HOMOGENEOUS DIRECTED GRAPHS. THE IMPRIMITIVE CASE

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Introduction

A relational system Γ is said to be *homogeneous* if any isomorphism $\alpha : A \to B$ between two of its finite substructures is induced by an automorphism of Γ. Assuming the language is finite, such structures are $\aleph_0$-categorical, and Lachlan has a very general theorem concerning the classification of the stable ones [4,7,10] which is a refinement (for this special case) of the results of [3]. Roughly speaking, the stable homogeneous structures for a fixed finite relational language fall into finitely many families, with the isomorphism type of the structures within a family determined by rather trivial numerical invariants. In particular, there are only countably many countable stable homogeneous structures for a finite relational language.

In certain cases all the homogeneous structures have been classified, though not as a result of any general theory. The homogeneous symmetric graphs or tournaments (directed graphs with any two vertices joined by an edge) were classified in [11] and [9] respectively. The methods of the second paper seem particularly interesting, as the nimbus of a general method seems dimly perceptible. I have shown recently that the same method can be used to classify the homogeneous directed graphs omitting the edgeless graph I_∞ on infinitely many vertices: the tournaments are of course those which omit I_2.

What of the homogeneous directed graphs in general? There are $2^{\aleph_0}$ known types which are freely generated by tournaments in the following sense. In the partial order of isomorphism types of finite tournaments ordered by embeddability, fix an infinite antichain $\mathcal{I}$ (one is exhibited in [6], which I follow here). For $\mathcal{X}$ an arbitrary subset of $\mathcal{I}$, form the closure $\mathcal{A}(\mathcal{X})$ of $\mathcal{X}$ with respect to free amalgamation, isomorphism, and substructure, where the free amalgamation of two directed graphs which agree on their common vertices is simply their union, pointwise and edgewise. Following Fraïssé, we associate to $\mathcal{A}(\mathcal{X})$ the $\mathcal{A}(\mathcal{X})$-generic homogeneous directed graph, from which $\mathcal{X}$ is easily recovered. In this way we find $2^{\aleph_0}$ countable homogeneous directed graphs. (In the future structures are assumed countable without further mention.)

And so it seems that Lachlan's theory cannot be extended to the unstable case; but actually this does not follow at all—not from these cardinality considerations. If one is to draw this sort of conclusion from such evidence then one must in particular regard the homogeneous directed graphs as intrinsically unclassifiable, while the opposite possibility—that they are all already known—is perfectly consistent with the evidence. I propose accordingly to work in this direction—an explicit classification of the homogeneous directed graphs—partly in order to lay to rest these cardinality considerations, which have lately reared their heads in more algebraic contexts as well [1,12,13]. This is not so say that one actually expects a smooth general theory of homogeneous structures for finite relational languages, only that sensible criteria for classifiability are wanted; and indeed a very sensible criterion has already been suggested by Lachlan. He proposes a Gentzen style entailment relation for finite sets $\mathcal{A}, \mathcal{B}$ of finite structures for a given language L: $\mathcal{A} \vdash \mathcal{B}$ means that any homogeneous L-structure embedding all the structures in $\mathcal{A}$ must also embed some structure in $\mathcal{B}$. Using Fraïssé's theory relating homogeneous structures and amalgamation classes, one sees that this relation is r.e., and that the problem of classifiability is expressed quite well by

$$(*) \qquad \text{Given } L\text{, is } \vdash \text{ recursive?}$$

This seems by far the most interesting problem in the area, and we known essentially nothing about it.

The goal of the present paper is quite modest. I will describe the known homogeneous directed graphs in some detail, checking homogeneity when it seems appropriate. They fall naturally into three families: deficient (omitting some 2-type), imprimitive (carrying a nontrivial $\emptyset$-definable equivalence relation), and free generated (in the sense described above, or in a dual sense), and there are in addition two more examples known which may be characterized by the 3-types they realize. The deficient examples were classified in the papers [11,9] referred to earlier. The imprimitive ones will be classified here.

There is one other topic which should be dealt with, at least in part, before attacking the primitive case directly. In [11] Lachlan classifies the homogeneous 2-tournaments (these are tournaments partitioned into two distinguished subsets). In dealing with directed graphs it may be convenient to deal with 3-tournaments, allowing in addition three 2-types to be realized between distinct components (as opposed to two realized in a a given component, up to symmetry). I have worked out the classification of the n-tournaments with an arbitrary number of cross types between components, for all n. This seems to be a natural problem to consider prior to tackling the homogeneous directed graphs, and the analysis suggests profitable lines of analysis for the latter problem, but I no longer expect the result to be directly applicable (that is, it may be usable, but it seems that there are better approaches). All of this will be explored in detail elsewhere.

1. The known homogeneous directed graphs

Our description of the known homogeneous directed graphs will be keyed to the following catalog.

I Deficient
- 1 I_n
- 2 $\vec{C}_3, \mathbb{Q}, \mathbb{Q}^*, T^\infty$

II Imprimitive
- 3 Wreathed (composition)
- 4 $\hat{T}$, for $T = I_1, \vec{C}_3, \mathbb{Q}$, or T^∞
- 5 $n * I_\infty$
- 6 Semi-generic

III Exceptional
- 7 $S(3)$
- 8 $\mathcal{P}$

IV Free
- 9 Generic omitting I_{n+1}
- 10 Generic omitting $\mathcal{T}$

Proofs of homogeneity will be given in §2. In the following discussion Γ is some countable, homogeneous, directed graph.

I. Deficient cases. There are three nontrivial 2-types, which will be denoted in two ways as convenience dictates:

$$x \to y \text{ or } y \in x'$$
$$x \leftarrow y \text{ or } y \in {}'x$$
$$x \perp y \text{ or } y \in x^\perp$$

If Γ omits one of these 2-types then it is said to be *deficient* and is then either edgeless (Case 1, $n \leq \infty$) or a tournament. The homogeneous tournaments as classified by Lachlan [9] are I_1, included in Case 1, the oriented triangle $\vec{C}_3$, the rational order $\mathbb{Q}$, the circular order $\mathbb{Q}^*$ described below, and the generic tournament T^∞.

