Universal Graphs with Forbidden Subgraphs

Gregory Cherlin

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Outline

Origins

Universality and Homogeneity

Universality without Homogeneity
Universal Graphs with Forbidden Subgraphs

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Outline
Origins
Universality and Homogeneity
Universality without Homogeneity

1 Origins

2 Universality and Homogeneity

3 Universality without Homogeneity
Origins

[**RADO64**]: Universal Graphs

- Universal (countable) graphs exist
- Universal locally finite graphs do not exist (de Bruijn)

[**KomPach91**] (survey): WHEN do universal graphs exist?
[RADO64]: Universal Graphs  
[KomPach91] (survey): WHEN do universal graphs exist?  
[ERDŐS-RÉNYI63]: Automorphisms  
- $\text{Aut}(\Gamma) = 1$ for $\Gamma$ random finite  
- $\text{Aut}(\Gamma)$ rich for $\Gamma$ random infinite  

*Thus there is a striking contrast . . . : while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.*
Origins

[RADO64]: Universal Graphs
[KomPach91] (survey): WHEN do universal graphs exist?

[ERDŐS-RÉNYI63]: Automorphisms

Thus there is a striking contrast . . . : while „almost all" finite graphs are asymmetric, „almost all" infinite graphs are symmetric.

[KPT05] Aut $\Gamma$ has fixed points $\iff$ Structural Ramsey
[Pes98] Aut($\mathbb{Q}$) has fixed points $\iff$ Ramsey
Universality

We follow Rado’s line (or Komjáth/Pach’s interpretation of it) ...
Origins

Universality and Homogeneity

Universality without Homogeneity
A \simeq B \iff A \sim B \text{ (conjugate under } \text{Aut}(\Gamma))
Homogeneity

**Definition (Homogeneity)**

\[ A \simeq B \iff A \sim B \]

**Consequences**

- **Universality** (modulo finite substructures)
- **Uniqueness** (modulo finite substructures)
- **Oligomorphic** (finitely many orbits on \( n \)-tuples)

As observed by Urysohn in 1924...
Homogeneity

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As observed by Urysohn in 1924 . . .
Urysohn 1924 (Letter)

“...[a] condition of homogeneity: the latter being, that it is possible to map the whole space onto itself... so as to carry an arbitrary finite set $M$ into an equally arbitrary set $M_1$, congruent to the set $M$.”

Ref: [Hušek08]
Urysohn 1924 (Letter)

“. . . [a] condition of homogeneity: the latter being, that it is possible to map the whole space onto itself . . . so as to carry an arbitrary finite set $M$ into an equally arbitrary set $M_1$, congruent to the set $M$.”

$U$: universal complete separable metric space
$U_Q$: universal rational-valued metric space
• $U_Q$ is a universal graph (edges: $d(u, v) = 1$)
cf. Moss, Cameron . . .
Limits of Homogeneity

Theorem (Lachlan/Woodrow 1980)

The homogeneous graphs are (up to complementation)

- $C_5, K_3 \otimes K_3$ (9 vertices)
- $m \cdot K_n$ ($m, n \leq \infty$)
- Generic $K_n$-free [Henson71]

However, a structural Ramsey theorem requires an order …
Theorem (Cherlin 2013)

The homogeneous ordered graphs are

- Generic linear extensions of homogeneous partial orders with edge relation “comparability” (cf. [Schmerl79])
- Generically ordered homogeneous graphs (cf. [LachWood80])
- Generically ordered homogeneous tournaments with edges “a → b ⇐⇒ a < b” (cf. [Lachlan84])
- Homogeneous permutations (cf. [Cameron03])
1. Origins

2. Universality and Homogeneity

3. Universality without Homogeneity
A Decision Problem

Survey: [KomjathPach91]
Narrowing the focus:
A Decision Problem

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Narrowing the focus:

**Problem**

$C$: finite set of finite, connected, forbidden subgraphs

Is there a universal $C$-free graph?
A Decision Problem

Survey: [KomjathPach91]

Narrowing the focus:

Problem

\[ C : \text{finite set of finite, connected, forbidden subgraphs} \]

Is there a universal \( C \)-free graph?

Variant

Forbid \textit{induced} subgraphs
Survey: [KomjathPach91]
Narrowing the focus:

**Problem**

\[ C : \text{finite set of finite, connected, forbidden subgraphs} \]

Is there a universal \( C \)-free graph?

**Variant**

Forbid induced subgraphs

- More general
- **Undecidable** via Wang’s domino problem
- for the brave . . .
What so special about SUBGRAPHS?

