GRAPHS OMITTING SUMS OF COMPLETE GRAPHS WITH ADDITIONAL REMARKS; NOT CHECKED (2022)

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Abstract. For every finite m and n, there is a finite set of countable $(m \cdot K_n)$-free graphs $\{G_1, \ldots, G_\ell\}$ with the following property: every countable $(m \cdot K_n)$-free graph embeds as an induced subgraph in at least one of the graphs G_i.

We make estimates of the sizes of such sets in various special cases.

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1. Introduction

P. Komjáth and J. Pach [8] have defined the complexity of a class Ω of graphs as the minimum cardinality κ such that there is a subset Ω_0 of Ω of cardinality κ which is cofinal in Ω, in the sense that every graph G in Ω embeds as a subgraph in one of the graph in Ω_0. We will prove the following.

Theorem 1. *If the graph G is a disjoint sum of complete graphs then the class of countable graphs strongly omitting G has finite complexity.*

This means that one can find finitely many countable graphs $G_1, \ldots, G_k$ in which the specified graph G does not embed as a subgraph, so that any countable graph with the same property embeds into at least one of the graphs G_i as an *induced* subgraph.

All graphs considered in this paper will countable (finite or countably infinite), so we will drop explicit reference to the cardinality in the following. If this constraint is relaxed, a range of other issues arise of a more set theoretic character arise.

Many similar problems have been considered in the graph theoretic literature, beginning with Rado's observation (implicit also in work of Fraïssé) that there is a graph containing an isomorphic copy of every graph as an induced subgraph. Most results of this type concern classes Ω of graphs which are determined by a bound κ on the number of vertices (especially, $\kappa = \aleph_0$, as here) and a class $\mathcal{C}$ of *forbidden subgraphs.*

As a rule the class $\mathcal{C}$ is taken to consist of connected graphs. Then the corresponding class Ω is closed under the formation of countable disjoint sums, and hence the complexity must be either uncountable or 1.

In such cases, the main question is whether the class Ω contains a graph G which is *universal* for Ω. This may be understood in either of two senses: G is universal in the ordinary sense if it embeds every graph in Ω as a subgraph, and G is strongly universal if it embeds every graph in Ω as an induced subgraph.

Some examples of classes Ω of graphs which do contain a strongly universal member are the following: the class of graphs omitting a specific finite complete graph K_n [10]; the class of graphs omitting a path of some specific length [9]; the class of graphs omitting all cycles of odd length, up to some specified finite bound [10]; and the class of graphs omitting all cycles of size at least n, for some specified n [10]. Similar classes which do not contain a universal member, even weakly (and hence have uncountable complexity) are the following: the class of graphs which do not contain an infinite complete subgraph (the complexity of this class is precisely $\aleph_1$ [6, Theorem 7, taking $\kappa = \aleph_0$]); the class of planar graphs [11]; the class of locally finite graphs (N. G. de Bruijn, reported in [12]); graphs omitting a cycle of size n (for $n \geq 4$) [2, 11], graphs which do not contain an infinite path [4]. We have also shown [3] that the class of graphs omitting a fixed finite set of cycles does not have a universal member unless the set of forbidden cycles is of the type already considered in [9], consisting of all of the cycles of odd length below some fixed bound.

At present there is no algorithm known for determining whether a class of graphs determined by a finite set of forbidden finite connected subgraphs has a universal member. When Ω is a class of graphs determined by a collection $\mathcal{C}$ of forbidden subgraphs which are not all connected, there tends to be no universal graph on trivial grounds: a necessary condition for the existence of such a graph is the *joint embedding property:* any two such graphs must embed jointly in a third. (This is a weaker condition than closure under direct sums.)

Consider for example the class Ω of graphs which strongly omit the disjoint sum $K_1 + K_2$ (an isolated vertex plus one edge). These are the graphs with no edges, together with the graph K_2, and so the complexity of this class is 2.

The natural question in this setting, for sets of constraints not assumed to be connected, would be whether the complexity is finite, countably infinite, or uncountable. In general, asking for finite complexity generalizes the universality problem to classes which do not necessarily have joint embedding.

P. Komjáth and J. Pach have shown previously that for any finite n the class Ω of graphs which do not embed $K_n + K_2$ has finite complexity, and have also treated the problem for graphs of the form $m \cdot K_3$ [8].

Our main theorem extends this result to the case of finitely many forbidden subgraphs which are finite sums of finite complete graphs. Our main theorem will be proved somewhat more generally, for graphs with a vertex coloring (not assumed to be related in any way to the graph structure). Even if we are interested only in the case of graphs, we find this detour necessary, as it allows for a certain inductive argument in which the number of colors increases. For similar reasons, the upper bounds obtained on the complexity of such classes are extraordinarily large—at each step in the induction the number of colors involved can increase substantially. To date virtually nothing is known about the rate of growth of the complexity as a function of the sizes of the forbidden subgraphs.

The method used in this paper is model-theoretic, involving a systematic use of amalgamation combined with changes of language to keep track of quantifier-free types over distinguished subsets, which we interpret as vertex colorings. Hence this can all be expressed directly in the language of vertex-colored graphs, and we have written the argument using purely graph-theoretic terminology.

Our proof actually applies to finite sums of graphs whose connected components are taken from a class $\mathcal{C}$ of finite connected graphs with the following two properties.

(1) Any connected induced subgraph of a graph in $\mathcal{C}$ is also in $\mathcal{C}$.
(2) For all finite sets of constraints $\mathcal{C}_0$ of the form (C, c) with $C \in \mathcal{C}$ and c a vertex coloring of C, there is a universal $\mathcal{C}_0$-free vertex colored graph.

The family $\mathcal{C}$ of all complete graphs has these properties: property (2) holds since free amalgamation suffices to build a homogeneous universal graph by the Fraïssé theory, and that accounts for our main result. Whether our more abstract version is in fact more general result is unclear.

An obvious candidate to consider would be the family $\mathcal{P}$ of finite paths, as forbidding a path gives a very well-behaved universal graph. However, already the path P_3 on three vertices fails the given condition.

First, take 3 colors $0, 1, 2 \in \mathbb{Z}/3\mathbb{Z}$ and forbid the paths of length 3 enriched by the colorings $(i+1, i, i+1)$—in other words a vertex of color i may have at most one neighbor of color $i + 1$. Consider an infinite path $(v_n \mid n \in \mathbb{Z})$ with v_n given color $n \mod 3$. This may be enriched in uncountably many ways by additional edges and it is easy to see that one cannot embed all of these enrichments as induced subgraphs of a single graph satisfying our constraints. So condition (2) fails if we require universality in the strong sense.

As it happens there is still a weakly universal graph for this particular set of constraints: namely take the generic model for the theory of a function f taking a disjoint union $A_0 \cup A_1 \cup A_2$ to itself with elements of type i going to elements of type $i + 1$, and use this to determine the graph structure between pairs A_i, A_{i+1}, while taking the induced graph on each A_i to be complete.

But if we pass on to five colors and forbid the configuration consisting of a vertex of color 0 adjacent to vertices of both of the additional colors, this has the effect of putting an additional unary predicate on the vertices of type 0, and beginning with an infinite path as in the preceding case this gives uncountably many expansions which cannot even be weakly embedded into a common countable extension with the same constraints.

So our method of proof breaks down here but one may still wonder the following.

Problem 1. *If $\mathcal{C}$ is a finite collection of finite paths, does the collection of $\mathcal{C}$-free graphs have finite complexity with respect to strong embedding?*

This is perhaps a more interesting question given that the natural method does not apply.

2. Preliminaries

A (simple, loopless) graph is a structure $G = (V, E)$, where V is the set of vertices and E is a binary relation on V which is irreflexive and symmetric. All graphs in this paper are assumed to be countable.

An embedding f from a graph $G_1 = (V_1, E_1)$ into a graph $G_2 = (V_2, E_2)$ is an isomorphism with a subgraph: $f : V_1 \to V_2$, and for x, y in V_1, if $(x, y) \in E_1$ then $(f(x), f(y)) \in E_2$. We say f is a *strong* embedding if it is isomorphic with an induced subgraph (that is, the converse holds).

Our main concern is with forbidden subgraphs.

Definition 2.1. For graphs G, H, we say that G *weakly omits* H if H cannot be strongly embedded into G, and G *strongly omits* H if H cannot be embedded into G (as a subgraph).

For a finite set $\mathcal{C}$ of graphs, we say that G omits $\mathcal{C}$ (weakly or strongly) if G omits each graph in $\mathcal{C}$ (weakly or strongly, respectively).

For a forbidden complete subgraph H the distinction between weak and strong omission falls away and we simply say that G omits H.

Notation 2.2. We write

$$H \hookrightarrow G$$

if the graph H embeds into the graph G (as a subgraph).

Let k be a fixed positive integer. We say that a graph G is *k-colored* provided the vertices of G are colored by k different colors. Here the coloring c is a map from the vertices to the colors with no further conditions imposed. We may write (G, c) for such a structure (without necessarily specifying what set is being used to represent the colors).

When colors are present, weak or strong embeddings are required to preserve colors (and by our conventions, every vertex carries some color in that case).

Definition 2.3. The *complexity* of a class Ω of graphs, or similar structures, is the minimum cardinality λ such that there is a subclass Ω_0 of Ω with cardinality λ such that every graph in Ω can be embedded in one of the graphs in Ω_0 as a subgraph. In the above definition if embedded is replaced by strongly embedded and subgraph is replaced by induced subgraph then the cardinality λ is called the strong complexity of Ω. We will say correspondingly that the subclass Ω_0 of Ω is cofinal or strongly cofinal in the class Ω.

A *direct sum* of graphs is their disjoint union, with no additional edges.

In the proofs of Lemma 3.2 and Proposition 6.3 below we will need Fraïssé's theory of homogeneous structures and amalgamation classes [5]. A structure Γ is homogeneous in Fraïssé's sense if any isomorphism f between two finite substructures of Γ can be extended to an automorphism of Γ .

If Γ is homogeneous then the class $\mathcal{A}_\Gamma$ of all finite structures which embed isomorphically into Γ is an *amalgamation class.* This means that $\mathcal{A}_\Gamma$ is closed under isomorphism and substructure, and has the *amalgamation property*: namely, for $A_0, A_1, A_2 \in \mathcal{A}_{\neg\mathcal{C}}$, whenever there are strong embeddings f_0, g_0 of A_0 into A_1, A_2 respectively, then there is some $A \in \mathcal{A}_\mathcal{C}$ and there are embeddings f_1, g_1 of A_1, A_2 into A with $f_1 f_0 = g_1 g_0$. Conversely, Fraïssé showed that any amalgamation class $\mathcal{A}$ of finite structures (with countably many isomorphism types) is $\mathcal{A}_\Gamma$ for a unique (countable) Γ.

One way to create universal graphs with forbidden subgraphs (or other structures) is to specify an amalgamation class of finite graphs and to let Γ be the corresponding countable homogeneous structure. Then Γ is in particular universal for the class of structures whose finite substructures lie in $\mathcal{A}$. (See [1] for a more detailed summary, with proof sketches.)

3. The main result

Theorem 1. *If the graph G is a disjoint sum of complete graphs then the class of graphs strongly omitting G has finite complexity.*

In fact our induction argument will prove the following stronger statement, allowing for vertex colorings as well. This strengthening is needed to make the induction work even in the case of interest here.

Theorem 2. *Let k be fixed, and let $\mathcal{C}$ be a finite set of sums of complete graphs, each equipped with a vertex coloring by k colors.*

Then there is a finite set of k-colored graphs strongly omitting the graphs in $\mathcal{C}$, which is strongly cofinal in the class of all such vertex colored graphs.