To form $\mathbb{Q}^*$ we can either partition $\mathbb{Q}$ into two dense subsets and reverse the arrows between elements in distinct subsets, or alternatively, place astronomers at all points lying at rational angles on a circle of large radius, equip them with telescopes enabling them to see halfway around in either direction, and draw arrows to the right as far as the eye can see; then each astronomer believes he lives on the rational line. This structure is mentioned in §6 of [2], and is studied in §4 of [15].

II. Imprimitive cases. If Γ is imprimitive then the nontrivial equivalence relation is the union of equality with either $\perp$ or its complement. Wreath products $\Gamma_1[\Gamma_2]$ are formed by taking Γ_1, Γ_2 with no 2-types in common, and replacing the points of Γ_1 by copies of Γ_2. In other words, if T is one of the four nontrivial homogeneous tournaments from Case 2, then we form $T[I_n]$ or $I_n[T]$ for $i < n \leq \infty$; the latter is more commonly called $n \cdot T$.

In all non-wreathed cases the equivalence relation will correspond to $\perp$. For T a tournament, the directed graph $\hat{T}$ is constructed as follows. Let $T^* = T \cup \{a\}$ where $a \to T$. Then $\hat{T}$ is the union of two copies T_1^*, T_2^* of T^*. For $x_1 \in T_1^*, y_2 \in T_2^*$, corresponding to $x, y \in T^*$, $x_1 \to y_2$ iff $y \to x$. Observe that $\perp$ has equivalence classes of size 2, any two of which form a 4-cycle $\vec{C}_4$. $\hat{I}_1 = \vec{C}_4$. One may also check that $\hat{C}_3$ is isomorphic with a graph on the nonzero points of the plain Y over the Galois field $\mathbb{F}_3$ with edges defined by: $x \to y$ iff $x \wedge y$ is equal to a fixed element of $\bigwedge^2 V$. (The exterior product is just the determinant of the matrix with columns x, y once bases are chosen; there is a similar structure on the nonzero points of the plane over $\mathbb{F}_q$, homogeneous for a binary language with $2(q-1)$ 2-types.) The graph $\hat{\mathbb{Q}}$ is a variant of $\mathbb{Q}^*$ in which each astronomer has an antipodal twin whom he cannot see. $\widehat{T^\infty}$ is generic subject to the constraints:

(1) $\perp$ gives rise to an equivalence relation with classes of size 2;
(2) The union of two $\perp$-classes is a copy of $\vec{C}_4$.

The graph $n * I_\infty$ is defined as the generic directed graph on which $\perp$ is an equivalence relation with n classes. For $n = \infty$ there is a variant which for lack of a more suggestive term we call *semi-generic.* The directed graph $\infty * I_\infty$ is generic for the constraint:

(1) $\perp$ gives rise to an equivalence relation.

To get the semi-generic variant we impose the further constraint.

2 For any pairs A_1, A_2 taken from distinct $\perp$-classes, the number of edges from A_1 to A_2 is even.

III. Exceptional homogeneous directed graphs. We can define the *myopic circular order* $S(3)$ most simply in terms of astronomers whose telescopes enable them to see 1/3 of their circular universe in each direction—leaving a third invisible. Alternatively, partition $\mathbb{Q}$ into three dense sets $\mathbb{Q}_i$ indexed by $i \in \mathbb{Z}/3\mathbb{Z}$, identify the types $\perp, \to, \leftarrow$ with $0, 1, 2$ respectively, and for $x \in \mathbb{Q}_i$, $y \in \mathbb{Q}_j$ distinct, assign to (x, y) the type $i - j + \mathrm{tp}_{\mathbb{Q}}(x, y)$.

The generic partially ordered set $\mathcal{P}$ needs no commentary.

IV. Freely generated homogeneous directed graphs. These are the graphs which are generic subject to a constraint of the form: Γ embeds no X from $\mathcal{X}$; here $\mathcal{X}$ is a class of deficient graphs of a given type. In (9) $\mathcal{X}$ is (I_{n+1}) and in (10) $\mathcal{X} = \mathcal{T}$ is a class of tournaments.

These are all the homogeneous graphs known to me, and I conjecture that in fact: only countably many are missing. (Just as in the imprimitive case the semi-generic directed graph appears unexpectedly, others could easily turn up.)

2. Proofs of homogeneity

For the homogeneity of $\mathbb{Q}^*$ see [2] or [9]. $\mathbb{Q}^*$ and $S(3)$ can be analyzed along similar lines: the astronomical description shows that the automorphism group is transitive, so we need only check that the expansion of the structure by a single parameter x is homogeneous, and up to a permutation of 2-types (and the removal of the element x) this expansion is just $\mathbb{Q}$ partitioned into 2 or 3 dense subsets, respectively. In the case of $S(3)$, identifying

$\perp, \to, \leftarrow$ with $0, 1, 2$ respectively, and letting $Q_i = \{y \mid \mathrm{tp}(x, y) = i\}$, we assign to $y \in Q_k$, $z \in Q_j$ the type $(i - j) + \mathrm{tp}(yz)$.

The homogeneity of wreath products of homogeneous structures in disjoint languages has been noted previously by Lachlan, if not earlier, and the existence of amalgamation classes corresponding to examples 8–10 is both straightforward and well known. It remains to discuss examples 4–6.

#4. Recall as a matter of notation that $\hat{T} = T_1^* \cup T_2^*$ with $T_i^* = \{a_i\} \cup T_i$ and $T_i \cong T$. It is quite easy to see that the structure imposed on $T_1 \cup T_2$ by (a_1, a_2) is homogeneous if (and only if) T is, as it consists of two copies of T with a definable isomorphism. As $\{a_2\} = a_1^\perp$, it suffices to see that $\hat{T}$ is transitive when $T \not\cong Qq^*$ is homogeneous. The following condition is sufficient for this, though not necessary:

(†)

For $x \in T$ there is an isomorphism $\alpha : {}'x \to x'$ such that for $y, z \to x$: $y \to \alpha z$ iff $\alpha y \to z$

This condition evidently holds for I_1, $\vec{C}_3$, and $\mathbb{Q}$; for $T = T^\infty$ and $x \in T$ the desired α comes from a back-and-forth construction.

