Metrically Homogeneous Graphs of Generic Type

Gregory Cherlin

July 22, 2015—Durham
The classification problem

Two dividing lines

Generic Type

Evidence
...Urysohn has managed to construct a complete metric space with a countable dense subset, which contains any other separable metric space isometrically, and furthermore satisfies a quite strong homogeneity condition; the latter being that one can take the whole space (isometrically) onto itself, so that an arbitrary finite set $M$ is carried over to an equally arbitrary finite set $M_1$ congruent to it.

[Alexandrov to Hausdorff (in German), 3.8.24]
... Urysohn has managed to construct a complete metric space with a countable dense subset, which contains any other separable metric space isometrically, and furthermore satisfies a quite strong homogeneity condition; the latter being that one can take the whole space (isometrically) onto itself, so that an arbitrary finite set \( M \) is carried over to an equally arbitrary finite set \( M_1 \) congruent to it.

[Alexandrov to Hausdorff (in German), 3.8.24]

\[
\mathbb{U} = \overline{\mathbb{U}_Q} \quad \mathbb{U}_Q = \lim_{\mathcal{F}} \mathbb{Q}\text{-metric}
\]
The Urysohn graph

$$U_Z = \lim_{\mathcal{F}} \mathbb{Z} \text{-metric}$$

$$\Gamma_Z: (a, b) \text{ is an edge iff } d(a, b) = 1.$$
The Urysohn graph

\[ \mathbb{U}_\mathbb{Z} = \lim_{\mathcal{F}} \mathbb{Z}\text{-metric} \]

\[ \Gamma_{\mathbb{Z}}: (a, b) \text{ is an edge iff } d(a, b) = 1. \]

Remark

The metric on \( \mathbb{U}_\mathbb{Z} \) is the path metric in \( \Gamma_{\mathbb{Z}} \).

Thus \( \mathbb{U}_\mathbb{Z} \) is a countable universal graph (allowing isometric embeddings in the connected case).
The Urysohn graph

\[ U_\mathbb{Z} = \lim_{\mathcal{F}} \mathbb{Z}\text{-metric} \]

\[ \Gamma_\mathbb{Z}: (a, b) \text{ is an edge iff } d(a, b) = 1. \]

Remark

*The metric on $U_\mathbb{Z}$ is the path metric in $\Gamma_\mathbb{Z}$.*

Thus $U_\mathbb{Z}$ is a countable universal graph (allowing isometric embeddings in the connected case).

*Local structure:* The induced graph $\Gamma_1$ on the neighbors of a vertex is the random graph.
The classification problem

**Metrically homogeneous graph** A connected graph $\Gamma$ whose associated metric space is homogeneous in Urysohn’s sense.

... *In the countable case, an answer to this question might be a step towards a classification of the distance homogeneous graphs.*

[Moss, Distanced Graphs, 1992]

... *the theory of infinite distance-transitive graphs is open. Not even the countable metrically homogeneous graphs have been determined.*

[Cameron, A census of infinite distance-transitive graphs, 1998]
What we have

- A natural division between the **special** and **generic** types;
- A **full** classification for the special types;
- A **conjectured** classification of generic type (nearly uniform);
- A reasonable amount of supporting evidence.
The classification problem

Two dividing lines

Generic Type

Evidence
Local type

$\Gamma_i$: The induced metric structure at distance $i$ from a basepoint.

Definitely homogeneous.
If there are some edges ($d(x, y) = 1$), and $\Gamma_i$ is connected, then it is a metrically homogeneous graph, with the graph metric as the induced metric.
Local type

$\Gamma_i$: The induced metric structure at distance $i$ from a basepoint.

Definitely homogeneous.
If there are some edges ($d(x, y) = 1$), and $\Gamma_i$ is connected, then it is a metrically homogeneous graph, with the graph metric as the induced metric.

$\Gamma_1$ is a homogeneous graph (possibly edgeless).
*We use the local structure as our main dividing line.*
### Definition

A metrically homogeneous graph has **exceptional local type** if $\Gamma_1$ is imprimitive, or does not contain an infinite independent set.