For the proof, we first define a parameter we call the *rank* of the constraint set $\mathcal{C}$. Then we will prove Theorem 2 by induction on the rank, with respect to the lexicographic order on ranks.

Definition 3.1. Let C be a finite graph and $\mathcal{C}$ a finite collection of graphs.

The *width* of the graph C is the number of connected components of C .

The width of the set $\mathcal{C}$ is the maximum width of its members.

The *height* of the set $\mathcal{C}$ is defined by

$$\text{height}(\mathcal{C}) = \max(|C| \mid C \in \mathcal{C} \text{ and } \text{width}(C) = \text{width}(\mathcal{C}).)$$

We define the rank of the set $\mathcal{C}$ as the ordered pair $(\text{width}(\mathcal{C}), \text{height}(\mathcal{C}))$. Ranks are compared lexicographically.

The case in which $\text{width}(\mathcal{C}) = 1$ ($\mathcal{C}$ is a collection of vertex colored cliques) is treated by a special argument in the following, by Fraïssé's method, as mentioned above.

Lemma 3.2. *Let $\mathcal{C}$ be an arbitrary collection of k-colored finite complete graphs and let $\mathcal{A}_{\neg\mathcal{C}}$ be the class of graphs omitting $\mathcal{C}$. Then the class $\mathcal{A}_{\neg\mathcal{C}}$ contains a strongly universal graph.*

Proof. We apply Fraïssé's theory of homogeneous structures and amalgamation classes [5]. The class $\mathcal{A}_{\neg\mathcal{C}}$ is an amalgamation class in Fraïssé's sense, using free amalgamation of graphs, and the corresponding homogeneous structure is strongly universal for $\mathcal{A}_{\neg\mathcal{C}}$. □

. Now we turn to the induction step. Suppose $\text{rank}(\mathcal{C}) = (w, h)$, where $w > 1$. Let

$$\mathcal{C}_0 = \{C \in \mathcal{C} \mid \text{width}(C) = w \text{ and } |C| = h\}.$$

Then $\text{rank}(\mathcal{C}) = \text{rank}(\mathcal{C}_0)$.

For $C \in \mathcal{C}_0$, choose X_C, a connected component of C . Let

$$\xi = \sum_{C \in \mathcal{C}_0} |X_C|.$$

Let V_0 be a set of ξ vertices.

For any subset $\mathcal{X} \subseteq \mathcal{C}_0$, let

$$\mathcal{H}_{\mathcal{X}} = \{H \mid V(H) \subseteq V_0,\ X_C \hookrightarrow H \text{ for all } C \in \mathcal{X}\}.$$

Clearly $\mathcal{H}_\mathcal{X}$ is finite, and its size only depends on $\mathcal{X}$ and on the choice of X_C for each $C \in \mathcal{X}$.

Let $\mathcal{L}$ be the class of k-colored graphs. For $H \in \mathcal{H}_X$, let $\mathcal{L}_H$ be the class of triples (G, c_0, f), where (G, c_0) is a k-colored graph, c_0 is the coloring function from $V(G)$ into $\{1, 2, \ldots, k\}$, and $f : H \to G$ is a strong embedding of H into G.

Let $\mathcal{L}'_H$ be the class of $(k \cdot 2^{|H|})$-colored graphs. There is a natural correspondence between $\mathcal{L}_H$ and $\mathcal{L}'_H$ as follows. Identify the set of $k \cdot 2^{|H|}$ colors in $\mathcal{L}'_H$ with the Cartesian product $\{1, 2, \ldots, k\} \times \mathcal{P}(H)$. For $(G, c_0, f) \in \mathcal{L}_H$, define a coloring function c on $V(G \backslash f[H])$ by

$$c(v) = (c_0(v), \{u \in H \,|\, (f(u), v) \in E(G)\}).$$

Associate $(G, c_0, f) \in \mathcal{L}_H$ to the induced subgraph $G \backslash f[H]$ equipped with the coloring c.

Conversely, given $(G, c) \in \mathcal{L}'_H$ with $c = (c_0, c_1)$, there is a unique structure $(G \cup H, c_0) \in \mathcal{L}_H$ with the following properties.

(1) G, H are disjoint induced subgraphs of $G \cup H$;
(2) c_0 is consistent with c_0 on G and with the coloring of H on H; and
(3) the edges in $G \times H$ are

$$\{(u, v) \in G \times H \,|\, v \in c_1(u)\}.$$

The point of the above construction is that our induction hypothesis can be applied to $\mathcal{L}'_H$.

Notation 3.3. For $(G, c) \in \mathcal{L}'_H$ we write $G \cup_{\mathcal{L}'_H} H$ for the corresponding expanded structure in $\mathcal{L}_H$.

We may also consider $G \cup_{\mathcal{L}'_H} H$ just as a k-colored graph.

The induction step is largely contained in the following:

Lemma 3.4. *For any finite k-colored graph H, there is a finite set $\mathcal{C}_H$ of $\mathcal{L}'_H$-structures such that the following hold.*

(1) $\mathcal{C}_H$ consists of $(k \times 2^{|H|})$-colored sums of finite complete graphs,
(2) $\mathrm{rank}(\mathcal{C}_H) < \mathrm{rank}(\mathcal{C})$*,*
(3) for $G \in \mathcal{L}'_H$, G strongly omits $\mathcal{C}_H$ if and only if $G \cup_{\mathcal{L}'_H} H$ strongly omits the union of $\mathcal{C}$ with

$$\{X_C \,|\, C \in \mathcal{C},\ X_C \text{ does not embed in } H\}$$

(as a k-colored graph).

The natural choice of $\mathcal{C}_H$ corresponding to condition (3) will satisfy conditions (1) and (2). We will check this in detail later. First, however, we complete the proof of Theorem 2 using the lemma.

Proof of Theorem 2. We proceed by induction on rank. We may assume the width w is greater than 1 and apply the construction described above.

Since $\mathrm{rank}(\mathcal{C}_H) < \mathrm{rank}(\mathcal{C})$, by the induction hypothesis there is a finite family of $(k \times 2^{|H|})$-colored graphs strongly omitting $\mathcal{C}_H$, which is strongly cofinal in the class of all such graphs. Let $\mathcal{G}_H$ be such a family.

Now define

$$\mathcal{G}_0 = \bigcup_{X \subseteq \mathcal{C}_0} \{G \cup_{\mathcal{L}'_H} H \mid H \in \mathcal{H}_{\mathcal{X}}, G \in \mathcal{G}_H\}.$$

We claim that $\mathcal{G}_0$ is strongly cofinal in the class of k-colored graphs strongly omitting $\mathcal{C}$. In fact, for any $X \subseteq \mathcal{C}_0$, any $H \in \mathcal{H}_X$, if $G \in \mathcal{G}_H$, then G strongly omits $\mathcal{C}_H$. Thus by (3), $G \cup_{\mathcal{L}'_H} H$ strongly omits $\mathcal{C}$. Hence any graph in $\mathcal{G}_0$ strongly omits $\mathcal{C}$.

Now we show that $\mathcal{G}_0$ is strongly cofinal in the class of graphs which strongly omit $\mathcal{C}$. Suppose Γ strongly omits $\mathcal{C}$. Let

$$\mathcal{X}_0 = \{C \in \mathcal{C}_0 \mid X_C \text{ embeds into } \Gamma\}$$

For $C \in \mathcal{X}_0$ choose an embedding $i_C : X_C \hookrightarrow \Gamma$ and let

$$H_0 = \bigcup \{i[X_C] \mid C \in \mathcal{X}_0\}$$

as an induced subgraph of Γ.

H_0 is isomorphic to a graph H in $\mathcal{H}_{\mathcal{X}_0}$. Let $f : H \to H_0$ be the isomorphism. Then $(\Gamma, f) \in \mathcal{L}_H$.

Let G be the associated structure in $\mathcal{L}'_H$ with the vertex set $V[\Gamma \backslash H_0]$. Thus $\Gamma \simeq G \cup_{\mathcal{L}'_H} H$ strongly omits $\mathcal{C}$ and also omits all X_C which do not embed in H. Hence by (3) of the lemma, G strongly omits $\mathcal{C}_H$. So G strongly embeds in some $G^* \in G_H$. Thus Γ strongly embeds in $G^* \cup_{\mathcal{L}'_H} H$. □

4. The class $\mathcal{C}_H$

Now let us prove Lemma 3.4 of Section 3.

Proof of Lemma 3.4. We fix the finite graph H. Let $\mathcal{C}_H$ be the union of the following three classes of $(k \times 2^{|H|})$-colored sums of finite complete graphs D.

$$\begin{aligned}\mathcal{C}_1 = \{D \,|\, &\text{For some } C \in \mathcal{C}\backslash\mathcal{C}_0,\\ &C \to D \cup_{\mathcal{L}'_H} H \text{ and the image of } C \text{ contains } D\}\\ \mathcal{C}_2 = \{D \,|\, &\text{For some } C \in \mathcal{C}_0 \text{ for which } X_C \hookrightarrow D \cup_{\mathcal{L}'_H} H,\\ &X_C \text{ does not embed in } H, \text{ and the image of } X_C \text{ contains } D\}\\ \mathcal{C}_3 = \{D \,|\, &\text{For some } C \in \mathcal{C}_0,\\ &C \hookrightarrow D \cup_{\mathcal{L}'_H} H \text{ and the image of } C \text{ contains } D \text{ and meets } H\}.\end{aligned}$$

Note that $\mathcal{C}_H$ depends on H and on X. We now show that $\mathcal{C}_H$ satisfies the conditions (1–3) of Lemma 3.4.

Ad 1: $\mathcal{C}_H$ consists of $(k \times 2^{|H|})$-colored sums of finite complete graphs.
This holds by definition.

Ad 2: $\text{rank}(\mathcal{C}_H) < \text{rank}(\mathcal{C})$.
For any $D \in \mathcal{C}_1$, there is some $C \in \mathcal{C}\backslash\mathcal{C}_0$ such that D embeds in C . But for $C \in \mathcal{C}\backslash\mathcal{C}_0$ we have $\text{rank}(C) < \text{rank}(\mathcal{C})$ and so $\text{rank}(\mathcal{C}_1) < \text{rank}(\mathcal{C})$.

If $D \in \mathcal{C}_2$ then there is some $C \in \mathcal{C}_0$ such that D embeds in X_C, so $\text{width}(D) = 1 < \text{width}(\mathcal{C})$. Hence $\text{rank}(\mathcal{C}_2) < \text{rank}(\mathcal{C})$.

If $D \in \mathcal{C}_3$, then D embeds properly into some $C \in \mathcal{C}_0$. So $\text{width}(D) \leq \text{width}(\mathcal{C})$ and $|D| < |C| = h$. Hence $\text{rank}(\mathcal{C}_3) < (w, h) = \text{rank}(\mathcal{C})$.

Ad 3: For $G \in \mathcal{L}'_H$, G strongly omits $\mathcal{C}_H$ if and only if $G \cup_{\mathcal{L}'_H} H$ strongly omits

$$\mathcal{C} \cup \{X_C \,|\, X_C \text{ does not embed in } H\}.$$

Suppose first that $G \in \mathcal{L}'_H$ and G strongly omits $\mathcal{C}_H$.

We show that $G \cup_{\mathcal{L}'_H} H$ omits X_C whenever X_C does not embed in H. Assume toward a contradiction that there is an embedding $i : X_C \to G \cup_{\mathcal{L}'_H} H$ so that $i[X_C] \cap G \neq \emptyset$. Let $D = i[X_C] \cap G$. Then $D \in \mathcal{C}_2$, which contradicts the assumption that G strongly omits $\mathcal{C}_2$.