To check the transitivity of $\hat{T}$, observe fist that there is a canonical involution $i \in \mathrm{Aut}\,\hat{T}$ defined by $x \perp i(x)$, so it suffices to find maps $\phi \in \mathrm{Aut}\,\hat{T}$ which take a_1 to any $x_1 \in T_1$. If x_1 corresponds to $x \in T$ then let α be as in (†) and define $\phi(a_i) = x_i$, $\phi(x_i) = a_{3-i}$, while for $y \to x \to z$:

$$\phi(y_i) = \alpha(y)_i \qquad\qquad \phi(z_i) = \alpha(z)_{3-i}$$

(†) expresses the condition that this is an automorphism of $\hat{T}$.

To see that (†) is not a necessary condition for transitivity, notice that if $\mathbb{Z}/n\mathbb{Z}$ is made into a directed graph by taking $x \to y$ to mean

$$(y - x) \in \{1, \dots, n-1\} \pmod{2}n$$

then $\mathbb{Z}/2n\mathbb{Z} \cong L(n-1)^*$, where $L(n-1)$ is the transitive tournament of order $n - 1$.

It will be useful later to know that $\widehat{\mathbb{Q}^*}$ is not homogeneous, and for this we check the failure of transitivity directly. On the one hand $a_1' = \mathbb{Q}^*$ by construction, while on the other hand, for $x_1 \in \mathbb{Q}_1^*$, $x_1' \setminus \{a_2\}$ is linearly ordered, by inspection.

#5. $n * I_\infty$

We must check that the class of finite directed graphs satisfying

(1) The union of $=$ and $\perp$ is an equivalence relation;
(2) this relation has at most n classes.

is an amalgamation class. It suffices to describe how to complete an amalgamation of $A \cup \{a_1\}$ with $A \cup \{a_2\}$ over A, by specifying the type of $a_1 a_2$ suitably.

We can take $a_1 \to a_2$ unless there is an obstruction of one of the following forms.

(1.1) $a_1 \perp b \perp a_2$, $b \in A$; or
(2.1) A has $n - 1$ $\perp$-classes, and there is no $b \in A$ with $b \perp a_1$ or $b \perp a_2$.

We can take $a_1 \perp a_2$ unless there is an obstruction of the form:

1.2 $a_i \perp c \not\perp a_j$, $\{i, j\} = \{1, 2\}$.

There cannot be both sorts of obstruction, so the amalgamation succeeds.

#5. The semi-generic $\perp$-imprimitive case.

We claim that the constraint (1) above can be combined with the constraint

(3) $|(A_1 \times A_2) \cap E|$ is even for A_1, A_2 two $\perp$-equivalent pairs (where E is the set of edges)

to give an amalgamation class of finite directed graphs. With the notation of the previous example, we must again specify the type of $a_1 a_2$.

We take $a_1 \to a_2$ unless there is either an obstruction of the form (1.1), or this choice yields:

(3.1) $b_1, b_2 \in$, $a_i \perp b_i$, and $\{((a_1, b_1) \times (a_2, b_2)) \cap E|$ is odd.

If (1.1) occurs then we take $a_1 \perp a_2$ and we have to check that $[(a_1, a_2) \times B] \cap E|$ is even for any $\perp$-equivalent pair B in A; this follows since $|[(a_i, b) \times B] \cap E|$ is even for $i = 1, 2$.

If case (1.1) does not apply but (3.1) does, then we take $a_2 \to a_1$ and constraint (1) is still satisfied, and moreover (3.1) is now false. What must still be checked is that for $a_i \perp c_i \in A$, that always $|[(a_1, c_1) \times (a_2, c_2)] \cap E|$ is even; for this it suffices to consider $(a_1, b_1) \times (a_2, b_2)$, $(a_1, b_1) \times (a_2, c_2)$, and $(b_1, c_1) \times (a_2, c_2)$.

This completes the description of the currently known examples. The next order of business is to show that the list of imprimitive types is complete.

3. Imprimitive homogeneous graphs with finite classes

Throughout the remainder of this article, Γ denotes an imprimitive homogeneous directed graph. As the nontrivial equivalence relation on Γ is the union of equality with either $\perp$ or its complement, and in the latter case Γ is necessarily a wreath product, we may assume the equivalence relation is

$$\text{“}=\text{”} \cup \text{“}\perp\text{”}$$

By a slight abuse of notation we will denote the equivalence relation also by $\perp$. The theorem we aim at is of course as follows.

Theorem 1. *If Γ is an imprimitive homogeneous directed graph, then Γ is one of the following.*

(1) a wreath product (composition) $T[I_n]$ or $I_n[T]$;
(2) $\hat{T}$, for $T = I_1$, $\vec{C}_3$, $\mathbb{Q}$, or T^∞;
(3) $n * I_\infty$;
(4) semi-generic for $\perp$ an equivalence relation.

As noted, we may take the equivalence relation on Γ to be (essentially) $\perp$. We consider first the case in which this relation has finite classes, of order $n < \infty$.

We can dispose of the case in which Γ is finite by reference to the list in [6] of all finite examples. So we may assume that Γ is infinite, and not a wreath product. Fix a $\perp$-class C, and find $x, y \in \Gamma \setminus C$ with

$$x \to y \qquad\qquad x' \cap C = y' \cap C$$

If $x' \cap C = \emptyset$ or C then it follows easily that Γ is wreathed.

Fix $a \in x' \cap C$.

If $|x' \cap C| = k$ with $1 < k < n$, then we can find $A \subseteq C$, $a \in A$, $A \neq x' \cap C$ and $z \in \Gamma \setminus C$ with $x \to z$ or $z \to x$ so that $z' \cap C = A$. Then axy and either axz or azx have the same type, a contradiction. We conclude that $k = 1$, and similarly that $n - k = 1$, $n = 2$. It then follows rapidly that $\Gamma = \hat{T}$ for some homogeneous T, and we checked in the previous section that this forces $T \ncong \mathbb{Q}^*$.

