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Introduction

In this paper we will prove the following theorem:

Theorem 1 *Let G be a simple L^*-group of even type with a weakly embedded subgroup. Then $G \cong \mathrm{PSL}_2(K)$ where K is an algebraically closed field of characteristic two.*

Here an L^*-group is a group of finite Morley rank such that every proper definable simple section of even type is an algebraic group (equivalently, a Chevalley group) over an algebraically closed field.

Theorem 1 follows from the following result in view of the main results of [7] and [8]:

Theorem 2 *Let G be a simple L^*-group of even type with a weakly embedded subgroup M. Then $C_G(A_1, A_2)$ is finite whenever A_1 and A_2 are two distinct conjugates of $\Omega_1(O_2{}^\circ(M))$.*

Our broader aim is to prove the following.

Conjecture 1 *Let G be a simple group of finite Morley rank of even type. Then G is algebraic.*

The proof we have in mind is modelled on the proof of the special case in which G is a simple K^*-group of even type. A K^*-group is a group of finite Morley rank in which every proper definable infinite simple section is algebraic (Chevalley) over an algebraically closed field. The difference between the two notions, K^* and L^*, is that L^*-groups are permitted, a priori, to have *simple definable degenerate sections* (with finite or even trivial Sylow 2-subgroup). Thus what we are proposing to do, ultimately, is a classification of certain simple groups of finite Morley rank in which we explicitly allow the Feit-Thompson Theorem to fail (rather badly), but arrive nonetheless at the desired result.

We view the result of the present paper as the main ingredient of the projected treatment of the L^* case, in the sense that it is this step which requires the greatest deviation from the methods used in the K^* situation. The next step would be the treatment of groups of finite Morley rank of even type with strongly closed abelian subgroups along the lines of [4], another case in which very substantial modifications are needed. Following on that case, the remainder of the proof, though quite long, would not deviate very strikingly from the analysis in the K^* case, in our opinion, though there are some additional points required.

Note (Fall 2004). At the present writing, the proof alluded to is the subject of a text in preparation in conjunction with Borovik [5]. The most striking deviations from the K^* case occur in the present paper and in the treatment of strongly closed abelian subgroups.

In the present article we will be working with a supposed counterexample to Theorem 2, and we will eventually arrive at a contradiction. Thus our standing assumption will be as follows.

$(\neg *)$ There exist A_1 and A_2, two distinct conjugates of $\Omega_1(O_2{}^\circ(M))$, such that $C_G(A_1, A_2)$ is infinite.

The condition $(\neg *)$ refers to the negation of the two conditions $(*)$ which were the respective starting points for [7] and [8]. However in these two papers the relevant condition $(*)$ is formulated in two different ways:

$\left(*_{[7]}\right)$ $C_G(A_1, A_2)$ is finite whenever A_1 and A_2 are two distinct conjugates of $\Omega_1(M)$.

$\left(*_{[8]}\right)$ $C_G(A_1, A_2)$ is finite whenever A_1 and A_2 are two distinct conjugates of $\Omega_1(O_2{}^\circ(M))$.

On the other hand, as explained in [7, p. 96], in the context of that paper (strong embedding) these two conditions are equivalent. This equivalence should not be assumed in the context of [8], however; it follows from the results of the present paper at a very late stage in the analysis. Accordingly, here we must adopt the formulation used in [8].

The proof of Theorem 2 follows the main strategy of Section 4 of [16], although the implementation of the ideas deviates substantially from that valuable source of inspiration, providing a cubist portrait of the original. The fact that this is possible can be traced back to the main result of [21], which will be reinterpreted as a theory of "good tori" (cf. Fact 1.27 below). We depend also on some useful general principles which were stressed in the introduction to [6], pp. 156–157, where they occur as Proposition 3.2, Proposition 3.4, Fact 4.19, and Fact 1.1, due variously to Borovik (personal communication), Poizat [19], and the first author [1]. (The point referred to as Proposition 3.4 in the introduction to [6] is mislabelled, and occurs in the body of [6] as Corollary 3.6.)