Now we show that $G \cup_{\mathcal{L}'_H} H$ strongly omits C for all $C \in \mathcal{C}$. Assume toward a contradiction that $i : C \to G \cup_{\mathcal{L}'_H} H$ is an embedding for some $C \in \mathcal{C}$. If $C \in \mathcal{C}\backslash\mathcal{C}_0$, let $D = i[C]\backslash G$. Then $D \in \mathcal{C}_1$, which contradicts that G strongly omits $\mathcal{C}_1$.

If $C \in \mathcal{C}_0$, then there are three cases.

1. If X_C does not embed in H, then $G \cup_{\mathcal{L}'_H} H$ omits X_C, as we have shown. So $G \cup_{\mathcal{L}'_H} H$ strongly omits C .

2. If $i[C] \cap H \neq \emptyset$, let $D = i[C] \cap G$. Then $D \in \mathcal{C}_3$, which contradicts that G strongly omits $\mathcal{C}_3$.

3. If X_C embeds in H but $i[C] \cap H = \emptyset$, then fix an embedding $j : X_C \to H$ and define $i^* : \mathcal{C} \to G \cup_{\mathcal{L}'_H} H$ by

$$i^*(x) = \begin{cases} i(x) & \text{if } x \in C \setminus X_C; \\ j(x) & \text{if } x \in X_C. \end{cases}$$

Let $D = i^*[C] \cap G = i[C \setminus X_C]$. Then $D \in \mathcal{C}_3$. This contradicts that G strongly omits $\mathcal{C}_3$.

Conversely, suppose $G \in \mathcal{L}'_H$ and $G \cup_{\mathcal{L}'_H} H$ (as a k-colored graph) strongly omits

$$\mathcal{C} \cup \{X_C \mid X_C \text{ does not embed in } H\}.$$

We prove now that G strongly omits $\mathcal{C}_H = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$.

If G does not strongly omit $\mathcal{C}_1$ or $\mathcal{C}_3$, then there are $D \in \mathcal{C}_1 \cup \mathcal{C}_3$ and an embedding $j_0 : D \to G$. Since $D \in \mathcal{C}_1 \cup \mathcal{C}_3$, there are $C \in \mathcal{C}$ and an embedding $i : C \to D \cup_{\mathcal{L}'_H} H$ such that $D \subseteq i[C]$.

With id_H the identity map on H, consider the map

$$j = j_0 \cup \mathrm{id}_H : D \cup_{\mathcal{L}'_H} H \to G \cup_{\mathcal{L}'_H} H$$

Hence $j \circ i : C \to G \cup_{\mathcal{L}'_H} H$ is an embedding, which contradicts that G strongly omits $\mathcal{C}$.

If G does not strongly omit $\mathcal{C}_2$, then there are $D \in \mathcal{C}_2$ and an embedding $j_0 : D \to G$. Since $D \in \mathcal{C}_2$, there are $C \in \mathcal{C}_0$ for which X_C does not embed in H and an embedding $i : X_C \to D \cup_{\mathcal{L}'_H} H$ such that $D \subseteq i[X_C]$. With id_H the identity map on H, let

$$j = j_0 \cup \mathrm{id}_H : D \cup_{\mathcal{L}'_H} H \to G \cup_{\mathcal{L}'_H} H.$$

Then $j \subseteq i : X_C \to G \cup_{\mathcal{L}'_H} H$ is an embedding. This contradicts that G omits X_C.

This completes the proof of clause (3). □

5. Complexity estimates

The method of the present paper gives an effective upper bound on the complexities of the classes of graphs considered here, but one which grows very rapidly (an exponential tower). One suspects the complexity does grow quite rapidly, but we do not have substantial lower bounds.

Here we always take complexity in the strong sense, that is we seek a finite set of representatives for the $\mathcal{C}$-free graphs up to strong embeddings (induced subgraphs, possibly vertex colored).

This is also a good point to recall that all our graphs are countable. As at point we work more directly with infinite graphs here this needs to be kept in mind.

5.1. **Qualitative upper bounds.** We will be interested in the following parameters associated with the family of forbidden structures $\mathcal{C}$ (which as usual we will take to be disjoint sums of complete graphs, with vertex colorings).

(1) The number of available vertex colors k (take $k = 1$ in the uncolored case).
(2) The number m of constraints which are disconnected.
(3) The maximum size N of a disconnected constraint.
(4) The maximum size s of a connected component in a disconnected constraint.

We take N and s to be zero when $m = 0$.

It will not be very damaging to use N as a bound for s, as far as purely qualitative estimates are concerned, as the result is a tower of exponentials of height roughly N. But in terms of these parameters our concern is with the following.

Notation 5.1. Let $f(k, m, N, s)$ denote the smallest bound c such that for any constraint set $\mathcal{C}$ consisting of sums of complete graphs with a vertex coloring, if its corresponding parameters are bounded by k, m, N, s respectively, then the complexity of the associated class of graphs is at most c.

In particular, $f(k, 0, N, s) = 1$.

Of course, one should check that this is even well-defined. We give that via an explicit estimate.

Proposition 5.2. *For $m > 0$ we have*

$$(\star) \qquad f(k, m, N, s) \leq 2^m 2^{\binom{ms}{2}} (k+1)^{ms} f(k', m', N-1, s)$$

with

$$k' = k \cdot 2^{ms}$$
$$m' = m \cdot (2^{ms} + 1)^N$$

We remark that if we apply this estimate recursively, when one gets down to $N = 0$ one may set $m = 0$ and halt—though we are forced to deal with increasingly large estimates for m along the way.

Proof. We follow the proof of finiteness with the modification that the subset $\mathcal{C}_0$ is replaced by the set $\mathcal{C}_0^*$ of all $C \in \mathcal{C}$ which are disconnected.[1]

[1]Or more precisely, those of maximal size which are disconnected, but for our purposes this seems like a marginal improvement.

Recall the construction: one selects a connected component X_C for each $C \in \mathcal{C}_0^*$, one sets

$$\xi = \sum_{C \in \mathcal{C}_0^*} |X_C|$$

and for each subset $\mathcal{X} \subseteq \mathcal{C}_0^*$ one considers the graphs H on a fixed set of ξ vertices which contain copies of all X_C for $C \in \mathcal{X}$. One then associates to each such choice of $\mathcal{X}$ and H a constraint set $\mathcal{C}_H = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ in an enriched language with $k \cdot 2^\xi$ colors in place of k.

Now $|\mathcal{C}_1| = m$ and $\xi \leq ms$. Thus we have 2^m subsets $\mathcal{X}$ to consider. Our estimate for the number of colored graphs H is

$$2^{\binom{ms}{2}}(k+1)^{ms}$$

with the first factor counting graphs on the set of ms vertices and the second factor counting the number of pairs (V, c) with V a subset of the fixed set of ms vertices, and c a coloring of the set V by k colors. This slightly overcounts the desired pairs (H, c).

It remains to show that the auxiliary set of constraints $\mathcal{C}_H$ which arises has its parameters bounded by $k', m', N-1, s$ respectively.

In view of the choice of $\mathcal{C}_0^*$, in the definition of $\mathcal{C}_H$ as $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, only the constraints in $\mathcal{C}_2$ can be disconnected. This accounts for the replacement of N by $N-1$. The value of k' comes directly from the construction and we may certainly retain the bound s. So it will suffice to check that the specified value for m' bounds the size of the constraint set $\mathcal{C}_2$.

One examines the definition again.

$$\begin{aligned}\mathcal{C}_2 = \{D \mid & \text{For some } C \in \mathcal{C}_0^* \text{ for which } X_C \hookrightarrow D \cup_{\mathcal{L}'_H} H, \\ & X_C \text{ does not embed in } H\text{, and the image of } X_C \text{ contains } D\}\end{aligned}$$

There are m choices for C. With C fixed, we need to pick out a proper subgraph and equip it with the extended coloring induced by H. Note that C already has a k-coloring, which will be retained. So this amounts to coloring C by $2^{ms}+1$ colors, with one additional color to indicate discarded vertices. All together this gives

$$m' \leq m \cdot (2^{ms}+1)^N \qquad \square$$

It seems plausible that this crude upper bound is not very far from the truth.

Notation 5.3. Let $\gamma(m, n)$ be the complexity of the class of graphs omitting $m \cdot K_n$.

We do not get a very clear upper bound for this other than as an iterated exponential of height mn, which seems excessive, though possible not far off.

Example 1. $\gamma(m, 1)$ is the number of graphs of order $m-1$, up to isomorphism.

One can analyze $\gamma(m, 2)$ in a somewhat similar fashion but in this case it is useful to work more model theoretically.

Definition 5.4. A $\mathcal{C}$-free graph Γ is *existentially closed* if any existential condition (with parameters) satisfied in a $\mathcal{C}$-free extension holds in Γ. In other words, given a finite induced subgraph A of Γ and a $\mathcal{C}$-free extension Γ' of Γ, any finite induced subgraph A' of Γ' which contains A can be embedded over A as an induced subgraph of Γ.

Since the existentially closed $\mathcal{C}$-free graphs are strongly cofinal in the $\mathcal{C}$-free graphs we can restrict our attention to such graphs.

Remark 5.5. *If $\mathcal{A}'$ is strongly cofinal in the class of structures $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{A}'$ have the same complexity.*

The existentially closed $\mathcal{C}$-free graphs tend to be quite well-behaved. The nicest case is the following.

Definition 5.6. A $\mathcal{C}$-free existentially closed graph is *maximal* if any $\mathcal{C}$-free existentially closed extension of it is isomorphic to it.

We note that this is one of the more extreme examples of our convention that all graphs are to be countable.

Remark 5.7. *The complexity of the class of $\mathcal{C}$-free graphs is at least the number of maximal existentially closed $\mathcal{C}$-free graphs.*

In the case of $\gamma(m, 1)$, the existentially closed graphs are the graphs of order $m-1$, and they are all maximal.

Now let us show something similar in the case of $\gamma(m, 2)$.

Lemma 5.8. *Let Γ be an existentially closed $(m \cdot K_2)$-free graph and let Γ_0 be the subgraph of Γ induced on the vertices of finite degree in Γ.*

Then the following hold.

(1) Γ has at most $m-1$ vertices of infinite degree.
(2) If there are k vertices of infinite degree, then Γ_1 omits the graph $(m-k) \cdot K_n$, and the vertex degrees in Γ_1 are at most. $2(m-k-1)$.
(3) There are finitely many non-isolated vertices in Γ_1, in fact fewer than $8(m-k-1)^2$ vertices.

Proof. Ad 1

This is clear and does not require existential completeness.

Ad 2 It is clear that the graph induced on the vertices of finite degree omits $(m-k)\cdot K_n$ as otherwise we choose an additional k edges involving points of infinite degree and embed $m \cdot K_n$.

Suppose now that v is a vertex of finite degree. We adjoin a new vertex v' adjacent to v and to no other vertices. If the extended graph is $(m \cdot K_n)$-free we contradict existential completeness, as v' does not embed over v and its neighbors into Γ. So $\Gamma \cup \{v'\}$ contains a copy of $m \cdot K_n$, or in other words $\Gamma \setminus \{v\}$ contains a copy A of $(m-1)\cdot K_n$.

We may suppose that each vertex of infinite degree in Γ lies on one of the edges of A and that those edges do not contain neighbors of v, by replacing some edges of A if necessary to achieve this. So we are left with a copy A' of $(m-k-1)\cdot K_n$ which does not contain vertices of infinite degree. If one of the edges at v in Γ is disjoint from A' then this creates a copy of $m \cdot K_n$ in Γ for a contradiction. So the neighbors of v lie in A' and thus the degree of v is at most $2(m-k-1)$.