4. $n * I_\infty$ WITH n FINITE

We have assumed that $\perp$ defines an equivalence relation on Γ, and we will assume throughout that Γ is not a wreath product. We now impose the condition

All $\perp$-classes are infinite.

We first take up the case in which $\Gamma/\perp$ is finite.

Lemma 4.1. *Suppose $|\Gamma/\perp| = n$ is finite. Then for distinct $\perp$-classes C_1, C_2 and $I \subseteq C_1$ finite, the set $I' \cap C_2$ is infinite.*

Proof. For $I \subseteq C_1$ finite, let $F(I)$ be the set of $\perp$-classes C^* in Γ other than C_1 for which $I' \cap C^*$ is finite. Suppose that for some such I, $F(I)$ is nonempty. Let $|I| = k$ be minimized.

As n is finite, there is a pair I_1, I_2 of disjoint k-subsets of C_1 with $F(I_1) = F(I_2)$. By homogeneity, $F(I_1) = F(I_2)$ for any pair of disjoint k-subsets of C_1, and hence $F(I)$ is independent of I for $I \subseteq C_1$ of order k. So we may set $F^*(C_1) = F(I)$ for any such I.

As Γ is not a wreath product, $\operatorname{Aut}\Gamma$ acts 2-transitively on the $\perp$-classes and therefore $F^*(C_1)$ consists of all $\perp$-classes other than C_1, that is I' is finite for $I \subseteq C_1$ of order k. Furthermore arguing as above, the size m of I' is bounded and hence, arguing as above, is constant.

Take $I_0 \subseteq C_1$ of order $k-1$, and set $J = I' \cap C_2$. Then J is infinite. Take $S \subseteq C_1 \setminus I_0$ of order $m+1$. For $s \in S$, we have$(I_0 \cup \{s\})'$ finite, that is $s' \cap J$ is finite.Therefore $'S \cap J$ is infinite.

Take $J_0 \subseteq J \cap' S$ of order k. Then J_0' contains S. But the value of k and m corresponding to C_1 should be the same for C_2, so this is a contradiction.

□

Corollary. *With the same notation, we have the following.*

1. *For any finite subsets I, I_1 of C_1 of the same order there is an automorphism of Γ taking I to I_1 and leaving C_2 invariant;*
2. *if $I, J \subseteq C_1$ are finite, then $I' \cap' J \cap C_2$ is infinite.*

Proof.

1. Take $a \in (I \cup I_1)' \cap C_2$. There is an automorphism taking $\{a\} \cup I$ to $\{a\} \cup I_1$.

2. Fix k arbitrary, and take $K \subseteq C_2$ of order k.

By the lemma (and dually) we may take $I_1 \subseteq K' \cap C_1$ and $J_1 \subseteq' K \cap C_1$ with $|I_1| = |I|$ and $|J_1| = |J|$. Thus $|I_1' \cap' J_1 \cap C_2| \geq k$.

By (1) the same applies to I, J. □

Lemma 4.2. *Suppose $|\Gamma/\perp| = n$ with $n \geq 3$ finite. Then x' is not a wreath product, for $x \in \Gamma$.*

Proof. We have supposed that Γ is not a wreath product. If the lemma fails, fix distinct $\perp$-classes C, C_1, C_2 with $x \in C$ and consider the finite tournament T_x on $(\Gamma \setminus C)/\perp$ with edge relation as in x'. By Lemma 4.1, for $x, y \in C$ the set $\{x, y\}'$ meets each equivalence class outside C, and thus $T_x = T_y$.

Therefore no automorphism of Γ carries the triple (C, C_1, C_2) to the triple (C, C_2, C_1).

Now let $A_1 = x' \cap C_1$, $A_2 = {}'x \cap C_1$, $B_1 = x' \cap C_2$, $B_2 = {}'x \cap C_2$. If there are pairs (a_1, b_2) and (a_2, b_1) with opposite orientation with $a_1 \in A_1$, $a_2 \in A_2$, $b_1 \in B_1$, $b_2 \in B_2$, then the map

$$x, a_1, b_2 \to x, b_1, a_2$$

is an isomorphism and hence is induced by an automorphism taking (C, C_1, C_2) to (C, C_2, C_1), for a contradiction.

Thus for such a_1, a_2, b_1, b_2 the orientation of (a_1, b_2) is the same as that of (a_2, b_1), and in particular is independent of the choice of $a_1 \in A_1$ and $b_2 \in B_2$.

Thus all points in A_1 realize the same type over C_2. It follows easily that all points in A_1 realize the same type over $\Gamma \setminus C_1$. By homogeneity, all points in C_1 realize the same type over $\Gamma \setminus C_1$, and again by homogeneity it follows that Γ is in fact a wreath product, for a contradiction. □

Proposition 4.3. *If Γ is an imprimitive homogeneous directed graph with equivalence relation $\perp$, with each $\perp$-class infinite, and with $\Gamma/\perp$ finite, then either Γ is a wreath product or Γ is $n * I_\infty$ with $n = |\Gamma/\perp|$.*

Proof. We proceed by induction on n, starting with $n = 1$. For the inductive step we suppose that $n > 1$ and that Γ is not a wreath product.

Claim 1. For $x \in \Gamma$, $x' \cong (n-1) * I_\infty$.

For $n = 2$, this is contained in Lemma 4.1. For $n > 2$, we deduce from Lemmas 4.1 and 4.2 that x' is an imprimitive homogeneous directed graph with equivalence relation $\perp$, with each $\perp$-class infinite, with $|x'/\perp| = n - 1$, and not a wreath product, so induction applies. The claim follows.

Claim 2. All finite directed graphs of the form

$$T \cup I$$

embed into Γ, where T is a tournament of order $n - 1$ and I is a $\perp$-class disjoint from the $\perp$-classes represented by T, and is indiscernible over T.