The basic idea of our analysis is to study conjugacy classes of certain tori, maximal tori in certain subgroups of type SL_2. A key technical result is the finiteness theorem given as Proposition 5.5, which is an application of Fact 1.37. This is a weak form of a conjugacy theorem, and in earlier treatments of similar material the conjugacy of Carter subgroups was used to derive analogous results. As we deal with groups of degenerate type which are not necessarily solvable, we lack both the Hall and the Carter theory where it was present in earlier analyses. Accordingly much of the first part of the present paper is devoted to providing a substitute for the missing conjugacy theorems. In Lemma 1.41 we prove a very general result on *generic covering* by conjugates, a rather formal one which is not in itself a conjugacy theorem, but does provide an important step toward such a theorem. To exploit this generic covering result, we need the theory of *good tori*, or more generally *rigid abelian groups,* as presented in §1. The technical Theorem 3, also given in §1, shows one important way in which "generic covering" can be converted into more precise information. It does not seem that results of this kind have been used in finite group theory or in algebraic group theory, though they are certainly valid in the latter case—in algebraic groups, conjugacy of maximal tori seems to cover this point in a more direct fashion.

The first three sections of the paper develop this and other general machinery, whose relevance is also seen in §3: Proposition 3.1 and, in a sharper form, Corollary 3.4. In the following three sections the relevant class $\mathcal{T}$ of good tori is introduced and the finiteness theorem, Proposition 5.5, is proved, along with a number of other results on this class. After that, for the next five sections we make the kind of analysis of the natural permutation representation of G that has been seen before—most clearly, in Jaligot's version [16]. The use of Proposition 5.5 as a starting point for this kind of analysis is new. One of the conclusions reached in the course of this analysis is that the class $\mathcal{T}$ consists in fact of a single conjugacy class under the action of M.

Finally, in the concluding section §12 we reach a contradiction using elements of order 3 in the manner of [16]. One can read [16] as using the theory of the Fitting subgroup to provide some explicit coset representatives, whereas we work directly with the appropriate cosets. The final contradiction we reach in our concluding argument is not the one of [16], but one may take as the underlying principle that if one is at all able to compute a sufficiently general part of the group operation in more than one way (which is of course the doubtful point), a contradiction of some sort is almost inevitable at this point.

1 Background

Fact 1.1 ([10, pp. 89, 92; Lemma 5.1]) *Let G be a group of finite Morley rank.*

1. *For $X \subseteq G$, $C_G(X) = C_G(d(X))$.*
2. *If B is a definable normal subgroup of G and $B \subseteq X \subseteq G$, then $d(X/B) = d(X)/B$.*
3. *If G is connected then G has Morley degree one.*

1.1 Nilpotent/Solvable

Fact 1.2 ([17], [10, §6.2]) *Let H be a nilpotent group of finite Morley rank.*

1. *$H = D * B$, where D and B are definable characteristic subgroups, with D divisible and B of bounded exponent. Moreover, $D \cap B$ is finite, and B is the direct sum of its p-torsion subgroups.*
2. *If H is divisible then $H = T \oplus N$ where T is the torsion subgroup of H and N is a torsion free subgroup. Moreover $T \leq Z(H)$.*

Fact 1.3 ([9]) *Let H be a solvable group of finite Morley rank, π a set of primes, and $N \lhd H$ a definable normal subgroup. Then the Hall π-subgroups of H are conjugate, and any Hall π-subgroup of H covers a Hall π-subgroup of H/N.*

Fact 1.4 ([10, Theorem 9.29];[15]) *For any set of primes π, the Hall π-subgroups of a connected solvable group of finite Morley rank are connected.*

Fact 1.5 ([16], Lemme 4.4; [3], Proposition 9.4) *Let $Q \rtimes X$ be a group of finite Morley rank where Q and X are definable. If Q is an abelian 2-group of bounded exponent and X is a $2^\perp$-group which centralizes the involutions of Q, then X centralizes Q.*

1.2 Elements of finite order

Fact 1.6 ([10, Exercise 11, p. 93]) *Let G be a group of finite Morley rank and H a normal definable subgroup. If $x \in G$ is such that $\overline{x} \in G/H$ is a p-element, then the coset xH contains a p-element.*

Definition 1.7 *A* p-torus *is a divisible abelian p-group. A* unipotent *group is a connected definable nilpotent group of bounded exponent.*

Fact 1.8 ([11]) *Let T be a p-torus in a group G of finite Morley rank. Then the index $[N_G(T) : C_G(T)]$ is finite. Moreover, there exists a uniform bound c such that for any p-torus contained in G, we have $[N_G(T) : C_G(T)] \leq c$*

Fact 1.9 (Borovik, cf. [7, Fact 2.14]) *Let $G = U \rtimes X$ be a group of finite Morley rank. with U and X definable. Let p be a prime number. Assume that U is a unipotent p-subgroup of G, and X is connected, solvable, and does not contain elements of order p. If the action of X on U is faithful, then X is divisible and abelian.*