A metrically homogeneous graph has **generic type** if $\Gamma_1$ is primitive, and two vertices at distance 2 have infinitely many common neighbors.
Definition
A metrically homogeneous graph has **exceptional local type** if $\Gamma_1$ is imprimitive, or does not contain an infinite independent set.

Theorem
The metrically homogeneous graphs of exceptional local type fall into the following classes.

- Diameter at most 2 (*homogeneous graph*)
- $n$-cycle
- Antipodal of diameter 3, finite
- Tree-like $T_{r,s}$, an $s$-branching tree of $r$-cliques; excluding the tree $T_{2,\infty}$. 
Definition

A metrically homogeneous graph has \textit{exceptional local type} if $\Gamma_1$ is imprimitive, or does not contain an infinite independent set.

A metrically homogeneous graph has \textit{generic type} if $\Gamma_1$ is primitive, and two vertices at distance 2 have infinitely many common neighbors.

Theorem

\textit{The only metrically homogeneous graph which is neither of exceptional local type nor generic type is the regular tree of infinite degree.}

$T_{2,\infty}$

\textbf{Major Tool} The Lachlan-Woodrow classification
The metrically homogeneous graphs of non-generic type are classified.
The classification problem

Two dividing lines

Generic Type

Evidence
A metrically homogeneous graph has **generic type** if $\Gamma_1$ is primitive, and two vertices at distance 2 have infinitely many common neighbors.
A metrically homogeneous graph has **generic type** if $\Gamma_1$ is primitive, and two vertices at distance 2 have infinitely many common neighbors.

$\Gamma_1$: Henson, Random, or an independent set. And for $u$, $v$ at distance 2, the common neighbors of $u$, $v$ give a graph isomorphic to $\Gamma_1$. 
The Generic Type Conjecture

(Via Fraïssé theory, go to amalgamation classes and forbidden substructures.)

Conjecture (Classification Conjecture Generic Type)

For diameter $\delta \geq 3$, either

$$A = A_\Delta \cap A_H$$

where $A$ is an amalgamation class determined by triangle constraints, and $A_H$ is an amalgamation class determined by Henson constraints; or

$$A = A_a \cap A_{H'}$$

where $A_a$ is an antipodal class and $A_{H'}$ is determined by antipodal Henson constraints.
Henson Classes

**Henson Graph** Forbid 1-cliques of order \( n \); or, dually, 2-cliques of order \( n \).

**For \( \delta \geq 3 \):** Forbid some \((1, \delta)\)-spaces.
Henson Classes

Henson Graph  Forbid 1-cliques of order $n$; or, dually, 2-cliques of order $n$.

For $\delta \geq 3$: Forbid some $(1, \delta)$-spaces.

*I’ll probably skip the antipodal variation . . .*
Triangle constraints

**Theorem**

Let $A_\Delta$ be an amalgamation class whose Fraïssé limit is a metrically homogeneous graph of generic type, and whose minimal constraints are of order at most 3. Then

$$A_\Delta = A_{k_1, k_2, c_0, c_1}$$

where $k_1, k_2$ are parameters controlling the forbidden triangles of small odd perimeter, and $c_0, c_1$ control the forbidden triangles of large perimeter (even or odd, respectively). Furthermore the parameters are subject to a certain collection of numerical constraints, e.g.

If $C = \min(c_0, c_1) \leq 2\delta + k_1$, then $C = 2k_1 + 2k_2 + 1$
The Generic Conjecture

Minimal constraints should be triangles or Henson constraints (and then we know everything)
Metrically Homogeneous Graphs of Generic Type

Gregory Cherlin

1. The classification problem
2. Two dividing lines
3. Generic Type
4. Evidence
Diameter 3 (with Amato, Macpherson)

If the conjecture holds in finite diameter then it holds in infinite diameter (hence: induction)

The bipartite case can be handled under an inductive hypothesis via *halving*.

Still to do: find an inductive treatment of the antipodal case, thereby reducing to the primitive case (via Smith’s theorem).
4-triviality

**Definition**

An amalgamation class is *4-trivial* if any forbidden structure of order 4 either contains a forbidden triangle or is a Henson constraint.

**Proposition**

*In a 4-trivial amalgamation class, the pattern of forbidden triangles is known.*
In the primitive generic type case, is the universal minimal flow the space of orders?