ has an element of order 4 with no fixed point

Then many orbits of length 4 ($\alpha^2$ has few fixed points). Take 5 such orbits and make the regular representation of $H = \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, with $\beta$ having exactly 4 orbits of length 5.
If $M$ has an element of order 4 with no fixed point

Then many orbits of length 4 ($\alpha^2$ has few fixed points). Take 5 such orbits and make the regular representation of $H = \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$, with $\beta$ having exactly 4 orbits of length 5. We still have $\tilde{0}$ fixed by $\langle \alpha \rangle$.

<table>
<thead>
<tr>
<th>$M \setminus G$</th>
<th>$[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$ (0)(1, 2, 4, 3)</td>
<td>(e, a, a^2, a^3)(b, ba, ba^2, ba^3) \cdots \in M</td>
</tr>
<tr>
<td>$\beta$ (0, 1, 2, 3, 4)</td>
<td>(1, b, b^2, b^3, b^4)(\cdots)(\cdots)(\cdots) \notin M</td>
</tr>
</tbody>
</table>
If $M$ has an element of order 4 with no fixed point

Then many orbits of length 4 ($\alpha^2$ has few fixed points). Take 5 such orbits and make the regular representation of $H = \mathbb{F}_5 \times \mathbb{F}_5^\times$, with $\beta$ having exactly 4 orbits of length 5. We still have $\tilde{0}$ fixed by $\langle \alpha \rangle$.

<table>
<thead>
<tr>
<th>$M \setminus G$</th>
<th>$[n]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>(0)(1,2,4,3)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>(0, 1, 2, 3, 4)</td>
</tr>
</tbody>
</table>

Finish as before, working mostly in $M \setminus G$. 
$M$ has no element of order 4?

Meanders . . .
Wander through the various possibilities for $M$, coming back to $M$ almost simple by the same method.
Then use the classification of finite simple groups (or rather an early result in that direction).
$M$ has no element of order 4?

Meanders . . .

Wander through the various possibilities for $M$, coming back to $M$ almost simple by the same method.

Then use the classification of finite simple groups (or rather an early result in that direction).

Exceptions occur:
E.g., $\text{Sym}(p)$ on $\text{AGL}(1, p)$
(and its restriction to $\text{Alt}(p)$).
Sym$(p)$ with stabilizer AGL$(1, p)$

This is the action on Sylow $p$-subgroups by conjugacy.
Sym(\(p\)) with stabilizer AGL(1, \(p\))

This is the action on Sylow \(p\)-subgroups by conjugacy. AGL(1, \(p\)) = \(C_p \rtimes C_{p-1}\); for \(p \geq 5\), a given \(C_{p-1}\) normalizes more than 1 \(p\)-Sylow. So AGL(1, \(p\)) acts on some orbits as on the affine line, with relational complexity 3.
Sym$(p)$ with stabilizer AGL$(1, p)$

This is the action on Sylow $p$-subgroups by conjugacy. $AGL(1, p) = C_p \rtimes C_{p-1}$; for $p \geq 5$, a given $C_{p-1}$ normalizes more than 1 $p$-Sylow. So $AGL(1, p)$ acts on some orbits as on the affine line, with relational complexity 3. (Similarly for $AGL(1, p) \cap Alt(p)$ once $p > 5$.)
Sporadic socle

Gill, Dalla Volta, Spiga, to appear.

**Theorem**

There are no primitive binary actions of almost simple groups with sporadic socle.
Sporadic socle

Gill, Dalla Volta, Spiga, to appear.

**Theorem**

*There are no primitive binary actions of almost simple groups with sporadic socle.*

Most actions are explicitly known. Computation will reach a certain distance (and rather far if supported by a rich range of theoretical tests).

Again, the “small stabilizer” case arises, and the fact that one just needs to understand *one* $M$-orbit can be very helpful.

Notably, $M = \text{Alt}_4 \times \text{Sym}_5$ in $\text{Co}_3$, $(5:4) \times \text{Alt}_5$ in Ru, where one finds $M \cap M^g = 2$-Sylow for some $g$. 
Sporadic socle

Gill, Dalla Volta, Spiga, to appear.

**Theorem**

There are no primitive binary actions of almost simple groups with sporadic socle.

Most actions are explicitly known. Computation will reach a certain distance (and rather far if supported by a rich range of theoretical tests).

Again, the “small stabilizer” case arises, and the fact that one just needs to understand one $M$-orbit can be very helpful.

Notably, $M = \text{Alt}_4 \times \text{Sym}_5$ in $\text{Co}_3$, $(5 : 4) \times \text{Alt}_5$ in $\text{Ru}$, where one finds $M \cap M^g = 2$-Sylow for some $g$.