1.3 Suzuki 2-groups

Definition 1.10 *A* Suzuki 2-group *is a pair (S, T) where S is a nilpotent 2-group of bounded exponent and T is an abelian group which acts on S by automorphisms, and which is transitive on the involutions of S.*

A Suzuki 2-group is said to be free *if T acts on S freely: for any $g \in S$ and $t \in T$, $g^t = g$ implies either $g = 1$ or $t = 1$.*

A Suzuki 2-group is said to be abelian *if S is abelian.*

A Suzuki 2-group is said to have finite Morley rank *if the structure (S, T) is of finite Morley rank.*

Fact 1.11 ([14]) *A free Suzuki 2-group of finite Morley rank is abelian.*

Remark 1.12 Davis and Nesin also show in [14] that if (S, T) is a free Suzuki 2-group, then S is not only commutative but homocyclic, and $(S/\Omega_i(S), T)$ is a free Suzuki 2-group as well. It follows from this that $\Omega_{i+1}(S)/\Omega_i(S)$ is an elementary abelian 2-group of rank $\mathrm{rk}\,(T)$.

1.4 Linearization

Definition 1.13 *Let G be a group of finite Morley rank, and K and H definable subgroups with H normalizing K. Then K is H-*minimal *if K is infinite and has no proper infinite definable H-invariant subgroup. Equivalently, K is a minimal connected nontrivial H-invariant definable subgroup.*

Fact 1.14 ([10, Theorem 9.5]) *Let $A \rtimes G$ be a connected group of finite Morley rank where G is definable, A is abelian and G-minimal, and $C_G(A) = 1$. Assume further that G has a definable infinite abelian normal subgroup H. Then $C_A(G) = 1$, H is central in G, $F = \mathbb{Z}[H]/ann_{\mathbb{Z}[H]}(A)$ is an interpretable algebraically closed field, A is a finite dimensional F-vector space, and the action of G on A is by vector space automorphisms; so $G \leq \mathrm{GL}_n(F)$ via this action, where n is the dimension. Furthermore, $H \leq Z(G) \leq Z(\mathrm{GL}_n(F))$.*

Fact 1.15 ([10, Theorem 9.7]) *Let $A \rtimes G$ be a group of finite Morley rank such that $C_G(A) = 1$. Let $H \lhd G_1 \lhd G$ be definable subgroups with G_1 connected and H infinite abelian. Assume also that A is G_1-minimal. Then $K = \mathbb{Z}[Z(G^\circ)]/ann_{\mathbb{Z}[Z(G^\circ)]}(A)$ is an interpretable algebraically closed field, A is a finite dimensional vector space over K, G acts on A as vector space automorphisms and H acts as scalars. In particular, $G \leq \mathrm{GL}_n(K)$ for some n, $H \leq Z(G)$ and $C_A(G) = 1$.*

Fact 1.16 ([19]) *If F is a field of finite Morley rank of characteristic $p \neq 0$, then every simple definable section of $\mathrm{GL}_n(F)$ is definably isomorphic to an algebraic group over F.*

1.5 Weak and strong embedding

Fact 1.17 ([2]) *Let G be a group of finite Morley rank, M a proper definable subgroup of G. M is weakly embedded if and only if the following hold:*

1. M has infinite Sylow 2-subgroups.

2. For any nontrivial unipotent 2-subgroup U and nontrivial 2-torus T in M, $N_G(U) \leq M$ and $N_G(T) \leq M$.

Corollary 1.18 *If G is a group of finite Morley rank and M a weakly embedded definable proper subgroup, then M contains a Sylow 2-subgroup of G.*

Fact 1.19 ([6]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup M. Then $M^\circ/{O_2}^\circ(M)$ is of degenerate type.*

1.6 Automorphisms

Fact 1.20 ([10, Theorem 8.4]) *Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where G and H are definable, G is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing H as a subgroup of $\mathrm{Aut}(G)$, we have $H \leq \mathrm{Inn}(G)\Gamma$ where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms.*

Remark 1.21 We will frequently use the special case of Fact 1.20 with $G = \mathrm{PSL}_2$. Here, as there are no nontrivial graph automorphisms, all definable actions induce inner automorphisms.