If $n = 2$ then Claim 1 suffices. So suppose $n > 2$. Fix $a, b \in T$ with $a \to b$, let $T_0 = T \setminus \{a, b\}$, and form an amalgamation diagram$K$ on a set of the form

$$T_1 \cup T_2 \cup \{a_1, a_2, b_1, b_2\} \cup I_1 \cup I_2$$

so that

- The type of (a_1, b_1) is left unspecified;
- $T_1 \cup I_1, T_2 \cup I_2 \cong T_0 \cup I$ by isomorphisms which extend to isomorphisms of $T_1 \cup \{a_1\}$ and $T_2 \cup \{b_1\}$ with $T_0 \cup \{a\}$, and of $T_1 \cup \{b_1\}$ and $T_2 \cup \{a_1\}$ with $T_0 \cup \{b\}$;
- $a_1 \perp a_2$, $b_1 \perp b_2$;
- $a_2 \to K \setminus \{a_1, a_2\}$, $b_2 \to K \setminus \{a_2, b_1, b_2\}$.

Viewing K as the amalgam of $K_1 = K \setminus \{b_1\}$ with $K_2 = K \setminus \{a_1\}$ over their common part, the elements a_2, b_2 prevent $a_1 \perp b_1$, and then one of the configurations $T_1 \cup \{a_1, b_1\}$ or $T_2 \cup \{a_1, b_1\}$ provides a copy of $T \cup I$ when the choice $a_1 \to b_1$ or $b_1 \to a_1$ is made. Therefore, to complete the proof of the claim, it suffices to check that K_1 and K_2 embed into Γ.

Now K_2 consists of the vertex a_2 dominating $K \setminus \{a_1, a_2\}$, and this embeds in Γ by Claim 1.

The configuration K_1 may be thought of as an amalgam in which the type of (a_1, a_2) is to be determined. As $n-1$ $\perp$-classes are already represented in $K \setminus \{a_1, a_2\}$, none of them containing a_1 or a_2, this amalgam forces a_1, a_2 to lie in the same $\perp$-class. Furthermore, $a_2 \to b_2$ and $b_2 \to a_1$, so the points a_1, a_2 cannot be identified. Thus to embed K_1 into Γ, it suffices to embed $K_1 \setminus \{a_1\}$ and $K_1 \setminus \{a_2\}$. This again follows from Claim 1 applied to a_2' and to b_2'.

This proves the second claim, and now we may prove the Proposition. We consider a finite configuration A contained in $n * I_\infty$, and we must embed it into Γ. We may suppose that A contains a tournament T of order n, which we use as a set of representatives for the $\perp$-classes. We suppose that the number of nontrivial $\perp$-classes in A is minimized. If this number is at most 1 then Claim 2 applies.

Suppose therefore that there are at least two nontrivial $\perp$-classes C_1, C_2 in A. Adjoint points to C_2 to ensure that the points of C_1 realized distinct types over C_2. Then view A as the amalgam of structures of the form $A_i = \{a_i\} \cup (A \setminus C_1)$ where a_i varies over C_1; the points of C_2 prevent any identifications of distinct a_i, and as $A \setminus C_1$ contains $n-1$ $\perp$-classes distinct from C_1, the elements a_i must all lie in the same $\perp$-class in the amalgam. Thus the result of this amalgamation is A.

The factors $A_i = \{a_i\} \cup (A \setminus C_1)$ have fewer nontrivial $\perp$-classes, hence embed in Γ by induction. □

5. The semi-generic case

Now we assume that the $\perp$-classes are infinite and that $\Gamma/\perp$ is infinite. We will refer to the extra constraint imposed on the semi-generic directed graph as the *parity constraint.*

Proposition 5.1. *If Γ is an imprimitive homogeneous directed graph with equivalence relation $\perp$, for which the $\perp$-classes are infinite, $\Gamma/\perp$ is infinite, Γ is not a wreath product, and the parity constraint is satisfied, then Γ is isomorphic with the semi-generic directed graph.*

We begin with two preliminary results.

Lemma 5.2. *Let Γ be an imprimitive homogeneous directed graph with equivalence relation $\perp$, for which the $\perp$-classes are infinite, $\Gamma/\perp$ is infinite, and Γ is not a wreath product, If C_1, C_2 are two $\perp$-classes and $a \in C_1$, then $a' \cap C_2$ and $'a \cap C_2$ are infinite.*

Proof. It suffices to consider $a' \cap C_2$. We show first that a' meets C_2.

As Γ is not a wreath product, there are arcs in both directions between C_1 and C_2.

Suppose a' does not meet C_2. Let (c, b) be any arc from C_2 to C_1. Then there is an automorphism carrying (c, a) to (c, b). But then C_1, C_2 go to C_2, C_1 and a goes to b, so b' does not meet C_2. Thus each vertex in C_1 either dominates or is dominated by C_2. As there is an automorphism switching C_1 and C_2, each vertex of C_2 either dominates or is dominated by C_1. If we now consider arcs (a_1, b_1) and (b_2, a_2) in both directions between C_1 and C_2, we reach a contradiction.

Thus a' meets C_2. Suppose now that $a' \cap C_2$ is finite and fix $b \in a' \cap C_2$. For each arc (x, y) between C_1 and C_2 there is an automorphism taking (a, b) to (x, y), and hence $x' \cap (C_1 \cup C_2)$ is finite. For $x \in C_1 \cup C_2$, let $R(x)$ be the set of points in $C_1 \cup C_2$ reachable from x, i.e., lying on an oriented path from x. This is a finite set. Then $R(a)$ is finite. Take $b \in C_2 \setminus R(a)$. Then (b, a) is an arc, so $a \in R(b)$, $R(a) \subseteq R(b) \setminus \{b\}$, and $|R(a)| < |R(b)$. However, there are arcs (a, c_2) and (b, c_1) with $c_1 \in C_1$, $c_2 \in C_2$, and an automorphism taking (a, c_2) to (b, c_1), so $|R(a)| = |R(b)|$, for a contradiction. □

Arguing as in the proof of Lemma 4.2, and making use of the previous lemma at the beginning, we may deduce the following.