Ad 3 If we have an graph of bounded degree d with more than $(m-1)(2d-1)$ edges then we may find m disjoint edges by successively removing one edge and all the adjacent edges, in groups of at most $1+2(d-1)$ edges.

Apply this to the graph induced on the vertices of finite degree. This graph contains at most $(m-k-1)(4(m-k-1)-1)$ edges. We may double this to estimate the number of vertices, and take a slightly coarser estimate for simplicity. □

Lemma 5.9. *Let Γ be an existentially closed $(m\cdot K_2)$-free graph. Then Γ is maximal.*

Thus the complexity of the family of $(m\cdot K_2)$-free graphs is the number of existentially closed $(m \cdot K_2)$-free graphs, up to isomorphism.

Proof. The second point will follow immediately by our earlier remarks once the first one is proved.

So consider an existentially closed $(m \cdot K_2)$-free graph Γ with k vertices of finite degree and let A be the graph induced on the non-isolated points of Γ_1 together with the k vertices of infinite degree.

If $v \notin A$ then v has finite degree and is an isolated vertex of Γ_1, so its neighbors all have infinite degree.

Claim 1. Let Γ' be a $(m \cdot K_2)$-free extension of Γ. Then any edge e of Γ' which is not an edge of Γ contains a vertex in Γ of infinite degree.

Let B be a subgraph of Γ of the form $m' \cdot K_2$ with m' maximal. If the edge e is disjoint from B then existential completeness yields an edge in Γ disjoint from B and a contradiction.

So e contains a vertex v of Γ. The other vertex v' of e lies outside Γ by hypothesis. If v has finite degree in Γ then we get a similar contradiction by considering v together with its neighbors, and v'. The claim follows.

Accordingly we consider the collection $\mathcal{A}_A$ of all finite graphs B containing A with the property that all edges of B not in A join vertices of $B \setminus A$ to one of the k distinguished vertices "of infinite degree." This odd terminology really means that we introduce constants denoting the vertices of A and among these we have names for the k vertices which had infinite degree in Γ.

Then $\mathcal{A}_A$ is an amalgamation class, with free amalgamation, so has a homogeneous Fraïssé limit Γ^*.

Claim 2. Γ^* is $(m\dot{K}_2)$-free.

This means that none of the structures in $\mathcal{A}_A$ contains a copy of $m \cdot K_2$. This is immediate since any edges involved in such a copy not lying in A would contain one of the k distinguished vertices of A and the copy could be replaced by a set of edges lying in Γ.

By our first claim the finite substructures of Γ lie in $\mathcal{A}_A$ and by existential completeness, amalgamation, and our second claim, the converse holds, and the isomorphism follows.

In particular any existentially closed expansion of Γ will have the same properties and be isomorphic to Γ^*. □

We remark in passing that in this particular case the maximality applies in a stronger sense, namely our graph Γ is isomorphic with any expansion which is $(m \cdot K_2)$-free, but we did not prove that. That would depend on a more explicit examination of the structure of Γ, which is highly degenerate, as we have already seen. Typically one does not expect this. In particular, existentially closed graphs tend to be connected, while our Γ has so few edges that it even has isolated points.

What one sees from the above is that the computation of $\gamma(m, 2)$ comes down to the determination of the number of structures which can occur as the distinguished finite subgraph A in the foregoing lemma, and this is largely a matter of determining the maximum size of A. We expect the complexity to be roughly the number of graphs of that size. At the moment our coarse estimate for that number is $8(m-1)^2$ (in other words, quadratic as a function of the size $2m$) and so the complexity would be bounded by something on the order of $2^{\binom{8(m-1)^2}{2}}$.

We can be more explicit about the actual meaning of the bound we need.

Definition 5.10. Let $\sigma(m,2)$ be the maximum size of a finite graph A with the following properties.

(1) A is $(m \cdot K_2)$-free.
(2) A contains no isolated vertices.
(3) For any vertex v of A, there is an embedding of $(m-1) \cdot K_2$ into $A \backslash \{v\}$.

The last condition expresses that A is maximal $(m \cdot K_2)$-free in the sense that the only way to expand A to a larger $(m \cdot K_n)$-free is by adding isolated vertices.

This suggests defining $\sigma(m,n)$ as well as the maximal size of a $(m \cdot K_n)$-free graph without isolated points such that any embedding into an $(m \cdot K_n)$-free graph adds no additional cliques of order n, but these notes are getting long enough without taking that up.

Lemma 5.11. $\sigma(m,2) = 3(m,-1)$

Proof. Letting $A = (m-1) \cdot K_3$ gives the lower bound $(m-1)$. We prove the upper bound.

Suppose A satisfies the relevant conditions. Then we may fix a subgraph (not necessarily induced) A_0 of A isomorphic to $(m-1) \cdot K_2$. So A_0 has $m-1$ connected components, each a pair of points joined by an edge.

Let $B = A \setminus A_0$. There are no edges in B. Therefore every vertex of B has at least one neighbor in A_0.

Claim 1. No component E of A_0 has more than one neighbor in B.

Suppose on the contrary that there are at least two vertices of B adjacent to some vertex of E. If each vertex of E has at least one neighbor in B it follows that there are two disjoint edges in $E \cup B$ and so after replacing E by two such edges we embed $m \cdot K_2$ in A, for a contradiction.

So we have a vertex $a \in E$ with at least two neighbors in B, and the other vertex of E has no neighbors in B. We now consider a subgraph A_1 of $A \setminus \{a\}$ isomorphic to $(m-1) \cdot K_2$ and the subgraph A^* of A on the vertices of $A_0 \cup A_1$ with the edges of A_0, of A_1, and all edges of the form (a,v) with $v \in B$.

Call the edges in A_0 or in $B \cup \{a\}$ type 0, and the edges in A_1 type 1. There may be some overlap (edges in both A_0 and A_1).

Let T be the connected component of a in A^*. On this component the types alternate except at the vertex a, where only type 0 occurs.

(Where there is overlap between the types, we have degenerate connected components of order 0.)

We show next that T is a tree. Suppose on the contrary that C is a minimal cycle in T. The cycle must contain the vertex a: otherwise, the cycle consists of edges alternately in A_0 and A_1 and in particular all vertices lie in A_0. However at least one such vertex v has another neighbor in T and the relevant edge can only have the form (a, b) with $b \in B$, so this forces $v = a$ and a contradiction.

So the cycle C has odd length $2k+1$ and contains the vertex a. There is a neighbor v of a not in the cycle C and the edge (a, v) together with the k edges of type 1 in C give $2k + 1$ edges which are disjoint from the edges of A_0 which are not in C. But at most k of the edges of A_0 lie in C and so we embed $m \cdot K_2$ into A, for a contradiction.

Thus T is a tree, and all vertices other than a have degree at most 2, so in fact it is a union of paths over a. Take a path P in T from a to a leaf, whose first edge joins a to a neighbor in B. Then among the edges on this path, at least as many are in A_1 as are in A_0. We also have another edge from a to a point of B. So we may replace the edges in A_0 on the path by the edges in A_1, together with the additional edge at a, and increase their number, taking the remaining in edges in A_0 to give an embedding of $m \cdot K_n$, for a contradiction. □

The upshot is that we expect the complexity of the class of graphs forbidding $m \cdot K_n$ to be of the order of magnitude of the number of graphs on $3(m-1)$ points. As a lower bound we can take the number of graphs on $3(m-1)$ points containing a copy of $(m-1) \cdot K_3$, and as there are few edges in $(m-1) \cdot K_3$ this is or the same order of magnitude as the number of graphs on $3(m-1)$ points. As an upper bound we multiply by m to allow for $k \leq m-1$ points of infinite degree and sum up the values over all of the fixed graphs A up to the maximum size $3(m-1)$, which amounts to a second factor of $3(m-1)$. Thus the relevant bounds are exponential in $[3(m-1)]^2$ and the most delicate point is $\sigma(m.2)$ connects data of order $2m$ with data of order $3(m-1)$, and one would like to know if $\sigma(m, n)$ is bounded by a linear function of mn, or at least a polynomial (along with many other things of a more theoretical and less combinatorial nature).

6. The low end: $(K_m + K_n)$-free graphs

A problem of a more elementary character, where one may expect more precise estimates, concerns the case of a single constraint with just two summand. $\mathcal{C} = \{K_m + K_n\}$.

6.1. $c(m, n)$.

Notation 6.1. Let $c(m, n)$ denote the complexity of the class of countable graphs omitting $K_m + K_n$.

We will generally take $m \geq n$ here. Note that

$$c(1, 1) = 1$$
$$c(m, 1) = 2$$

with the relevant graphs in the second case being the generic graph omitting K_m and the graph K_m itself.

We give some small values explicitly.

Proposition 6.2. *$c(m, n)$ has the following values for $n \leq 2$ and $m \geq n$.*

n	m	$c(m, n)$	m	$c(m.n)$
1	1	1	> 1	2
2	2	2	> 2	4

Proof. It suffices to describe the relevant existentially closed graphs, and to check that they are maximal and cofinal in the corresponding class.

For graphs omitting $K_m + K_1$ we take K_m and the generic graph omitting K_m, the latter dropped if $m = 1$. Everything is clear in this case.

For graphs omitting $K_2 + K_2$ we take a triangle and a star of infinite degree, and add infinitely many isolated vertices. These are clearly maximal existentially closed, and conversely any triangle free graph in the class has at most non-trivial connected component, which is a star (or a single edge), and existential completeness forces the center to have infinitely many neighbors and non-neighbors.

For graphs omitting $K_m + K_2$ with $m > 2$ we take the following four graphs.

(1) Generic omitting K_m.
(2) The generic graph of the form $K_m \cup B$ where B is an independent set of vertices with no vertex in A in an m-clique.
(3) The generic graph of the form $K_{m-1} \cup B$ with B an independent set of vertices.
(4) K_{m+1} together with infinitely many isolated vertices.

Graphs (2,3) are easy enough to describe explicitly but constructing them via Fraïss'e theory makes the application of general theory more straightforward.

We need to see that the graphs (2, 3) are well-defined, maximal existentially closed, and cofinal in the $(K_m + K_2)$-free graphs containing K_m but not K_{m+1}.

In each case we have a clique A of order m or $m-1$ which is treated as a set of constants and we consider the class $\mathcal{A}_A$ of finite extensions of A with the stated conditions. These form amalgamation classes under free join so the corresponding generic graphs exist. From the description of $\mathcal{A}_A$, the resulting graphs are $(K_m + K_2)$-free.

To prove maximality and existential completeness, it suffices to show that any expansion of one of these graphs satisfies the same conditions (has the same finite substructures).

In case (1), because we are taking $m > 2$, there is an infinite set of pairwise disjoint edges, and hence no copy of K_m can be added.

In case (2), because we are taking $m > 2$, every vertex of $A = K_m$ has infinite degree and hence there can be no other m-clique in a (K_m+K_2)-free extension.

In case (3) there are infinitely many expansions of the given clique K_{m-1} to copies of K_m so in any (K_m+K_2)-free extension the remaining vertices form an independent set.

Case (4) is clear.

There proves everything except cofinality of the class, for the case in which Γ is an existentially closed graph containing a clique K of order m but none of order $m+1$. If that clique is unique then the graph embeds into the graph (2). So suppose there is a second clique K' of order m. Then $A = K \cap K'$ must have order $m-1$, and thus $K \cup K'$ is a copy of K_{m+1} with one edge removed. It follows that there is no edge disjoint from A, as otherwise it will be disjoint from at least one of K, K'. Thus there is an embedding into the generic graph of type (3).