Fact 1.22 ([10, Exercise 13, p. 78]) *Let α be a definable involutory automorphism of a connected group of finite Morley rank G, whose centralizer in G is finite. Then α acts by inversion on G.*

1.7 L-groups

Definition 1.23 *1. An L-group is a group of finite Morley rank such that every definable simple section of even type is an algebraic group (equivalently, a Chevalley group) over an algebraically closed field.*

2. For G a group of finite Morley rank, $B(G)$ denotes the subgroup generated by all 2-unipotent subgroups of G.

Fact 1.24 ([6, Lemma 3.12]) *Let H be a connected L-group of even type with a weakly embedded subgroup M. Then*

$$H \cong L \times D$$

where $L = B(H) \cong \mathrm{SL}_2(F)$, with F algebraically closed of characteristic two, and $D = C_H(L)$ is a subgroup of degenerate type. $M^\circ \cap L$ is a Borel subgroup of L and $D \leq M$.

1.8 Fields, good tori, rigid abelian groups

Definition 1.25 *A definable divisible abelian group T of finite Morley rank is a* good torus *if every definable subgroup of T is the definable closure of its torsion.*

Lemma 1.26

1. *If T is a good torus and $T_0 \leq T$ is definable and connected, then T_0 is a good torus.*
2. *Let G be a group of finite Morley rank which is an extension of a definable good torus by a good torus. Then G is a good torus.*
3. *The product of two normal definable good tori in a group of finite Morley rank is a good torus.*
4. *A connected group of finite Morley rank which is finite-by-(good torus) is a good torus.*

Proof. The first point is clear.

Let us prove (2). Note that the assumptions imply that G is connected. Let A be the normal definable good torus in G such that G/A is also a good torus. As G is connected, $A = d(\mathrm{Tor}(A))$, and A is divisible abelian, Facts 1.1 (1) and 1.8 imply that $A \leq Z(G)$. Hence G is nilpotent. Since G is divisible-by-divisible, it follows from Fact 1.2 (1) that G is divisible. The assumptions imply that $d(\mathrm{Tor}(G)) \geq A$. Since G/A is a good torus we then have $G = d(\mathrm{Tor}(G))$, in view of Fact 1.3. Now Fact 1.2 (2) together with Fact 1.1 (1) yields that G is abelian. Finally, if H is a definable subgroup of G, $HA/A = d(\mathrm{Tor}(HA/A))$ by the assumption that G/A is a good torus. By Fact 1.3, the torsion subgroup of H covers the torsion subgroup of $H/H \cap A$, so $\mathrm{Tor}(H)$ covers $\mathrm{Tor}(HA/A)$. By Fact 1.1 (2) we find $HA/A = d(\mathrm{Tor}(H))A/A$. But HA/A is definably isomorphic to $H/H \cap A$, and $H \cap A = d(\mathrm{Tor}(H \cap A))$. So we have $H = d(\mathrm{Tor}(H))$.

Item (3) follows from (2). The last point is immediate. □

The next point is a reformulation of the main result of [21].

Fact 1.27 ([7]) *Let F be a field of finite Morley rank and of nonzero characteristic. Then $F^\times$ is a good torus.*

For our purposes, the saturation hypothesis in the next result is harmless, as we are always free to pass to an elementary extension. However, it is also unnecessary, as good tori remain good in elementary extensions (see [12, Appendix]), a result which will not be used here. The reader who would prefer a "saturation-free" argument can invoke the aforementioned result, and systematically ignore our occasional remarks on this subject.

Fact 1.28 ([7, Lemma 3.12]) *Let D be a good torus in an $\aleph_0$-saturated structure. Then every uniformly definable collection of subgroups of D is finite.*

Corollary 1.29 *Let G be an $\aleph_0$-saturated group of finite Morley rank and D be a definable good torus in G which is covered by a uniformly definable family $\mathcal{F}$ of definable subgroups of G. Then D is contained in one of the subgroups in $\mathcal{F}$.*

Proof. The family $\{H \cap D : H \in \mathcal{F}\}$ is also uniformly definable, hence finite. □

The following notion extends that of good torus.

Definition 1.30 *An abelian group of finite Morley rank is said to be* rigid *if its connected component is a good torus.*

Lemma 1.31 *Let G be a group of finite Morley rank and A a definable, rigid abelian subgroup of G. Then*

1. *A° is divisible abelian.*
2. *Every definable subgroup of A is the definable closure of its torsion.*
3. *$N_G{}^\circ(A) = C_G{}^\circ(A)$.*

Proof.

The first statement is an immediate consequence of the definitions of a rigid abelian group and of a good torus.

As for the second statement, let B be a definable subgroup of A. Since B is abelian, Fact 1.6 implies that B/B° is covered by a finite subgroup of B. The conclusion follows from this and the definition of a good torus.