Lemma 5.3. *Let Γ be an imprimitive homogeneous directed graph with equivalence relation $\perp$, for which the $\perp$-classes are infinite, $\Gamma/\perp$ is infinite, and Γ is not a wreath product, Then for $x \in \Gamma$, x' is not a wreath product.*

After these preparations, we may now prove the following family of assertions for all n. Then clause $(2.n)$, with n varying, gives the proposition.

Lemma 5.4. *Let Γ be an imprimitive homogeneous directed graph in which $\perp$ gives an equivalence relation, not a wreath product, with $\Gamma/\perp$ infinite, and satisfying the parity constraint. If K is a finite directed graph in which $\perp$ gives an equivalence relation, and one of the following applies, then K embeds into Γ.*

(1.n) $K = T \cup I$ *with* T *a tournament of order* n *and* I *a* $\perp$*-class disjoint from* T.
(2.n) K *obeys the parity constraint and* $|K/\perp| = n$.

Proof.

Claim 1. Condition $(1.n)$ implies condition $(2.(n+1))$.

As we have the parity constraint in Γ, this is immediate: in $(2.(n+1))$, amalgamating the factors of K with a unique nontrivial $\perp$-class can only give K.

It suffices therefore to prove condition $(1.n)$ for all n, and for this we proceed inductively. More precisely, we prove for each n, that for all Γ satisfying our hypotheses, condition $(1.n)$ holds.

If $n = 1$, then Lemma 5.2 suffices.

We now suppose $n > 1$ and assume condition $(1.(n-1))$ and hence also $(2.n)$. We have $K = T \cup I$ with $|T| = n$ and I a $\perp$-class disjoint from the $\perp$-classes of T.

We fix two points $u, v \in T$ and let $A = K \setminus \{u, v\}$. Let p, q be the desired types of u over u over A and of v over A, respectively. Let $B = A_1 \cup A_2$ be the union of two copies of A, arranged so that corresponding elements of A_1, A_2 are in the same $\perp$-class, and B satisfies the parity constraint. Let pq denote the type over B given by p on A_1 and q on A_2, and define qp similarly. Adjoin two more vertices x, y to one of the $\perp$-classes of B so that Bxy satisfies the parity constraint, and extend pq, qp to types r_1, r_2 over Bxy which cannot be realized in a single $\perp$-class under the parity constraint.

By condition $(2.n)$, the types r_1 and r_2 are realized in Γ, and hence for any embedding of Bxy into Γ, there are realizations a_1, a_2 of r_1, r_2 in Γ over Bxy. By the choice of r_1, r_2 they lie in distinct $\perp$-classes, that is $a_1 \to a_2$ or $a_2 \to a_1$. Then correspondingly either $A_1 a_1 a_2$ or $A_2 a_1 a_2$ is a realization of K. □

6. The case of $\infty * I_\infty$

We will treat the final case in a similar but more elaborate fashion. Our goal is the following.

Proposition 6.1. *Let Γ be a homogeneous directed graph in which $\perp$ gives an equivalence relation with infinitely many infinite classes, and suppose that Γ does not satisfy the parity constraint (in particular, Γ is not a wreath product). Then $\Gamma \cong \infty * I_\infty$.*

We record the contents of Lemmas 5.2 and 5.3 for the case at hand.

Lemma 6.2. *Let Γ be a homogeneous directed graph in which $\perp$ gives an equivalence relation with infinitely many infinite classes, and suppose that Γ does not satisfy the parity constraint. Then for $x \in \Gamma$, x' is a homogeneous directed graph on which $\perp$ is an equivalence relation with infinitely many classes, each infinite, and is not a wreath product.*

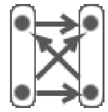
The proof of the proposition will be based on the following notions.

Definition 6.3. If $\mathcal{A}$ is an amalgamation class of finite directed graphs in which $\perp$ defines an equivalence relation, let $\mathcal{A}^*$ be the class of all finite directed graphs K such that an arbitrary extension $K \cup I$ of K with I an additional $\perp$-class will belong to $\mathcal{A}$.

Definition 6.4. An amalgamation class $\mathcal{A}$ of finite directed graphs in which $\perp$ gives an equivalence relation will be called *robust* iff it satisfies the following conditions.

- An independent set I_n on n vertices belongs to $\mathcal{A}$ for all n.
- A linearly ordered tournament L_n on n vertices belongs to $\mathcal{A}$ for all n.
- Some $A \in \mathcal{A}$ violates the parity constraint.

The latter condition can be expressed more concretely by saying that $\mathcal{A}$ contains the particular directed graph $O = C_1 \cup C_2$ on four vertices pictured below, consisting of two $\perp$-classes C_1, C_2 of order 2 with three arcs in one direction and one in the reverse direction.



This allows us to state the main point in the proof of Proposition 6.1 very concisely.

Lemma 6.5. *Let $\mathcal{A}$ be a robust amalgamation class of finite directed graphs on which $\perp$ is an equivalence relation. Then $\mathcal{A}^*$ is also a robust amalgamation class.*

We first deduce the proposition from the lemma, and then prove the lemma.

Proof of Proposition 6.1. It suffices to show that any finite directed graph K on which $\perp$ is an equivalence relation belongs to the amalgamation class $\mathcal{A}$ associated with Γ. We prove this for all such Γ simultaneously, by induction on the number n of $\perp$-classes in K. If there is only one $\perp$-class the claim is obvious, so we suppose $n > 1$. We write

$$K = J \cup I$$

with I one of the $\perp$-classes in K.

By Lemma 6.5 the associated amalgamation class $\mathcal{A}^*$ is robust, so the induction hypothesis applies, and $J \in \mathcal{A}^*$. By the definition of $\mathcal{A}^*$, we have $K \in \mathcal{A}$. □

Proof of Lemma 6.5. For terminological convenience we refer both to the amalgamation class $\mathcal{A}$ and the corresponding homogeneous directed graph Γ below.