**Observation** There are relatively few transitive binary actions as well, apparently and this can be remarkably useful in exploiting knowledge about the point stabilizer.
1. Structures and Permutation Groups

2. History

3. Questions, Examples

4. Very small $\rho$

5. Some natural examples
$k$-sets

$k$-sets under $\text{Sym}(n)$: $\lceil \log_2 k \rceil + 2$

(Remains bounded as $n \to \infty$.)
**k-sets**

**k-sets under** $\text{Sym}(n)$:  $\lfloor \log_2 k \rfloor + 2$  
*(Remains bounded as $n \to \infty$)*

**k-sets under** $\text{Alt}(n)$:

\[
\begin{cases}
  n - 1 & \text{if } k = 1 \\
  n - 2 & \text{if } k = 2 \text{ or } n = 2(k + 1) \\
  n - 3 & \text{otherwise}
\end{cases}
\]
$k$-sets under $\text{Sym}(n)$: $\lceil \log_2 k \rceil + 2$

$(Remains bounded as n \to \infty.)$

$k$-sets under $\text{Alt}(n)$:

$$\begin{cases} 
  n - 1 & \text{if } k = 1 \\
  n - 2 & \text{if } k = 2 \text{ or } n = 2(k + 1) \\
  n - 3 & \text{otherwise}
\end{cases}$$

Why?
Relational spectrum

Spectrum: $\text{Sym}(20)$ on 4-tuples: (2–4)
Spectrum: $\text{Alt}(20)$ on 4-tuples: (2–4, 8–17). Both pieces derived from the action of $\text{Sym}(20)$
Spectrum: Sym(20) on 4-tuples: (2–4)
Spectrum: Alt(20) on 4-tuples: (2–4, 8–17). Both pieces derived from the action of Sym(20)
Above $\rho^+ = \rho(k\text{-sets}, \text{Sym}(n))$ the relational spectrum for Alt(n) on k-sets comes from sequences of k-sets which just separate points in [n].
Namely $(X_1, \ldots, X_r)$ and its image under an odd permutation.
Relational spectrum

Spectrum: $\text{Sym}(20)$ on 4-tuples: $(2–4)$
Spectrum: $\text{Alt}(20)$ on 4-tuples: $(2–4,8–17)$. *Both pieces derived from the action of $\text{Sym}(20)$*

Above $\rho^+ = \rho(k\text{-sets, } \text{Sym}(n))$ the relational spectrum for $\text{Alt}(n)$ on $k$-sets comes from sequences of $k$-sets which *just separate points* in $[n]$.

Namely $(X_1, \ldots, X_r)$ and its image under an odd permutation.

**Question**

What is the longest sequence of $k$-sets which just separates points in $[n]$?
Proposition

Suppose there is a sequence of $k$-sets of length $r$ which just separates points in $[n]$. Then there is a numerical partition of $n$ into a sum of $n - r$ terms $n = \sum n_i$ with the following splitting property: if $n_i \geq 2$ and $n_i$ is replaced by $(1, n_i - 1)$ then some subsum involving exactly one of these two terms sums to $k$. 
Proposition

Suppose there is a sequence of $k$-sets of length $r$ which just separates points in $[n]$. Then there is a numerical partition of $n$ into a sum of $n - r$ terms $n = \sum n_i$ with the following splitting property: if $n_i \geq 2$ and $n_i$ is replaced by $(1, n_i - 1)$ then some subsum involving exactly one of these two terms sums to $k$.

Application: Look for the shortest sum with the splitting property:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$n$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$(n - 1) + 1$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$2(k + 1)$</td>
<td>2</td>
</tr>
<tr>
<td>Else</td>
<td>$(k - 1) + (k - 1) + ( \cdots )$</td>
<td>3</td>
</tr>
</tbody>
</table>

Then reverse the analysis.
Proposition

Suppose there is a sequence of $k$-sets of length $r$ which just separates points in $[n]$. Then there is a numerical partition of $n$ into a sum of $n - r$ terms $n = \sum n_i$ with the following splitting property: if $n_i \geq 2$ and $n_i$ is replaced by $(1, n_i - 1)$ then some subsum involving exactly one of these two terms sums to $k$.

The analysis: If we omit $X_i$, there is a pair $(a_i, b_i)$ no longer separated.
This makes an acyclic graph with $r$ edges, so $n - r$ components. The sizes of the components are the $n_i$. 
Just separating sequences

**Proposition**

Suppose there is a sequence of $k$-sets of length $r$ which just separates points in $[n]$. Then there is a numerical partition of $n$ into a sum of $n - r$ terms $n = \sum n_i$ with the following splitting property: if $n_i \geq 2$ and $n_i$ is replaced by $(1, n_i - 1)$ then some subsum involving exactly one of these two terms sums to $k$.

The analysis: If we omit $X_i$, there is a pair $(a_i, b_i)$ no longer separated.

This makes an acyclic graph with $r$ edges, so $n - r$ components. The sizes of the components are the $n_i$.

To reverse, use stars and make the $k$-sets correspondingly (and check).
Cohorts?