We proceed for the third statement as follows. We have $N_G{}^\circ(A) \leq N_G(A^\circ)$. By Fact 1.8 and the definition of a good torus, we then have $N_G{}^\circ(A^\circ) = C_G{}^\circ(A^\circ)$. Since A is abelian, again $A = A^\circ A_0$ where A_0 is a finite group. Furthermore, as A° is divisible, the set of torsion elements of any fixed exponent is finite, and we may choose A_0 to be characteristic in A. It follows that $N_G{}^\circ(A)$ centralizes A_0, and thus A. $\square$

The same argument as in the proof of Fact 1.28 shows *mutatis mutandis* the following statement.

Lemma 1.32 *Let A be a rigid abelian group in an $\aleph_0$-saturated structure. Then every uniformly definable collection of subgroups of A is finite.*

Definition 1.33 *Let $H \leq K$ be groups interpreted in a third group G, which has finite Morley rank. We say that H is* continuously characteristic *in K (relative to G) if H is X-invariant for every connected subgroup X of* $\mathrm{Aut}(K)$ *for which $K \rtimes X$ is interpreted in G (in other words, the group X together with its K-action is interpreted in G).*

Typically we are concerned with the case $H \leq K \leq G$; in this case, if H is continuously characteristic in K then it is $N_G{}^\circ(K)$-invariant, which is usually the point.

Lemma 1.34 *Let G be a group of finite Morley rank, $H \lhd K$ a pair of connected definable subgroups of G with K/H a good torus. Then H contains a nontrivial connected, definable, continuously characteristic subgroup of K.*

Proof.

If the commutator subgroup K' is nontrivial, then as it is definable, connected, and contained in H, it will do. Assume therefore that K is abelian.

If K is not divisible then for some prime p the annihilator of p in K is infinite (Fact 1.2), and its connected component is contained in H. So we may suppose K is divisible.

Now in view of Lemma 1.26 (3), K contains a unique maximal good subtorus T, and T is centralized by any connected group of automorphisms of K interpreted in G.

Hence every good subtorus of K is continuously characteristic in K. If H contains a good torus, we are done.

Suppose therefore that H contains no good torus. Let T be a minimal infinite definable subgroup of H. If T contains torsion, then T is the definable closure of its torsion, and by minimality T is then a good torus, a contradiction. So T is torsion-free. Now K contains a maximal torsion-free definable divisible abelian subgroup T_0, whose image in K/H is a torsion-free subgroup of a good torus (Fact 1.6), hence trivial. Thus $T_0 \leq H$ is the desired continuously characteristic subgroup. □

The following general conjugacy theorem was noticed only recently. It simplifies the considerations of Section 5 substantially. Possibly some of the other arguments relating to conjugation or generic covering can also be simplified by fuller use of this result.

Fact 1.35 ([12]) *Let G be a group of finite Morley rank. Then any two maximal good tori in G are conjugate.*

The proof goes largely by induction on rank, making use of the following result, which is useful in its own right.

Fact 1.36 (Generic Covering) *Let G be a connected group of finite Morley rank, T a good torus in G, and $H = C^\circ(T)$. Then $\bigcup H^G$ is generic in G.*

The conjugacy theorem has the following corollary, which makes use also of Lemma 1.28. It is proved in [12] without the hypothesis of saturation, but we may limit ourselves here to this weaker version, which is sufficient for our purposes.

Fact 1.37 ([12]) *Let G be an $\aleph_0$-saturated group of finite Morley rank, and $\mathcal{F}$ a uniformly definable family of good tori contained in G. Then under the action of G, the groups in $\mathcal{F}$ fall into finitely many conjugacy classes.*

We will use this result to prove Proposition 5.5 below. Our original proof of that special case was more complicated, but motivated the proof in [12] of the more general statement.

1.9 Genericity

Fact 1.38 ([13, Lemma 3.3]) *Let G be a connected group of finite Morley rank and B a definable subgroup of G of finite index in its normalizer. Assume that there is a definable subset X of B, not generic in B, such that $B \cap B^g \subseteq X$ whenever $g \in G \setminus N_G(B)$. Then $\cup_{g \in G} B^g$ is generic in G.*

Definition 1.39 *Let H be a definable subgroup of a group of finite Morley rank G, not necessarily connected. Then we say that H is* generically disjoint from its conjugates *in G if $H \setminus \bigcup_{g \in G \setminus N(H)} H^g$ is generic in H.*

Note that since we allow H to be disconnected, it may meet a conjugate in a generic subset, while still being generically disjoint from its conjugates.