This completes the verification for the case $m > 2$, $n = 2$. Note also that two of these graphs fall away when $m = 2$, giving the exceptional case again. □

We add a further value.

n	m	$c(m,n)$	m	$c(m.n)$
1	1	1	>1	2
2	2	2	>2	4
3	3	4		

Proposition 6.3.

$$c(3,3) = 4.$$

Proof. We first perform an analysis a priori of maximal existentially closed graphs in this class, then construct them.

A vertex v is said to be *special* if it lies in infinitely many triangles intersecting pairwise in the vertex v. Note that there is at most one special vertex. Similarly, an edge is *special* if it has infinitely many extensions to a triangle.

Claim 1. Any two cliques which are either special, or are triangles, meet.

This is immediate as otherwise we find a pair of disjoint triangles extending the given data.

Case 1. If G contains a clique K of order 5 then all triangles of G lie in this clique.

Case 2. Suppose G contains a clique K of order 4 but no clique of order 5.

Then any triangle in G has at least an edge in K. We claim that for any two disjoint edges in K, exactly one of them is special.

Let e, e' be two such edges and suppose that e' is not special. Then in the extension of G by a new common neighbor of the vertices of e', there is an embedding of $K_3 + K_3$, which must consist of the new triangle and a second triangle T in G disjoint from e, and hence meeting K in e.

If neither e nor e' is special then we also have a triangle T' meeting K in e'. Then T, T' must meet in a vertex extending K to a clique of order 5, which is a contradiction. So at least one of e, e' is special, and we know at most one is.

It follows that there are exactly three special edges in K. These may form either a *star* or a *triangle* in K. We may call these cases $2a$ and $2b$.

Furthermore, in this situation all triangles not contained in K meet K in one of the special edges.

Case 3. Suppose that G contains no clique of order 4.

By existential completeness, G does contain some triangle T.

Claim 2. For any triangle T in G there is a triangle T' which meets T in exactly one vertex.

We extend G to a graph G' with a new vertex v adjacent to the vertices of T. As this configuration is not realized in G, there must be a triangle disjoint from one of the triangles containing v, and hence disjoint from the corresponding edge of T. But T' must meet T, so the intersection is a single vertex. This proves the claim.

Claim 3. G contains a special vertex.

We begin with the configuration T, T' consisting of two triangles with a common vertex v.

Suppose that v is not special. Then a maximal family of triangles in G disjoint over v will be finite. It follows that the extension which adds two vertices to make a new triangle over v is not $(K_3 + K_3)$-free. So there must be a triangle T'' in G disjoint from this triangle, which means that it does not contain v. But T'' must meet T and T'.

If T'' is contained in $T \cup T'$ it gives rise to a clique of order 4 and a contradiction. So T'' consists of a vertex $a \in T$, $b \in T'$, and c outside $T \cup T'$. Then $T^* = (a, b, v)$ form a triangle whose edges lie in three other triangles involving three distinct vertices outside T^*. We know however that there is another triangle meeting T^* in exactly one vertex, which must also contain the vertex outside T^* joined to the oppose edge. This again produces a clique of order 4 and a contradiction.

Thus the vertex v is special, and the claim is proved. In particular every triangle contains the vertex v.

Next we claim that the configurations identified as cases 1, 2a, 2b, and 3 correspond to unique possibilities for the existentially closed graph G. So we return to a closer examination of each case.

Case 1. G contains a clique K of order 5.

We treat K as a set of distinguished vertices and consider the family $\mathcal{A}_K$ of finite extensions of K whose triangles are all contained in K. This forms an amalgamation class under free join and hence corresponds to a generic graph of this type.

Any $(K_3 + K_3)$-free extension will have the same property and thus this graph is maximal and existentially closed.

Case 2. Suppose G contains a clique K of order 4 but no clique of order 5.

We know then that there are three edges designated as special in K, forming either a star (case $2a$) or a triangle (case $2b$). In each case we consider the class $\mathcal{A}_K$ of finite extensions in which all triangles not contained in K meet K in a special edge. These form an amalgamation class under free join, so the corresponding generic graph exists in each case. In this graph the edges designated as special in K are in fact special, since the corresponding finite extensions of K all lie in the class $\mathcal{A}_K$.

Now any extension of the generic graph G will again omit K_5 and thus embed again into G, and the maximality and existential completeness of G follow.

Case 3. G contains a special vertex v.

Then all triangles of G contain v. We consider the class $\mathcal{A}_v$ of finite graphs containing the vertex v whose triangles all contain v. This is an amalgamation class under free join and thus the corresponding generic graph exists and is $(K_3 + K_3)$-free. Furthermore the vertex v is in fact special in the generic graph, or any graph extending it, from which existential completeness and maximality again follow.

We have thus constructed four maximal existentially closed $(K_3 + K_3)$-free graphs, one for each case identified, and we showed along the way that any existentially closed $(K_3 + K_3)$-free graph falls under one of the cases and embeds (is in fact isomorphic to) the corresponding generic graph.

It follows that the complexity of the class of $(K_3 + K_3)$-free graphs is 4. □

6.2. $m > n \geq 3$**:** $c'_k(m, n)$**.** With $n \geq 3$ and $m > n$ the values of $c(m, n)$ are no longer bounded, and the situation becomes more complex. We could also allow $m = n$ but the details would vary, so we consider the more typical case. We refine our definitions a bit to reflect the kind of analysis we have been making. We will work for a time in the context of $c(m, n)$ and then specialize to the case $n = 3$.

Notation 6.4.

$c_k(m, n)$ is the complexity of the class of (K_m+K_n)-free graphs which contain a clique of order k but none of order $k + 1$.

$c'_k(m, n)$ is the number of maximal existentially closed $(K_m + K_n)$-free graphs containing a clique of order k but none of order $k + 1$; $c'(m, n)$ is defined similarly.

We have

$$\sum_k c'_k(m.n) = c'(m, n) \leq c(m, n) \leq \sum_k c_k(m, n)$$

and conjecturally $c'(m, n) = c(m, n)$. The sum $\sum_k c(m, n)$ is an overcount as one should restrict attention to existentially closed graphs; for example, for many values of k we will have $c'_k(m, n) = 0$ but $c_k(m, n) > 0$.

Lemma 6.5. *If* $m \geq n$ *then*

(1) $c'_k(m, n) = 0$ *for* $k < m - 1$ *; also for* $k = m$ *if* $m = n$.
(2) $c'_k(m, n) = 0$ *for* $k \geq m + n$.

Proof.

Ad 1

If Γ is a graph containing no clique of order $m-1$ then the disjoint sum of Γ with a clique of order $m-1$ is (K_m+K_n)-free, so Γ cannot be existentially closed. Thus $c_k(m,n)=0$ for $k<m-1$.

The same argument applies if Γ contains no clique of order m, when $m=n$.

Ad 2

Since K_m+K_n is a subgraph of K_{m+n} this is clear. □

So the sum of interest in general is

$$\sum_{m-1\le k\le m+n-1} c'_k(m,n).$$

We record some extreme values.

Lemma 6.6. *For $m\ge n$ we have*

$$c'_{m+n-1}(m,n)=1$$

For $m>n$ we have

$$c_{m-1}(m,n)=1$$

Proof. Suppose first that Γ contains a clique K of order $m+n-1$ and is (K_m+K_n)-free. Then every clique of order m is contained in K. There is a generic graph G containing K and satisfying this property, so any existentially closed graph containing K is isomorphic to G and is maximal. This shows that $c'_{m+n-1}(m,n)=1$.

Now suppose $m>n$ and let G be the generic K_m-free graph. Then G contains an infinite disjoint sum of copies of K_n and hence any (K_m+K_n)-free extension of G omits K_m and embeds into G. It follows that G is maximal existentially closed. □

Our intention in the ret of our analysis is to look more closely into $c(m,3)$ for $m>3$, verifying that this is

$$\sum_{m-1\le k\le m+2} c'_k(m,3)$$

and that the growth rate is on the order of 2^{m^2}.

So far we have $c'_{m-1}(m,3)=c'_{m+2}(m,3)=1$ and it remains to examine $c'_m(m,3)$ and $c'_{m+1}(m,3)$.

We continue to work somewhat generally, in the sense that we consider $c'_{m+n-2}(m,n)$ and $c'_{m+n-3}(m,n)$ before specializing to $n=3$.

6.3. **Estimating $c'_{m+n-2}(m,n)$ and evaluating $c'_{m+1}(m,3)$.**

Definition 6.7. A clique K in a graph G is *m-special* if it lies in infinitely many cliques of order m with pairwise intersection K. As a degenerate case, we consider m-cliques to be m-special (with the "infinitely many" extensions all coinciding with K).

Remark 6.8. *In a $(K_m + K_n)$-free graph, all m-special cliques meet all n-special cliques.*

Evaluating $c'_{m+n-2}(m,n)$

Suppose $m > n$ and G is an existentially closed (K_m+K_n)-free graph with a clique K of order $m+n-2$ and no clique of order $m+n-1$.

Claim 1. If $A \subseteq K$ has order $n-1$ then either A is n-special or $(K\backslash A)$ is m-special.

We suppose A is not n-special. Then adding a new vertex adjacent to the vertices of A gives an extension of G which is not $(K_m + K_n)$-free, so there is a clique K' of order m in G disjoint from A. It must contain $K \setminus A$ as otherwise we have an embedding of $K_m + K_n$. So $K' = (K\backslash A) \cup \{v\}$ for some vertex $v \notin K$. If $K\backslash A$ is not m-special then similarly A is contained in a clique $K''$of order m of the form $A \cup \{u\}$ with $u \notin K$. If $u = v$ then $K \cup \{v\}$ is a clique of order $m+n-1$, for a contradiction. But if $u \neq v$ then we have an embedding of $K_m + K_n$. So the claim follows.

Now view K as a structure in which predicates distinguish the $(m-1)$-subsets which are m-special and the $(n-1)$-subsets which are n-special. The structure on K satisfies the following constraints.

(1) If K is partitioned into sets (A,B) of sizes $m-1$, $n-1$, then exactly one of A, B is special in the appropriate sense.
(2) Any $(n-1)$-subset of an m-special $(m-1)$-set is n-special.

Conversely, suppose K is given with predicates picking out $(m-1)$-subsets and $(n-1)$-subsets, which we will refer to as the m-special and n-special subsets of K. We then consider the amalgamation class $\mathcal{A}_K$ consisting of extensions of K for which any n-clique not contained in K contains an n-special $(n-1)$-set and any m-clique not contained in K contains an m-special $(m-1)$-set. Here amalgamation is given by joint embedding. Let G_K be the corresponding generic graph (with its additional structure).

Then G_K is (K_m+K_n)-free: it is clear that any embedding of K_m+K_n into G_K cannot involve a copy of K_m or K_n in K, and thus involves disjoint special cliques of sizes $m-1$ and $n-1$ in K, which partition K, and this contradicts one of our conditions.

Consider an extension Γ of G_K which is (K_m+K_n)-free and let A be an m-clique or an n-clique. Consideration of $A \cup K$ shows that $A \cap K$ is then an $(m-1)$-clique or $(n-1)$-clique respectively. If it is not one of the designated special cliques, its complement is, and then in G_K the complement is actually special. Then together with A one gets an embedding of K_m+K_n into Γ, and a contradiction. Thus the extension Γ satisfies the conditions imposed on G_K and accordingly embeds into G_K. It follows that G_K is existentially closed as a (K_m+K_n)-free graph and is maximal.

We may draw the following consequence.