That $\mathcal{A}^*$ is an amalgamation class follows on purely formal grounds. If $J_1, J_2 \in \mathcal{A}^*$ but no amalgam J of J_1, J_2 lies in $\mathcal{A}^*$, then for each such amalgam $J = J_1 \cup J_2$ there is an extension $K_J = J \cup I_J$ of the specified form which is not in $\mathcal{A}$. Letting $I = \cup_J I_J$ we then have $J_1 \cup I, J_2 \cup I \in \mathcal{A}$ by hypothesis, and hence some amalgam $J \cup I$ of the two also belongs to $\mathcal{A}$. But as I contains I_J, this gives a contradiction.

So the main point is to check is that tournaments of the forms $J = I_n$, L_n, or O all belong to $\mathcal{A}^*$. We begin with some considerations that apply equally well in all three cases.

We consider an extension $K = J \cup I$ of J by an additional $\perp$-class I. The claim is that K lies in $\mathcal{A}$. If $|I| \geq 3$ and I realizes at least two distinct types over J, then we can write K as $K_1 \cup K_2$ with $K_1 = J \cup I_1$, $K_2 = J \cup I_2$, satisfying the following conditions.

- $|I_1 \cap I_2| = 1$.
- $|I_1|, |I_2| > 1$.
- No points in $I_1 \setminus I_2$ and $I_2 \setminus I_1$ realize the same type over J.

Then K is the unique amalgam of K_1, K_2 over their common part, so if $K_1, K_2 \in \mathcal{A}$ we get $K \in \mathcal{A}$ as well.

Thus it remains to consider two cases.

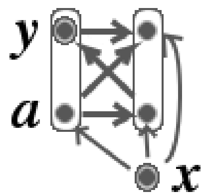
(1) I is indiscernible over J.
(2) $|I| = 2$.

Our treatment of these cases will be built up gradually.

Claim 1. For $x \in \Gamma$, O embeds into x'.

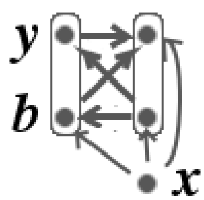
We assume the contrary, toward a contradiction. Then by Lemma 6.2 and Proposition 5.1, x' is isomorphic to the semi-generic imprimitive directed graph. That is, any finite directed graph in which $\perp$ gives an equivalence relation and the parity constraint is satisfied must embed into x', and, in particular, into Γ.

We make a direct construction. First form the following amalgamation diagram.



Both factors of this diagram embed into Γ since the one omitting y satisfies the parity constraint, and the one omitting x is O. So there is an amalgam in Γ in which there is an arc connecting x and y. Since we have ruled out $x \to y$ by hypothesis, we conclude that $y \to x$ in the amalgam.

We have marked an additional vertex a because we next amalgamate this configuration with a similar configuration containing a vertex b in place of a, as shown.



This second configuration satisfies the parity constraint, so embeds into Γ.

Now we amalgamate the two factors shown, that is we determine the type of the pair (a, b). Evidently we must have $a \perp b$. So omitting y, we have $x \to O$.

This proves the claim.

Note that in what follows we can no longer assume that every finite directed graph on which $\perp$ gives an equivalence relation, and which respects the parity constraint, necessarily embeds into Γ, as this followed only when we assumed that x' does not contain O.

Claim 2. If K is a finite directed graph on which $\perp$ is an equivalence relation with two classes, then K embeds in Γ.

Let K consist of the two $\perp$-classes I, J. Extending J if necessary, we may suppose that all elements of I realize distinct types over J. Then we can easily reduce to the case $|I| = 2$.

Similarly, with $|I| = 2$, we can reduce to the following two cases.

(1) $|I| = |J| = 2$.
(2) $|I| = 2$, J is indiscernible over I.

If $|I| = |J| = 2$ and the parity constraint is not satisfied by K, then $K \cong O$ which embeds in Γ by assumption.

If the parity constraint is satisfied, and I realizes distinct types over J, then we add an additional point a to I and express K as the union of two copies of O containing $\{a\} \cup J$. This can be viewed as an amalgamation diagram with unique solution K.

There remains the case in which $|I| = |J| = 2$ and $I \to J$. If this does not embed in Γ, then any two vertices a_1, a_2 in one $\perp$-class have at most one common neighbor in the other.

Take two $\perp$-classes C_1, C_2, $a_1, a_2, a_3 \in C_1$ distinct, and let $J = a_1' \cap C_2$. By Lemma 5.2, J is infinite. Then a_2', a_3' meet J in at most 1 vertex each. Take a pair $b_1, b_2 \in J \setminus (a_2' \cup a_3')$. Then $a_1, a_2 \in \{b_1, b_2\}'$, a contradiction.

This disposes of all cases in which $|I| = |J| = 2$.

Now fix $\perp$-classes C_1, C_2, take $a \in C_1$, and set $A = a^{\perp} = C_1 \setminus \{a\}$, $B = a' \cap C_2$. The structure (A, B), in which A and B are named and the relation $\to$ between A and B is given, is a homogeneous structure. It may be viewed as a bipartite graph with the two sides distinguished, taking the relation $\to$ from A to B as the edge relation, and $\leftarrow$ as the non-edge relation. Then up to bipartite complementation—or in therms of the digraph, up to orientation—(A, B) is either complete, a perfect matching, or generic [5].

By what we have already proved in the case $|I = |J| = 2$, (A, B) is neither complete nor a perfect matching. Hence (A, B) is generic, and the claim follows.

Claim 3. Every configuration $K = xLI$ of the following form embeds into Γ.

- $L \cong L_n$ with first element a.
- $x \perp a$, $x \to LI \setminus \{a\}$.
- I is a $\perp$-class not meeting the $\perp$-classes of L.

If $|L| = 1$ this is covered by Claim 2. In general we proceed by induction on $n = |L|$.