The following lemma generalizes Fact 1.38 and it has the same proof.

Lemma 1.40 *Let G be a connected group of finite Morley rank and B a definable subgroup of G of finite index in its normalizer which is generically disjoint from its conjugates in G. Then $\cup_{g\in G}B^g$ is generic in G.*

Proof. Let $X = \cup_{g\in G\setminus N(B)} B^g$.

We use the following mapping

$$\begin{array}{cccc} \Psi & : & G\times(B\setminus X) & \longrightarrow & \bigcup_{g\in G}(B\setminus X)^g \\ & & (g,b) & \longmapsto & b^g \end{array}$$

to compare $\mathrm{rk}\,(G)$ to $\mathrm{rk}\,(\bigcup_{g\in G} B^g)$.

For $b_1,\ b_2 \in (B\setminus X)$ and $g_1,\ g_2\in G$, if $b_1^{g_1} = b_2^{g_2}$ then $b_1 \in B\cap B^{g_1g_2^{-1}}$, and the choice of b_1 and the definition of X imply that $g_1g_2^{-1}\in N_G(B)$. It follows that the fiber of $b_1^{g_1}$ is the set $\{(g_2, b_1^{g_1g_2^{-1}}) : g_1g_2^{-1}\in N_G(B)\}$. This set corresponds bijectively and definably to a coset of $N_G(B)$. Hence the rank of the fibers of Ψ is $\mathrm{rk}\,(N_G(B))$. It follows, using general properties of rank, that we have

$$\mathrm{rk}\,(G) + \mathrm{rk}\,(B\setminus X) = \mathrm{rk}\,(\bigcup_{g\in G}(B\setminus X)^g) + \mathrm{rk}\,(N_G(B))$$

But B is of finite index in its normalizer and $B\setminus X$ is generic in B. As a result, $\mathrm{rk}\,(B\setminus X) = \mathrm{rk}\,(B) = \mathrm{rk}\,(N_G(B))$ and

$$\mathrm{rk}\,(G) = \mathrm{rk}\,(\bigcup_{g\in G}(B\setminus X)^g) \le \mathrm{rk}\,(\bigcup_{g\in G} B^g)$$

□

Lemma 1.41 *Let G be a connected group of finite Morley rank, and let K_1, H_1, H_2 be definable subgroups of G, with $K_1 \le H_1$. Suppose in addition:*

1. *${N_G}^\circ(K_1) = {K_1}^\circ$;*
2. *K_1 is generically disjoint from its conjugates in G;*
3. *$\bigcup_{g\in G} H_2^g$ is generic in G.*

Then the conjugates of H_2 in G generically cover H_1, in the sense that $H_1\cap\bigcup_{g\in G} H_2^g$ is generic in H_1.

Proof.

Let $X = K_1\setminus\bigcup_{g\in G\setminus N(K_1)} K_1^g$. Then X is $N(K_1)$-invariant, and indeed $N(K_1)$ is the full stabilizer of X in G, while distinct conjugates of X are pairwise disjoint. By hypothesis, $\mathrm{rk}\,(X) = \mathrm{rk}\,(K_1)$. It follows that for $Y\subseteq X$ $N(K_1)$-invariant, and for any definable group H containing K_1, we have $\mathrm{rk}\,(\bigcup_{g\in H} Y^g) = \mathrm{rk}\,(Y) + \mathrm{rk}\,(H/N_H(K_1)) = \mathrm{rk}\,(H) + \mathrm{rk}\,(Y) - \mathrm{rk}\,(K_1)$ and thus $\bigcup_{g\in H} Y^g$ is generic in H if and only if Y is generic in X.

Taking $H = H_1$ here, we see that it will suffice to show that $Y = X\cap\bigcup_{g\in G} H_2^g$ is generic in K_1.