Lemma 6.9. *For $m > n$, c'_{m+n-2} is the number of uniform $(n-1)$-hypergraphs on $(m+n-2)$ vertices with the property that all hyperedges meet, counted up to isomorphism.*

Proof. We need to classify the configurations K on $(m+n-2)$ vertices equipped with a designation of the special $(m-1)$-sets and $(n-1)$-sets, satisfying the conditions (1,2) above.

In view of the complementarity condition, it suffices to consider the hypergraph on K whose edges are the non-special $(n-1)$-sets. In terms of this hypergraph our conditions become

No two hyperedges are disjoint.

This gives the result. □

In particular with $n = 2$, $m > 2$ we find $c'_m(m, 2) = 2$, and with $n = 3$, $m > 3$ we find $c'_{m+1}(m, 3) = m+2$ as there are $(m+1)$ possible stars (including the degenerate cases with 0 edges or 1 edge) and also the possibility of a triangle.

For $n > 3$ one construction involves taking a fixed point as common to the distinguished (non-special) $n-1$-sets and then viewing the structure as an $(n-2)$-hypergraph on the remaining points, a slight overcount for various reasons—it neglects the reduction to isomorphism types and the possibility that the hyperedges have other points in common. In any case this gives a lower bound in the range of $2^{\binom{m+n-3}{n-2}}$ and an upper bound in the range of $2^{\binom{m+n-2}{n-1}}$, so a decent estimate for the rate of growth once n is not terribly small (e.g., 3).

As far as $c(m, 3)$ is concerned we have the following, so far.

k	$m-1$	m	$m+1$	$m+2$
$c'_k(m,n)$	1	??	$m+2$	1

To evaluate $c_m(m, 3)$ we will look more generally at $c'_{m+n-3}(m, n)$.

The expectation of course is that $c(m, 3) = m + 4 + c'_m(m, 3)$ with $c'_m(m, 3)$ making the dominant contribution. Up to this point we have checked the relevant cofinality conditions and that all of the graphs considered are needed, that is, they are maximal existentially complete in the full class of $(K_m + K_n)$-free graphs.

7. Estimating $c'_{m+n-3}(m, n)$

We suppose that

$$m > n \geq 3$$

and that G is an existentially closed $(K_m + K_n)$-free graph with some clique K of order $m + n - 3$ and no clique of order $m + n - 2$. Set

$$N = m + n - 3$$

We consider not only m-special cliques and n-special cliques, but also N-special cliques. We remark that any two cliques of order N must meet any clique of order $n' \geq n$ in at least $n' - 2$ vertices, as otherwise their union has at least $N + 3 = m + n$ vertices and contains a copy of $K_m + K_n$.

We first consider the cases in which there is an N-special clique of order $N - 1$ or $N - 2$.

7.1. Case 1. There is an N-special clique K_0 of order $N - 2$.

Then any clique K' of order n must meet K_0 is at least $n - 2$ vertices, as otherwise we find an extension K_1 of order N of K_0 meeting K' in fewer than $n - 2$ vertices.

So we consider the amalgamation class $\mathcal{A}_{K_0}$ of extensions of K_0 such that each n-clique meets K_0 in at least $n - 2$ points. This is closed under free join. Let G_{K_0} be the corresponding generic graph. Observe that G_{K_0} is $(K_m + K_n)$-free since in any embedding there are at least $(m - 2) + (n - 2)$ vertices in K_0, while $|K_0| = m = n - 5$.

We claim that any $(K_m + K_n)$-free extension of G_{K_0} will have the same properties. The one that needs to be checked is that there is no clique K^* of order $N+1$. If there is one, we can extend K_0 to a clique K_1 of order N so that the intersection has order at most $N - 2 < |K^*| - 2$ for a contradiction.

It follows that G_{K_0} is a maximal existentially closed $(K_m + K_n)$-free graph.

Thus this case contributes only 1 to the value of $c'_{m+n-3}(m, n)$.

7.2. N-special $(N - 1)$-cliques and balanced N-cliques.

Definition 7.1. Let G be a $(K_m + K_n)$-free graph and A an N-clique in G. We will say that G is *balanced* if for every partition (A_1, A_2) into parts of sizes $m - 1, n - 2$ or $m - 2, n - 1$ respectively, either A_1 is m-special or A_2 is n-special.

As m-special and n-special cliques must meet, in a balanced N-clique exactly one part of each of the relevant partitions is special in the appropriate sense.

We will identify a contribution of precisely $m+2$ to $c'_m(m,3)$ in the case in which there is a balanced N-clique, via a more general structural analysis.

First however we will show that if there are no N-special $(N-2)$-cliques, but at least two N-special $(N-1)$ cliques, then the union of the N-special $(N-2)$-cliques is a balanced N-clique.

Lemma 7.2. *Let G be an existentially complete (K_m+K_n)-free graph with no N-special $(N-2)$-clique. Then the union of the N-special $(N-1)$-cliques is a clique.*

Proof. Let A_1, A_2 be two distinct N-special $(N-1)$-cliques, and A their intersection. So $A_i = A \cup \{v_i\}$ for some vertex v_i, $i=1$ or 2.

We consider an extension of G in which A is extended to a clique of order N by two new vertices. As A is not N-special there is an embedding of K_m+K_n into the extension, with one of the two cliques, or order m or n, containing a vertex outside G and thus meeting G in a subset of A or order at least $m-2$ or $n-2$. The other clique must then contain at least 3 vertices outside A and hence lies in G. In particular G contains a clique K of order n with at least three vertices outside A. On the other hand K must meet A_1 and A_2 in at least $n-2$ points and so K contains the vertices v_1, v_2. Thus $A_1 \cup A_2$ is a clique.

The lemma follows. □

Lemma 7.3. *Let G be an existentially complete (K_m+K_n)-free graph with no N-special $(N-2)$-clique and no $(N+1)$-clique, and with at least two N-special $(N-1)$-cliques.*

Then the union A of the N-special $(N-1)$-cliques is a balanced N-clique.

Proof. The argument for partitions of shape $(m-1, n-2)$ or $(m-2, n-1)$ is similar, so we treat the former case.

So let (A_1, A_2) be a partition of shape $(m-1, n-2)$ and suppose toward a contradiction that neither of the two parts is special in the appropriate sense.

Extending G by one new vertex adjacent to the vertices of A_1 to make a new clique of order m, the extension cannot be (K_m+K_n)-free and thus there is a clique K_2 of order n in G disjoint from A_1. But K_2 meets A in at least $n-2$ vertices, so $K_2 \cap A = A_2$.

Similarly, extending A_2 to a new clique of order n, there must be a clique K_1 of order m in G disjoint from A_2 and containing at least $m-2$ vertices of A_1.

Furthermore the cliques K_1, K_2 have some common vertex v, so if K_1 contains A_1 then we have a clique $A \cup \{v\}$ of order $N+1$, for a contradiction. So $A_1 \setminus K_1$ consists of one vertex a, and $K_1 \setminus A$ contains 2 vertices.

On the other hand A contains at least two N-special subsets of size $N-1$. So there is some N-special $(N-1)$-clique contained in A which omits a vertex of $(K_1 \cup K_2) \cap A$. This may be extended to an N-clique containing no additional vertex of $K_1 \cup K_2$. Hence it misses at least three vertices of one of the cliques K_1, K_2, giving an embedding of $K_m + K_n$ into G, and a contradiction.

Making a similar argument for the shape $(m-2, n-1)$, this proves the lemma. □

Going forward, we will be supposing only that we have a balanced N-clique A. In Table 1 we summarize the conditions imposed on A by the ambient graph G, that is, with the m-special and n-special cliques in A distinguished. We note that in case $m = n+1$, for cliques of size $m-2 = n-1$, either or both of the two notions might apply.

(1) For a partition of A into parts (A_1, A_2) of sizes $m-1, n-2$ or $m-2, n-1$ respectively, either A_1 is m-special or A_2 is n-special, but not both.
(2) Any m-special clique and any n-special clique have a vertex in common.
(3) If the clique A' is m-special of order $m-k$ ($k = 1$ or 2) then its subsets of order $n-k'$ are n-special for $k' \leq k$.

Table 1. The m-special, n-special structure on A

The third condition follows directly from the definitions. Note that $m-2 \geq n-1$.

7.3. **Case 2. There is a balanced N-clique: Constructions.** We proceed next to construct the appropriate generic graphs of each type for any finite structure A of order N satisfying the conditions of Table 1.

Lemma 7.4. *Let $m \geq n \geq 3$. Let A be a finite structure of order $N = m+n-3$ which consists of a complete graph on A together with two families of designated subsets, called m-special and n-special cliques, where the m-special cliques have orders $m-1$ or $m-2$ and the*

n-special cliques have orders $n-1$ or $n-2$. Assume the conditions of Table 1. Then there is an existentially complete (K_m+K_n)-free graph G_A, unique up to isomorphism, which induces the given structure on A. Furthermore G_A is maximal existentially complete.

Proof.

Claim 1. G_A is (K_m+K_n)-free.

Any m-clique or n-clique has at most two vertices outside A, so an embedding of K_m+K_n into G_A cannot involve a component embedded into A. Accordingly the intersections with A are m-special and n-special respectively, and meet.

Claim 2. In any (K_m+K_n)-free expansion of G_A, any m-clique or n-clique K not contained in A meets A in an m-special or n-special clique (in the sense of A), respectively.

K contains at most 2 vertices outside A, so the complementarity condition applies, and if $K \cap A$ is not special in the appropriate sense, then its complement in A is special in the other sense, and an embedding of K_m+K_n results, for a contradiction.

The same argument applies to our original graph G and shows that G is isomorphic to G_A. It also follows now that G_A is maximal existentially complete. One should note also the following.

Claim 3. In G_A, the induced structure on A is as given.

This follows from the preceding claims. □

Next we would like need to count the structures A associated with balanced N-cliques, up to isomorphism, or estimate their number.

7.4. **Balanced N-cliques as finite structures: Counting.** We refer to the conditions of Table 1. For the case $n=3$ we will arrive at $m+2$ distinct structures, but we first proceed more generally.

All of this is more easily understood—at least for the purposes of counting—in terms of the structure on the m-special cliques alone.

So suppose we have a structure A with a distinguished family of "m-special" cliques of orders m and $m-1$. We let A^* be the expansion of A by the family of cliques of orders $n-1$ and $n-2$ whose complements are not m-special, and call those cliques n-special. We claim that A^* will satisfy the axioms given above if and only if A satisfies the following axioms.

Lemma 7.5. *The conditions given in Tables 1 and 2 are equivalent.*

(1) If K is an m-special clique of order $m-2$ then any extension of K by an additional vertex is m-special.
(2) If K_1 and K_2 are m-special then their intersection has order at least $m-n+1$.

TABLE 2. The m-special structure on A

Proof. More explicitly, if A is a structure of order $N = m+n-3$ given in terms of a distinguished family of m-special cliques of orders $m-1$ and $m-2$, and the complementarity property is used to expand to A^* with the corresponding family of n-special cliques, then we claim that the properties of Table 1 hold in the structure A^* if and only if the properties of Table 2 hold in the structure A. Since the complementarity property defines the relationship between A and A^* we may set that condition aside.

Now consider property (2) for A^*, which we decode as follows: any m-special clique K_1 meets the complement of any clique K_2 of order $m-1$ or $m-2$ which is not m-special. In other words, if K_1 is contained in K_2 then K_2 must be special. This is the first condition in Table 2.