- Sample Theorems
- Conjectures
- **Underlying Theory** [CheSheShi97]
## Sample Theorems

<table>
<thead>
<tr>
<th>Who, When</th>
<th>What</th>
<th>Which</th>
</tr>
</thead>
<tbody>
<tr>
<td>KMP88</td>
<td>Forbid a long path</td>
<td>∃</td>
</tr>
<tr>
<td></td>
<td>No short odd cycles</td>
<td>&quot;</td>
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<tr>
<td>ChShe07</td>
<td>Tree</td>
<td></td>
</tr>
<tr>
<td>ChShi96</td>
<td>Set of cycles</td>
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</tr>
<tr>
<td>ChSheShi97</td>
<td>Hom-closed set</td>
<td></td>
</tr>
<tr>
<td>FürKom97</td>
<td>2-connected</td>
<td></td>
</tr>
<tr>
<td>Kom99,ChTal07</td>
<td>2 blocks</td>
<td></td>
</tr>
</tbody>
</table>

Forbid a long path, No short odd cycles, Tree, Set of cycles, Hom-closed set, 2-connected, 2 blocks: \( \min(m, n) \leq 5 \) not \((5, 5)\)!
Conjectures (1 Constraint)

Conjectures on existence of universal $C$-free graphs

1 (Solidity) Blocks of $C$ should be complete

2 (Block-Path) After pruning trees, $C$ should become a block-path
Conjectures on existence of universal $C$-free graphs

1 (*Solidity*) Blocks of $C$ should be complete

2 (*Block-Path*) After pruning trees, $C$ should become a block-path

**Theorem (ChShe, in prep)**

*If the constraint $C$ is a block path, and a universal $C$-free graph exists then $C$ has complete blocks.*

**Corollary**

$(2) \implies (1)$
Methods

- Pruning
- Algebraic Closure (+ Füredi-Komjáth method)
Non-Definition — $a$ is $C$-algebraic over $X$ if forbidding $C$ bounds the number of vertices like $a$. 
Non-Definition — $a$ is $\mathcal{C}$-algebraic over $X$ if forbidding $\mathcal{C}$ bounds the number of vertices like $a$.  

There are two ways to be algebraic:
- Obviously
- Or by transitivity
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Example
Let $C$ contain a star (i.e., we bound the vertex degrees). Then

- Obviously algebraic means *neighbor*
- Algebraic means *in the connected component*
Non-Definition — $a$ is $C$-algebraic over $X$ if forbidding $C$ bounds the number of vertices like $a$.

There are two ways to be algebraic:

- Obviously
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Example (cont.)

- Forbidding $C_4$ makes a common neighbor unique. This can be iterated.
- Forbidding $C$, 2-connected but not complete, with $a, b$ non-adjacent, makes $\bar{a}$ unique over $C \setminus \{a, b\}$, where $\bar{a}$ results by setting $a = b$. 
Theorem

Let $C$ be a finite set of finite connected forbidden subgraphs with all blocks complete. Then the following are equivalent.

- There is a universal $C$-free graph with oligomorphic automorphism group;
- The algebraic closure of a vertex is always finite.

The halting problem for the relation obviously algebraic in
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- There is a universal $C$-free graph with oligomorphic automorphism group;
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The halting problem for the relation obviously algebraic in

Example

If $C$ contains a star, decidable:

- Algebraic closure = connected component
- Oligomorphic iff some path forbidden
Pruning

The first method of pruning:

- For a tree, remove its leaves.
- Generally, remove a minimal block-leaf (or more generally, a “corner”)

**Lemma**

If $C$ prunes to $C'$, then a universal $C$-free graph will contain a universal $C'$-free graph. So we may argue inductively.
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Applications: from trees to near-paths (by treating the minimal case).
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Applications: from trees to near-paths (by treating the minimal case).
— And probably . . .
A tentative Result

Theorem (CheShe, in progress)

Let $C$ be a block-path with $\ell \geq 6$ blocks, all complete, of sizes $m_i = |B_i| \geq 3$ all $i$, and allowing a universal $C$-free graph. Then up to reversal the sequence $(m_i)$ is one of: $(4, 4, 3^*)$, $(3, m, 3^*)$, $(m, 3^*)$

Is the end in sight? Not yet —
Problem

C: $K_n$ plus $n$ paths, 1 at each vertex. Is there a universal $C$-free graph?
A Problem for Graph Theorists

Problem

C: $K_n$ plus $n$ paths, 1 at each vertex. Is there a universal $C$-free graph?

Is this a problem for graph theorists?

Think about $\text{acl}_C$ . . . Menger’s theorem?