A Census of Homogeneous finite dimensional Permutation Structures
(After Sam Braunfeld)

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MFO
Overview
- Cameron’s Homogeneous Permutations
- Structure of the Lattice of $\emptyset$-definable equivalence relations
- The current Census (after Braunfeld): problems and conjectures
- 2-constraint and 3-constraint

Details
- Genericity criterion
- Representation Theorem: generalized ultrametric spaces
- The role of distributivity
- 2-constrained classes
1 Overview

2 Details
CAMERON 2002: Homogeneous permutations.
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What is a permutation? 
\((A; <_1, <_2)\)
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2-dimensional diagrams:

A plot of the permutation 134652
(Waton)
1. **Trivial**: $|A| = 1$

2. **Nontrivial primitive**: $<_2 = _1^\pm$ or $<_1, _2$ independent (generic)

3. **Imprimitive**: $(\mathbb{Q}^2; <_1, E_1)$ lexicographic realized as $(\mathbb{Q}^2, <_1, _2)$ in one of two ways.

**Problem I.** The $n$-dimensional case.

**Remark.** All homogeneous ordered graphs have an obvious source; to what extent does adding an order to a language lead to new examples?

**Problem II.** When does a countable universal permutation exist for a family determined by finitely many constraints? (More relevant to the study of permutation pattern classes, but we leave it aside.)
Classification of homogeneous permutations

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Higher dimensions: first census

1. **Trivial**
2. **Nontrivial primitive:** apart from restrictions $<j=<_i^\pm$, no other variations known.
3. **Imprimitive:** Lexicographic $\mathbb{Q}^k$, up to $k = 2^{n-1}$ (with the corresponding chain of equivalence relations definable).

What else?
Higher dimensions: second census

Sam Braunfeld’s examples:

**Theorem**

*Any finite distributive lattice can occur as the lattice of all \(\emptyset\)-definable equivalence relations in a finite dimensional permutation structure.*

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**Problem**

*Normal subgroup structure of the automorphism groups; there is a metric element, as we shall see.*
Example

\((\mathbb{Q} \times \mathbb{Q}, E_1, E_2)\) (product, boolean lattice with two atoms). Extends to \((\mathbb{Q}^2, E_1, E_2, <^*, <^*)\) by generically ordering the quotient \(\mathbb{Q}q^2 / E_i\). This allows a change of language to \((\mathbb{Q}^2, <_1, <'_1, <_2, <'_2)\).
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This is a difficult example to understand abstractly, and does not give a good model for the proof of the representation theorem (as far as I know).
New Census

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   (a) Define equivalence relations.
   (b) Impose convexity conditions on them.
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**Problem**

*Classify the 3-constrained examples explicitly!*

*Remark.* The same problem arose in the case of metrically homogeneous graphs. In that case the solution is a family of examples which is uniformly definable in Presburger arithmetic.
There is no obvious parallel to look for in the present case.
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Genericity Criterion

Theorem (Cameron)

*If all 3-types are realized by a homogeneous permutation then it is generic.*
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Conjecture

This holds for all homogeneous finite dimensional permutation structures.

What is known?
Genericity Criterion

Proposition

Suppose $k$, $n$ satisfy the following condition.

$$\frac{k!}{(k - \ell)!} > n \cdot 2^\ell$$

$$\ell = \left\lfloor \frac{k}{2} \right\rfloor$$

Then any homogeneous $n$-dimensional permutation structure which realizes all $(k - 1)$-types is generic.

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A less numerical version of the argument pushes $k - 1$ down to 3 when $n = 3$, confirming the conjecture in this case.
Genericity Criterion: Proof

Proof.

We show that any structure of order \( k \) is the unique amalgam of two substructures of order \( k - 1 \). The numerical condition allows us to choose \( \ell \) pairs of indices \((i, j)\) such that for any one of the \( n \) orders, one of these pairs is non-adjacent with respect to that order.
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Then we can add $\ell - 1$ points so that every pair becomes non-adjacent with respect to every order, and view the extended structure on $k + \ell - 1$ points as the unique amalgam resulting from factors of order

$$k + (\ell - 1) - \ell = k - 1$$

(remove one point from each of the $\ell$ pairs).
Let $\Lambda$ be a finite distributive lattice.

(1) A $\Lambda$-metric space is $\Lambda$-valued with triangle inequality

$$d \leq d'' \lor d'''$$

(Corresponds to: $E_\lambda(x, y) \iff d(x, y) \leq \lambda$.)
Realization of lattices

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2. Canonical amalgamation:
   \[ d(a_1, a_2) = \bigwedge (d(a_1, x) \lor d(a_2, x)) \]

Is this strong?—If $\emptyset$ is meet irreducible.
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**Lemma**

*If $\emptyset$ is meet irreducible, then the universal homogeneous $\Lambda$-metric space has an expansion by linear orders to a homogeneous structure in which all meet irreducible equivalence relations are convex with respect to at least one such.*
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If $\emptyset$ is not meet irreducible, replace $\Lambda$ by $\Lambda' = [\emptyset', \Lambda]$ and then factor out $E_\emptyset$.

The last step is admittedly not very plausible: $E_\emptyset$ is not convex and it is hard to see what structure is inherited by the quotient, or why it should be homogeneous . . . .
Is distributivity necessary?

**Lemma**

*If \( \Gamma \) is a non-trivial homogeneous \( n \)-dimensional permutations structure, then any proper inclusion \( F < E \) in the lattice of \( \emptyset \)-definable equivalence relations has infinite index.*

*If \( \Gamma \) is a homogeneous structure in a language with equivalence relations satisfying this infinite index condition, then the lattice is distributive.*
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This does not prove the necessity of distributivity: maybe the reduct to the language of equivalence relations is not homogeneous!

But it makes it very plausible . . . .
Proof of distributivity

Compare the \((x, u)\) to the path \((x, y, u)\), noting that \(d(x, y) \leq e \land f\). How do we get the factors? An analog of Neumann's Lemma.
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2-Constrained Classes

### Proposition

*If $\Gamma$ is 2-constrained then it is of standard primitive type: that is, we impose a set of conditions $<j\mp <j'$ and nothing else.*

(If 2-constraints determine the 3-constraints then similarly.)
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(If 2-constraints determine the 3-constraints then similarly.)

Proof.

This says from \( p, q, r \) we get the 2-type \( \text{majority}(p, q, r) \).