Now consider the third and last condition in Table 1: if K_1 is m-special and B is a subset of appropriate cardinality ($n-1$ if $|K_1| = n-1$, and $n-1$ or $n-2$ if $|K_1| = n-2$), then $K_2 = B^c$ is not m-special. In other words, if K_1, K_2 are m-special, we require $|K_1 \backslash K_2| < n-1$ or $|K_1 \backslash K_2| < n-2$ according as $|K_1| = m-1$ or $m-2$, that is

$$|K_1 \backslash K_2| < |K_1| - (m-n)$$

and $|K_1 \cap K_2| \geq m-n+1$. □

We can make the presentation in Table 2 more manageable still by passing to complements in a more literal sense: that is, by taking the cliques of order $n-1$ or $n-2$ which are *not* n-special as distinguished. Then our conditions become the following.

Taking this as our point of departure, we next count, or estimate, the number of isomorphism types of structures on $N = m+n-3$-vertices satisfying our conditions. So we consider a family of subsets of orders $n-1$ and $n-2$, which we refer to as *distinguished*. Recall that these are the ones which are not n-special under the embedding into G_A—or, to put matters another way. We can set that interpretation aside now and work directly with the axioms.

(1) Downward closure: if a clique of order $n-1$ is not n-special then any $(n-2)$-clique contained in it is not n-special.
(2) If K_1 and K_2 are not n-special then their union has order at most $2(n-2)$. More explicitly, we have the following.
  - If K_1 has order $n-1$ and K_2 has order $n-2$ they meet;
  - if K_1, K_2 both have order $n-2$ then their intersection contains at least 2 vertices.

TABLE 3. The non-n-special structure on A

Namely, the family of distinguished cliques in A is closed downward and satisfies the following intersection condition for K_1, K_2 distinguished.

If $|K_1| = n-1$ and $|K_2| = n-2$ then K_1, K_2 meet;

If $|K_1| = |K_2| = n-1$, then K_1 and K_2 meet in at least two vertices.

For a general lower bound we can consider the structures obtained by distinguishing any set of $(n-2)$-cliques, and no $(n-1)$-cliques. So here we are simply counting $(n-2)$-uniform hypergraphs on $m+n-3$ vertices.

Another way to meet our conditions is as follows, which may well be typical of cases with many $(n-1)$-cliques distinguished (many $m-2$-cliques are special).

Definition 7.6. Let A be a set of N vertices and A_0 a set of $2n-4$ vertices. Choose a family of subsets of A of orders $n-1$ and $n-2$, to be called m-special, all of which are contained in A_0, and are closed downward.

Of course this second construction is a very limited one as the parameter m does not enter in.

Our main interest now is in the case $n = 3$, where the situation degenerates.

Lemma 7.7. *Under the assumptions of Case 2a, with $n = 3$, the structure A is a clique on m vertices with one of the following structures.*

(1) There is no m-special clique of order $m-2$ and the structure on A is fully determined by the choice of m-special $(m-1)$-sets (or 3-special vertices); or
(2) there is a unique m-special clique of order $m-2$, and the m-special cliques of order $m-1$ are those which contain it, with the rest determined by complementarity.

In particular there are $m+2$ maximal existentially complete $(K_m + K_3)$-graphs of type $2a$.

Proof. Here $N = m + n - 3 = m$.

We work with the complementary point of view as in Table 3. So we have a distinguished family of edges (pairs of vertices) and vertices which is downward closed, and our intersection condition includes the constraint that all distinguished vertices lie on all distinguished edges. So either there is exactly one distinguished edge with its vertices as the distinguished vertices, or there are no distinguished edges and an arbitrary set of distinguished vertices.

This gives $1 + (m + 1) = m + 2$ structures, whose more natural description is recovered by passing to the complements. □

If one moves on to $n = 4$, $N = m + 1$, then we need to select a family of pairs and triples, downward closed, with all pairs and triples meeting, and all triples having a pair in common.

On the one hand we can have no triples and an arbitrary graph on m points, giving the dominant term. With exactly one triple T the additional edges will consist of pairs (u, v) with $v \in T$, $u \notin T$, and so we need to count sequences a, b, c with

$$0 \leq a \leq b \leq c; \qquad\qquad a + b + c \leq m + 3$$

giving roughly $m^3/36$ possibilities.

With $k > 1$ triples sharing a common edge E the additional edges consist of a vertex in E and a vertex in the remaining $(m+1)-(k+2) = m - k - 1$ points, corresponding to pairs a, b with $0 \leq a \leq b$ and $a + b \leq m - k - 1$, summing the count over $2 \leq k \leq m - 1$, for approximately $m^3/12$ structures. Finally we have the exceptional case in which the triangles have no common edge. In this case their union has order 4 and there are 3 or 4 triangles. If there are additional edges then there are three triangles, with one common vertex, and the additional edges contain this vertex. All together these exceptional case contribute only $1 + (m - 2) = m - 1$ additional structures.

One expects similar behavior for larger n, but this would involve looking more closely at the consequences of the intersection conditions, a topic which has been extensively studied in combinatorics.

8. Beyond balance

Lemma 8.1. *Let G be an existentially complete $(K_m + K_n)$-free graph with maximal clique size $N = m+n-3$. If two cliques A_1, A_2 of order N have intersection A of order $N - 1$, then A is N-special.*

Proof. Let $A_i = A \cup \{v_i\}$ for $i = 1, 2$.

We take an extension of G by a new vertex v extending A to a clique of order N. It suffices to check that the extension is $(K_m + K_n)$-free.

Otherwise, one has an embedding of $K_m + K_n$ into the extension, which means that one of K_m, K_n corresponds to a clique K contained in $A \cup \{v\}$ and the other corresponds to a clique K' contained in G and disjoint from K.

If $K \cup K'$ does not contain $A_1 \cup A_2$ then we can replace v by an element of $A_1 \cup A_2$ and embed $K_m + K_n$ into G, for a contradiction. But $K \cap G \subseteq A$, so K' contains the two vertices v_1, v_2 and hence $A_1 \cup A_2$ is a clique of order $N + 1$, a contradiction. □

Lemma 8.2. *Let G be an existentially complete $(K_m + K_n)$-free graph with maximal clique size $N = m + n - 3$ and with no N-special clique of order $N - 2$. If two N-cliques A_1, A_2 have an intersection A of order $N - 2$, then A is contained in two distinct N-special $(N - 1)$-cliques.*

Proof. Let $A_i = A \cup \{u_i, v_i\}$.

Claim 1. If $A \cup \{u_1\}$ is not N-special, then there is an edge between v_1 and one of u_2, v_2.

We form an extension of G by a new vertex v extending $A \cup \{u_1\}$ to a new N-clique. If $A \cup \{u_1\}$ is not special then the extension is not $(K_m + K_n)$-free and we have an embedding of $K_m + K_n$ as two disjoint cliques K, K' where we may suppose that K contains the vertex v and hence $K \cap G \subseteq A \cup \{u_1\}$, while K' embeds into G. If the vertex v_1 or both vertices u_2, v_2 lie outside $K \cup K'$ then we may replace the vertex v by v_1, or v and u_1 by u_2, v_2, to embed $K_m + K_n$ into G, for a contradiction.

So K' must contain v_1 and one of u_2, v_2, proving the claim. We may choose notation so that

$$(v_1, v_2) \text{ is an edge.}$$

As G contains no $(N + 1)$-clique we find that (v_1, u_2) is not an edge.

If $A \cup \{v_1\}$ and $A \cup \{v_2\}$ are N-special then our lemma holds. Otherwise, the previous claim shows that there is another edge between $\{u_1, v_1\}$ and $\{u_2, v_2\}$ and as there is no triangle on these four points we must have a 4-cycle (u_1, v_1, v_2, u_2).

Now we will reach a contradiction by showing that A is N-special.

So adjoin two new vertices w_1, w_2 extending A to an N-clique. It suffices to show that the extension is $(K_m + K_n)$-free.

Otherwise, we embed $K_m + K_n$ into the extension as $K + K'$ with K meeting $\{w_1, w_2\}$ and we find $K \subseteq A \cup \{w_1, w_2\}$, $K' \subseteq G$.

Let $C = (A_1 \cup A_2) \setminus A$, a 4-cycle. Now $(K \cup K') \cap C = K' \cap C$ is a clique, and hence misses a pair of vertices $b_1, b_2 \in C$ joined by an edge. Then we can replace the vertices of $K \setminus G$ by one or both of b_1, b_2 and embed $K_m + K_n$ into G. □

Lemma 8.3. *Let G be an existentially complete $(K_m + K_n)$-free graph with maximal clique size $N = m + n - 3$, with no N-special clique of order N_2, and with at most one N-special clique of order $N-1$. If there is more than one N-clique in G then the intersection of all N-cliques is an N-special clique A of order $N - 1$.*

Proof. By Lemma 8.2 and our hypotheses, any two N-cliques have $N-1$ vertices in common. By Lemma 8.1 these intersections are N-special, hence coincide, by our hypotheses. The lemma follows. □

Now continuing our analysis for the case $N = m+n-3$, and setting aside cases 1 and 2, our assumptions are

(1) The maximal clique size is $N = m + n - 3$.
(2) There is no N-special clique of order $N - 2$.
(3) There is no balanced N-clique.

By Lemma 7.3 we have at most one N-special clique of order $N - 1$ and thus we come down to the following possibilities, keeping Lemma 8.1 in mind.

- There is exactly one N-special clique of order $N - 1$, and the N-cliques all contain it.
- There is exactly one N-clique.

Here we are getting into the weeds.

At the moment we are not making good use of the assumption that there is no balanced N-clique and as a result it is possible that there will be some overlap between the cases previously considered and the remaining cases, which would be a point worth checking, if one wants to make some definite claim about the exact value of $c(m, 3)$.

8.1. Case 3. One N-special clique A of order $N - 1$.

To recapitulate, in Case 3 we have the following conditions.

(1) G is existentially complete (K_m+K_n)-free with maximal clique size $N = m + n - 3$, where $m > n \geq 3$.
(2) The intersection A of all N-cliques in G is an N-special clique of order $N - 1$.

It is possible here that one of the N-cliques is balanced and we might want to exclude such cases to avoid overlap with cases treated previously. (Or rearrange the cases, eventually.)

In Case 3 we examine the m-special and n-special cliques in A. The cliques of order $m-1$ or $n-1$ are m-special and n-special, respectively, so our concern is with the cliques of orders $m-2$ and $n-2$, and notably with the partitions of A as (A_1, A_2) with $|A_1| = m-2$ and $|A_2| = n-2$.

Lemma 8.4. *Under the assumptions of Case 3, if (A_1, A_2) is a partition of A into sets of orders $m-2, n-2$ respectively, where A_1 is not m-special and A_2 is not n-special, then there is a unique vertex v^* outside A for which there are an m-clique K_1 and an n-clique K_2 such that*

$$K_i \cap A = A_1 (i = 1, 2)$$
$$v^* \in K_1 \cap K_2$$

Furthermore, there are at most two such vertices.

We do not claim that the various vertices of this type corresponding to different partitions are necessarily distinct.

Proof.

Claim 1. There are cliques K_1, K_2 of orders m, n respectively so that $K_i \cap A = A_i$ for $i = 1, 2$.

To find K_1 we consider the extension of G by two new vertices extending A_2 to an n-clique. As we suppose A_2 is not n-special there is an embedding of $K_m + K_n$ into this extension, and this is of the form $K_1 + K_1'$ where K_1' contains one of the new vertices and hence is the extension of A_2 by these vertices. Thus $K_1 \subseteq G$ and $K_1 \cap A \subseteq A_1$. If K_1 misses a point of A_1 then after extending A to an N-clique A^* by a vertex outside $A \cup K_1$, we can replace the two new vertices in K' by two vertices of A^* and embed $K_m + K_n$ into G, for a contradiction.