Set $Z = \bigcup_{g\in G} H_2^g$. If $X\setminus Z$ is generic in K_1, then $\bigcup_{g\in G}(X\setminus Z)^g$ is generic in G, by our first remark, with $H = G$. But $\bigcup_{g\in G}(X\setminus Z)^g \subseteq (G\setminus Z)$, and Z is generic in G by hypothesis. As G is connected, this is a contradiction. Thus $X\setminus Z$ is nongeneric in K_1, and $X\cap Z$ is generic in K_1, as desired. □

Fact 1.42 ([7, Lemma 4.6]) *Let G be a connected group of finite Morley rank. Assume that B is a good torus which is of finite index in $N_G(B)$. Then the set $\mathcal{B} = \cup_{g \in G} B^g$ is generic in G.*

Fact 1.43 ([7, Cor. 4.8]) *Let $A \rtimes G$ be a group of finite Morley rank where G and A are definable. Assume that A is connected and elementary abelian of exponent two, that G is connected of degenerate type, and that G acts faithfully on A. If B is a Borel subgroup of G then $B \cap (\cup_{g \in G \setminus N_G(B)} B^g)$ is not generic in B, and the Borel subgroups of G are conjugate in G.*

The next lemma encapsulates a form of genericity argument, based on the irreducibility of connected groups, which has played a significant role since the first analysis of "bad groups".

Fact 1.44 ([13, Lemma 3.4]) *Let G be a connected group of finite Morley rank, and B a proper definable connected subgroup of finite index in its normalizer in G, such that $\cup_{g \in G} B^g$ is generic in G. Assume that $x \in N_G(B) \setminus B$ is of finite order modulo B, and let $\langle x \rangle B$ be the setwise product ($\bigcup \{yB : y \in \langle x \rangle\}$). Then the following definable subset X of xB is generic in xB:*

$$X = \{x_1 \in xB : x_1 \in (\langle x \rangle B)^g \textit{ for some } g \in G \setminus N_G(B)\}$$

For the remainder of this subsection, we turn from criteria for genericity to consequences of genericity, aiming at Theorem 3 below.

Fact 1.45 ([18]) *Let $\mathcal{C}$ be a uniformly definable family of subgroups in a group G of finite Morley rank. Then the indices $[H : H^\circ]$ ($H \in \mathcal{C}$) are uniformly bounded.*

Lemma 1.46 *Let G be a group of finite Morley rank and $\mathcal{F} = \{D_i : i \in I\}$ be a uniformly definable family of subgroups of G whose connected components are divisible groups. Then $\mathcal{F}^\circ = \{{D_i}^\circ : i \in I\}$ is a uniformly definable family.*

Proof. By Fact 1.45 there exists $m \in \mathbb{N}$ such that $|D_i : {D_i}^\circ| \leq m$ for every $i \in I$. As a result, for any $i \in I$, if $x \in D_i$ then $x^{m!} \in {D_i}^\circ$. Since by hypothesis, ${D_i}^\circ$ is divisible, we conclude that ${D_i}^\circ = D_i^{m!}$ for every $i \in I$. □

Lemma 1.47 *Let A be a group of finite Morley rank with A/A° cylic and $\mathcal{C}$ a finite set of proper definable subgroups of A. Then* $\mathrm{rk}\,(A \setminus \bigcup \mathcal{C}) = \mathrm{rk}\,(A)$.

Proof. As A/A° is cyclic, there is a coset C of A° in A which is disjoint from every proper subgroup of A containing A°. The intersection of this coset with finitely many definable subgroups of A which do not contain A° is nongeneric in that coset. That is, $\mathrm{rk}\,(C \setminus \bigcup \mathcal{C}) = \mathrm{rk}\,(C) = \mathrm{rk}\,(A)$. □

Corollary 1.48 *Let G be an $\aleph_0$-saturated group of finite Morley rank and D be a definable rigid abelian subgroup with D/D° cyclic. If D is covered by a uniformly definable family $\mathcal{F}$ of definable subgroups of G, then D is contained in one of the subgroups in $\mathcal{F}$.*

Proof. Consider the family $\{H \cap D : H \in \mathcal{F}\}$, which is finite by Lemma 1.32. □

Theorem 3 *Let G be an $\aleph_0$-saturated group of finite Morley rank, and $\mathcal{C}$ a uniformly definable family of rigid abelian subgroups such that $\bigcup \mathcal{C}$ is generic in G. Then there exists $T \in \mathcal{C}$ such that T° is a maximal good torus in $C(T)$; or equivalently, if $T \leq \tilde{T}$ with $\tilde{T}$ rigid abelian, then $[\tilde{T} : T] < \infty$.*

Proof. $\mathcal{C}$ is a definable subset of G^{eq}. We may treat its defining parameters as constants, and assume that it is 0-definable.