We make an amalgamation of the form xLI_1I_2 where for a, b the first two elements of L, the type of (a, b) remains to be determined, and where I_1, I_2 are copies of I chosen so that either xLI_1 or xLI_2 will be an isomorphic copy of xLI once the orientation of the arc (a, b) is chosen. Thus I_1 realizes the type of I over xL, and I_2 realize the type resulting when the parameters a, b are switched.

The parameter x ensures that there is an arc between a and b. Since the completed amalgam must contain the desired configuration, it suffices now to check that the factors of this amalgamation obtained by omitting a or b embed into Γ.

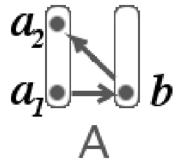
For the factor omitting b, this holds by induction hypothesis.

For the factor omitting a, we note that $x \to LI_1I_2 \setminus \{a\}$ and thus it suffices to find the factor $(L \setminus \{a\})I_1I_2$ in x'; this is given by the induction hypothesis applied within x'.

Claim 4. $I_n, L_n \in \mathcal{A}^*$ for all n.

Claim 2 covers I_n and Claim 3 covers L_n, dropping the parameter x.

Claim 5. The configuration $A = (a_1, a_2, b)$ with $a_1 \perp a_2$ and $a_1 \to b \to a_2$ belongs to $\mathcal{A}^*$.

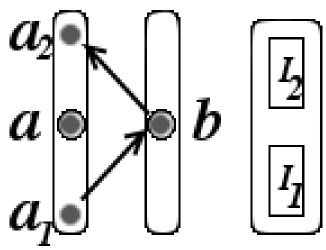


Consider $K = A \cup I$ with I an additional $\perp$-class. We must embed K in Γ.

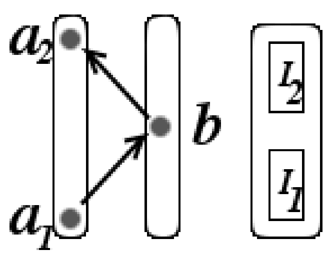
We form an amalgamation (a_1aa_2, b, xI_1I_2) with three $\perp$-classes a_1aa_2, b, and xI_1I_2, leaving the type (a, b) to be determined by the amalgam in such a way that an arc $b \to a$ makes a_1baI_1 isomorphic to K, while an arc $a \to b$ makes aba_2I_2 isomorphic to K. Furthermore, we take $a_1aa_2b \to x$.

Since any completion of the amalgamation diagram described must contain a copy of K, it must be shown that suitable factors embed into Γ.

Our specifications determine the type of a_1 over I_1, a_2 over I_2, and b over both I_1 and I_2, but leave open the type of a_1 over I_2 and a_2 over I_1. We will take advantage of the latitude this affords.



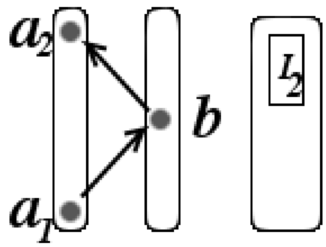
The factor omitting b embeds in Γ by Claim 2, regardless of how the unspecified types are filled in. So the task is to embed some form of the factor omitting a into Γ.



We may view this as an amalgamation diagram in which the type of a_2 over I_1 is to be determined. Our only constraint here is that a_2 and I_1 should lie in different $\perp$-classes, and this is satisfied in any amalgam. So it suffices to embed the factors omitting a_2 or I_1 into Γ.

The factor omitting a_2 is an extension of $a_1 b \cong L_2$ by $I_1 I_2$, and is available since $L_2 \in \mathcal{A}^*$.

So we come down to the factor omitting I_1, or rather any form of that factor in which I_2 occupies a third $\perp$-class.



Here the type of a_1 over I_2 may be anything, so we treat this configuration again as an amalgamation in which that type is to be determined; the parameter a_2 ensures $a_1 \not\perp I_2$.

So we again pass to the factors of the diagram displayed. As before, the factor omitting a_1 is available since $L_2 \in \mathcal{A}^*$, whatever its precise form. And the factor omitting I_2 is available since Γ contains O.

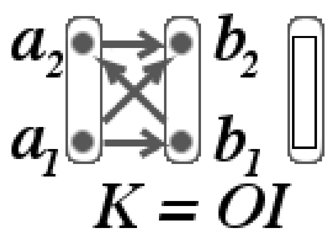
Claim 6. O belongs to $\mathcal{A}^*$.

We consider the homogeneous directed graph Γ^* corresponding to the amalgamation class $\mathcal{A}^* \subseteq \mathcal{A}$. Here $\perp$ is an equivalence relation with infinitely many infinite classes, and by the previous claim, Γ^* is not a wreath product. So if O does not embed in Γ^* we are left with one possibility.

- Γ^* is semi-generic.

Thus any finite configuration satisfying the parity constraint will embed into Γ^*.

We now consider a configuration $K = O \cup I$ with I an additional $\perp$-class.



As usual we may reduce to the case in which I is either indiscernible over O or of order 2.

If $I \cup \{a_1, a_2\}$ or $I \cup \{b_1, b_2\}$ respects the parity constraint then that configuration embeds in Γ^* and hence belongs to $\mathcal{A}^*$, which implies that the full configuration $K = OI$ belongs to $\mathcal{A}$, as required.

This holds in particular if I is indiscernible over O. So we come down to the following case.

- $|I| = 2$; and
- $I \cup \{a_1, a_2\}, I \cup \{b_1, b_2\} \cong O$.

In this case, if $I = \{c_1, c_2\}$, add a third point c to I and treat OIc as the unique amalgam of Oc_1c and Oc_2c. Here we need only choose c so that the factors omitting c_1, c_2 both embed in Γ, and for this it suffices to ensure that $\{a_1, a_2\} \cup \{c_1, c\}$ and $\{b_1, b_2\} \cup \{c_2, c\}$ both obey the parity constraint, which we may achieve by choosing the type of c appropriately.

The proof of the lemma is now complete. We dealt with I_n and L_n in Claim 4, and we just dealt with O.

□

As we have seen, Proposition 6.1 follows from Lemma 6.5, and this completes the proof of Theorem 1.

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