This gives the clique K_1.

For the clique K_2 we begin similarly with an extension by two new vertices adjacent to the vertices of A_1, and an embedding of $K_m + K_n$ into the extension of the form $K_2' + K_2$ where K_2' contains a new vertex and thus lies in A. Again $K_2' \cup K_2$ must cover A, so K_2 must cover A_2. Furthermore $K_2' \cup K_2$ must contain two vertices outside A, and they must lie in K_2. Since $|K_2| \geq n$, after shrinking K_2 if necessary we find a clique of order n whose intersection with A is A_2.

This proves the claim.

The cliques K_1, K_2 must meet, so they contain at least one vertex v^* lying outside A. Furthermore $(K_1 \cap K_2) \cup A$ is a clique, so the intersection consists of one vertex. That is

$$K_1 \cap K_2 = \{v^*\}$$

Claim 2. The vertex v^* is unique.

Suppose on the contrary that K_1^* is an m-clique containing A_1 and K_2^* is an m-clique containing A_2 with $K_1^* \cap K_2^* = \{w^*\}$ and $w^* \neq v^*$. Then $K_1 \cap K_2^*$ and $K_1^* \cap K_2$ both meet, and at least one does not contain v^*. Therefore after replacing the pair (K_1^*, K_2^*) by either (K_1, K_2^*) or (K_1^*, K_2), we may suppose that w^* belongs to K_1 or to K_2, and hence (v^*, w^*) is an edge, for a contradiction. This proves the claim, and the lemma. ☐

Notation 8.5. For (A_1, A_2) a partition of A of shape $(m-2, n-2)$ for which A_1 is not m-special and A_2 is not n-special, let v_{A_1,A_2} be the vertex afforded by the previous lemma.

Let A^* be the union of A with the set of vertices v_{A_1,A_2} associated to such partitions.

We view A^* as equipped with the following data.

(1) A^* carries the graph structure induced from G, with the subset A distinguished. In particular, A is a clique.
(2) The distinguished family of subsets of A of order $m-2$ which are m-special, and the distinguished family of subsets of A of order $n-2$ which are n-special.
(3) For each vertex v of A^*, the non-empty collection of partitions (A_1, A_2) of shape $m-2, n-2$ with neither part distinguished, for which $v = v_{A_1,A_2}$.

We mention some properties of A^*.

(1) In any complementary pair of shape $(m-2, n-2)$ at most one of the two cliques is distinguished.
(2) If a subset of order $m-2$in A is distinguished, then its subsets of order $n-2$ are distinguished.

This implies the following intersection condition: any two distinguished $(m-2)$-sets must contain at least $m-n+1$ common vertices.

We then have the following constraints on G.

Lemma 8.6. *Under the assumptions of Case 3, G satisfies the following conditions.*

(1) Every m-clique meets A in at least $m-2$ elements; every n-clique meets A in at least $n-2$ elements.
(2) If an m clique K meets A in a set A_1 of $m-2$ elements, then either A_1 is m-special, or neither the vertex v_{A_1,A_2} is defined with $A_2 = K \setminus A_1$, and this vertex belongs to K; similarly for n-cliques.

Proof. As G is $(K_m + K_n)$-free and A is N-special, the first point is clear.

Now suppose that K is an m-clique and $A_1 = A \cap K$ has $m-2$ elements, and is not m-special. The complement is certainly not n-special, as otherwise we embed $K_m + K_n$ directly, so the vertex v_{A_1,A_2} is defined, and is associated with a pair of cliques K_1, K_2 with K_2 an n-clique containing A_2. Since K must meet K_2 and the vertex v_{A_1,A_2} is uniquely determined, it must lie in K. □

Now we associate to the finite structure A^* the class $\mathcal{A}_{A^*}$ of finite extensions of A^* (as graphs) respecting the conditions of the previous lemma. This is an amalgamation class with respect to free join and thus there is a generic graph G_{A^*} associated with A^*.

Lemma 8.7. *For A^* as described, the graph G_{A^*} is maximal existentially complete $(K_m + K_n)$-free and induces the original structure on A^*.*

Proof.

Claim 1. G_{A^*} is $(K_m + K_n)$-free.

Suppose toward a contradiction that K_1, K_2 are cliques of orders m, n respectively which are disjoint. In particular K_1, K_2 induce a partition (A_1, A_2) of A of shape $m-2, n-2$.

Since A_1, A_2 cannot both be distinguished in A^*, one of the cliques is required to contain a vertex of the form v_{A_1,A_2}, and for this we must have neither part distinguished, and hence both cliques contain v_{A_1,A_2}, for a contradiction.

The claim follows.

The designated cliques in A^* become special in G_{A^*} since the relevant finite structures satisfy the constraints imposed. In the case of cliques of order $m-2$, this depends on the condition of downward closure.

Claim 2. In any $(K_m + K_n)$-free extension of G_{A^*}, the structurer imposed on A^* is as given.

We claim that no other clique in A of order $m-2$ or $n-2$ can become special. Suppose for example that A_2 is a clique of order $n-2$ which is not distinguished in A^*, and $A_1 = A \setminus A_2$. If A_1 is distinguished then it is m-special in G_{A^*} and so A_2 has no extension to an n-clique not in A. If neither part is distinguished in A^* then the vertex v_{A_1,A_2} prevents A_2 from becoming special.

The rest is clear and the claim follows.

From this it follows that the $(K_m + K_n)$-free extensions of G_{A^*} embed into G_{A^*}, that G_{A^*} is existentially complete, and that it is maximal.

The lemma follows. □

It is possible that this construction will bring us back to an earlier case. If A_0 is a subset of A of order $N-2$ and all its $(m-2)$-subsets are distinguished then A_0 will become N-special in G_{A^*}. On the other hand, this will uniquely determine the structure on A^*.

It remains to estimate the number of structures A^* available. Here we distinguish some $(m-2)$-sets in A, close downward to $(n-2)$-sets, distinguish additional $(n-2)$-sets, and on the set of complementary pairs of shape $(m-2, n-2)$ without distinguished parts we impose an equivalence relation corresponding to the condition

$$v_{A_1,A_2} = v_{A'_1,A'_2}$$

If we use just one of our degrees of freedom we have the following constructions.

(1) pick an $(m-2)$-uniform hypergraph on A withe property that any two hyperedges contain at least $m-n+1$ common vertices, and close downward;
(2) pick an $(n-2)$-uniform hypergraph on A;
(3) impose an equivalence relation on the set of all partitions of shape $(m-2, n-2)$.

The first two constructions give roughly $2^{\binom{m+n}{n}}$ structures while the third is counted up to isomorphism by the partition function $p(\binom{m+n-4}{n-2})$ which is approximately

$$\frac{1}{\sqrt{48}\binom{m+n-4}{n-2}} e^{\pi\sqrt{\frac{2}{3}\binom{m+n-4}{n-2}}}.$$

This is considerably smaller.

We come back finally to the contribution to $c'_m(m, 3)$. By the intersection condition there is at most one distinguished set of order $m-2$. So we have the following possibilities.

(1) There is a distinguished set of order $m-2$. The distinguished vertices are its members and there are no vertices v_{A_1,A_2}.
(2) There is no distinguished set of order $m-2$. There are $k \le m-1$ distinguished vertices and an equivalence relation is imposed on the remaining $m-1-k$ vertices.

Thus there is one structure of the first kind and there are $\sum_{k\le m-1} p(k)$ structures of the second kind.

Thus the contribution to $c'_m(m, 3)$ (ignoring overlap) is

$$1 + \sum_{k\le m-1} p(k).$$

This appears to come out to approximately $\sqrt{\frac{6m}{\pi}}p(m)$ using the standard approximation for $p(x)$, of about the same order of magnitude as $p(m)$.

8.2. **Case 4. A unique N-clique.** The hypotheses now are.

(1) G is existentially complete $(K_m + K_n)$-free with maximal clique size $N = m + n - 3$, where $m > n \geq 3$.
(2) There is a unique N-clique A in G.

Lemma 8.8. *Under the assumptions of Case 4 the N-clique A is balanced.*

Proof.

Claim 1. If K, K' are cliques in G and

$$|(K \cup K') \cap A| \geq N - 1,$$

then $K \cap K' \subseteq A$.

Otherwise, let $v \in (K \cap K') \setminus A$. Then $[(K \cup K') \cap A] \cup \{v\}$ is a clique of order at least N, contradiction our case assumption.

Now let (A_1, A_2) be a partition of A of type $(m-1, n-2)$ or $(m-2, n-1)$. If A_1 is not m-special and A_2 is not n-special then our usual construction produces cliques K_1, K_2 of orders m and n respectively with $K_1 \cap A \subseteq A_1$, $K_2 \cap A \subseteq A_2$. Furthermore if the type of the partition is $(m-1, n-2)$ then $K_2 \supseteq A_2$ and $|A_1 \setminus K_1| \leq 1$, while if the type of the partition is $(m-2, n-1)$ then $K_1 \supseteq A_1$ and $|A_2 \setminus K_2| \leq 1$. In either case $(K_1 \cup K_2) \cap A \geq N - 1$ and then our first claim forces K_1, K_2 to be disjoint, for a contradiction. □

As we have treated the balanced case earlier there is no need to deal separately with this case.

At this point we have identified a cofinal collection of maximal existentially complete graphs determining $c'_m(m, 3)$ and counted the numbers in each case, though we have also noticed some overlap between the contributions and there may be more.

The contributions were as follows.

Case 1	Case 2	Case 3	Case 4
1	$m+2$	$\sum_{k \leq m-1} p(k) + 1$	0

There is an overlap of at least 1 so at the moment we have

$$\sum_{k \leq m-1} p(k) + 1 \leq c'_m(m, 3) \leq \sum_{k \leq m-1} p(k) + m + 3,$$

with p the partition function. To get the exact number it would suffice to review the cases again and check the overlap.

So the complexity $c(m,3)$ is

$$c'_m(m,3) + m + 4 \approx \sum_{k \le m-1} p(k),$$

of about the same order of magnitude as $p(m)$.

The count is dominated by existentially complete $(K_m + K_3)$-free graphs with maximal clique size $N = m + n - 3$ having infinitely many N-cliques, all of which are extensions of a fixed $(N-1)$-clique by a single vertex.

9. Concluding remarks

The last few sections, preceding, ought really to deal with some more abstract issues, concerning the model companion T^* of the theory T of graphs omitting a collection of sums of complete graphs. We paid a good deal of attention to the algebraic closure of the empty set without actually mentioning it or giving any relevant theory. A more systematic approach would go as follows.

Conjecture 1. *In any model of T^* we have the following.*

(1) *The algebraic closure of any set is its union with the algebraic closure of the empty set.*
(2) *The algebraic closure of the empty set is finite, of bounded size (and, most likely, not very large, related to bounds on the sizes of maximal families of sets with intersection properties).*

Furthermore, the completions of T^* are given by specifying the algebraic closure of the empty set. and are $\aleph_0$-categorical.

The $\aleph_0$-categoricity would follow from the other statements, via local finiteness of the algebraic closure operation, modulo some generalities about types in T^*. All of this together would imply that the complexity of this class is the number of completions of T^*, and that all the existentially complete models are maximal existentially complete.

This analysis is not necessarily limited to forbidden subgraphs of the specified forms, but then the algebraic closure operator would become less trivial and we would require a local finiteness condition very much dependent on the structure of the forbidden subgraphs, as has already been seen in the case of connected constraints, where T^* is complete and the question is one of universality.

However all of that lies outside the possible scope of these notes.

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