This is a mechanism whereby low complexity for one group in a cohort may result in high complexity for smaller groups. But low complexity is not that common. We will look at a more delicate case.
Sym\((2n)\) and Alt\((2n)\) on partitions: shape \(n \times 2\)

(2017-18, with Wiscons)

\[\rho^+(n \times 2) : n\]
Sym(2n) and Alt(2n) on partitions: shape $n \times 2$

(2017-18, with Wiscons)

$\rho^+(n \times 2) : n$

Möbius Band

Edge-colored graph: connected, but any two edge colors have small components.
Sym(2n) and Alt(2n) on partitions: shape $n \times 2$

(2017-18, with Wiscons)

$$ \rho^+(n \times 2) : n $$

$$ \rho^-(n \times 2) : \begin{cases} 
  n + 1 & n = 3 \\
  n & n = 2, 4; \text{ or odd; or a multiple of 6} \\
  n - 1 & n > 6 \text{ even, not a multiple of 6}
\end{cases} $$

(or so it seems)
Sym(2n) and Alt(2n) on partitions: shape $n \times 2$

(2017-18, with Wiscons)

$$\rho^+(n \times 2) : n$$

$$\begin{align*}
\rho^-(n \times 2) : & \begin{cases} 
n + 1 & n = 3 \\ 
n & n = 2, 4; \text{ or odd; or a multiple of 6} \\ 
n - 1 & n > 6 \text{ even, not a multiple of 6} 
\end{cases}
\end{align*}$$

(or so it seems)

Some of this follows by direct inheritance from Sym(n):

- Inheritance for $n$ odd: $\rho^- \geq \rho^+$ because when $n = n_1 + n_2$, one of the parts is odd (Möbius band)
- Sequences of partitions just separating points: $n - 1$ if $n > 2$. 
Independent partitions of shape $n \times k$

Maximum sequences of partitions of shape $n \times k$ which just separate points.

$$
\begin{cases}
  n(k - 1) & \text{if } m = n = 2 \\
  n(k - 1) - 1 & \text{if } \min(n, k) = 2 \text{ and } \max(n, k) > 2 \\
  n(k - 1) - 2 & \text{if } n, k > 2
\end{cases}
$$
Independent partitions of shape $n \times k$

Maximum sequences of partitions of shape $n \times k$ which just separate points.

$$
\begin{align*}
&\begin{cases}
  n(k - 1) & \text{if } m = n = 2 \\
  n(k - 1) - 1 & \text{if } \min(n, k) = 2 \text{ and } \max(n, k) > 2 \\
  n(k - 1) - 2 & \text{if } n, k > 2
\end{cases}
\end{align*}
$$

$nk = \sum n_i$. The splitting condition:

*If $n_i \geq 2$ then the sum with $n_i$ split to $1 + (n_i - 1)$ can be rearranged into $n$ sums equal to $k$ (with $1, (n_i - 1)$ separated).*

Examples (Optimal)

- $k^{n-2}(k - 1)^21^2$
- $k^{n-1}1k$
- $(k - 1)^n21^{n-2}$
- $(k + 1)1^{(n-1)k-1}$

- $n + 2$
- $n + k - 1$
- $2n - 1$
- $(n - 1)k$
Conjecture

The relational complexity of $\text{Alt}(nk)$ on shape $n \times k$ is well approximated by $n(k - 1) - 2$ (and should always be at least that).

The relational complexity of $\text{Sym}(nk)$ on shape $n \times k$ is typically much less (but not for $k = 2$).
For $\text{Alt}(2k)$ we expect $2k - 3$.

Examples: $2 \times k$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$2k - 3$</th>
<th>$\rho^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>$\geq 9$</td>
</tr>
</tbody>
</table>
Shape $2 \times k$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$2k - 3$</th>
<th>$\rho^-$</th>
<th>$\rho^+$ (L.B.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3 (2)</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5 (4)</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>7</td>
<td>4 (4)</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>$\geq 9$</td>
<td>6 (4)</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td>$\geq 5$ (4)</td>
</tr>
</tbody>
</table>

For $\text{Sym}(2k)$ there is a lower bound applying to the point stabilizer, namely

$$2[\log_2 k]$$

This may possibly be the true value for the point stabilizer when $k$ is odd.
Problems I

Problem

Show that the relational complexity of $\text{Sym}(nk)$ acting on cosets of $\text{Sym}(k) \wr \text{Sym}(n)$ has relational complexity going to infinity with $n$. 
Problems I

Problem

Show that the relational complexity of $\text{Sym}(nk)$ acting on cosets of $\text{Sym}(k) \wr \text{Sym}(n)$ has relational complexity going to infinity with $n$.

Problem

Let $\rho_0(G) = \min(\rho(X, G) \mid \text{primitive})$. Is this uniformly bounded for $G$ simple? If so, what is the minimum bound holding for almost all such $G$?
Problem

Show that

$$\lim_{n \to \infty} \frac{\rho^+(n \times k)}{n} = c_k$$

for some explicit constant $c_k$ ($<< k$?).
Problem

*Show that*

\[
\lim_{n \to \infty} \rho^+(n \times k)/n = c_k
\]

for some explicit constant \(c_k\) (<< \(k\)).

Problem

*Determine the relational complexity of \([n \choose k]^d\)*

\((k = 1: \text{Saracino.})\)