We first make some adjustments to $\mathcal{C}$. By Fact 1.45, there is a finite bound $m = m(\mathcal{C})$ on $[A : A^\circ]$ for $A \in \mathcal{C}$. Thus $m!A = A^\circ$ for $A \in \mathcal{C}$, and A° is uniformly definable from A. It follows easily that the set $\mathcal{C}' = \{A^\circ\langle a\rangle : A \in \mathcal{C}, a \in A\}$ is another uniformly definable family such that $\bigcup \mathcal{C}' = \bigcup \mathcal{C}$, and for each $A \in \mathcal{C}'$ we have A/A° cyclic. Since each group in $\mathcal{C}'$ is a subgroup of finite index in some group in $\mathcal{C}$, if we prove our claim for $\mathcal{C}'$ then it follows for $\mathcal{C}$. So from now on we will write $\mathcal{C}$ for $\mathcal{C}'$.

Suppose towards a contradiction that for every $T \in \mathcal{C}$ there exists $\tilde{T}$, a rigid abelian subgroup of G, such that $T \leq \tilde{T}$ and $[\tilde{T} : T] = \infty$. We may suppose further that $\tilde{T} = T \cdot \tilde{T}^\circ$.

Then $\tilde{T}$ is defined by a formula $\varphi_{\tilde{T}}(x, \overline{a})$ with parameters from G. We associate to any formula $\varphi(x, \overline{y})$ the family

$$\mathcal{C}_\varphi = \{\varphi(G, \overline{g}) : \varphi(G, \overline{g}) \text{ is an abelian group}\}$$

where $\overline{g}$ varies over G. As the set

$$\{\overline{g} \in G^{l(\overline{g})} : G \models \{x : \varphi(x, \overline{g})\} \text{ is an abelian group}\}$$

is 0-definable, the family $\mathcal{C}_\varphi$ is a 0-definable subset of G^{eq}. In particular $\tilde{T} \in \mathcal{C}_{\varphi_{\tilde{T}}}$. So far, all we have done is to put the group $\tilde{T}$ into a family of abelian groups which is uniformly definable over $\emptyset$. However the family need not consist exclusively of *rigid* abelian groups, so we refine this further.

Take m_φ so that $[A : A^\circ] \leq m_\varphi$ for $A \in \mathcal{C}_\varphi$. Let us introduce the abbreviation "$B \ll_\varphi A$" to stand for the condition

$$[A : m_\varphi!A] \leq m_\varphi \ \&\ A = B \cdot m_\varphi!A \ \&\ m_\varphi!B < m_\varphi!A$$

This will be applied only when $A \in \mathcal{C}_\varphi$ and $B \in \mathcal{C}$, in which case $m_\varphi!B = B^\circ$, and the condition is equivalent to

$$m_\varphi!A = A^\circ \ \&\ A = B \cdot A^\circ \ \&\ [A : B] = \infty$$

As actually phrased, however, it is clearly first order.

Let $\mathcal{C}^*_\varphi$ be

$$\{A \in \mathcal{C}_\varphi : \exists B \in \mathcal{C}\ B \ll_\varphi A\}$$

Then for $A \in \mathcal{C}^*_\varphi$, we have $A^\circ = m_\varphi!A$, and the quotient A/A° is cyclic. In particular $\deg(A)$ is uniformly definable from A for $A \in \mathcal{C}^*_\varphi$.

Since $\tilde{T}$ is rigid, the set of intersections $\{\tilde{T} \cap A : A \in \mathcal{C}^*_{\varphi_{\tilde{T}}} \cup \mathcal{C}\}$ is finite (Lemma 1.32), of size $k_{\tilde{T}}$, say. For any finite k and any formula $\varphi(x, \overline{y})$, we may consider the family

$$\mathcal{C}_{\varphi,k} = \{A \in \mathcal{C}^*_\varphi : |\{A \cap B : B \in \mathcal{C}^*_\varphi \cup \mathcal{C}\}| \leq k\}$$

The family $\mathcal{C}_{\varphi,k}$ is uniformly definable over $\emptyset$ (i.e., 0-definable as a subset of G^{eq}) since $\mathcal{C} \cup \mathcal{C}^*_\varphi$ is, and k is fixed.

By our choice of $k_{\tilde{T}}$, we have $\tilde{T} \in \mathcal{C}_{\varphi_{\tilde{T}}, k_{\tilde{T}}}$.

The preceding discussion may be summarized as follows:

($*$) For every $T \in \mathcal{C}$ there exists a finite number k, a formula φ, and some $A \in \mathcal{C}_{\varphi,k}$ such that $T \ll_\varphi A$.

We claim next that condition ($*$) holds uniformly: there exist finitely many pairs of the form $(\varphi_1, k_1), \ldots, (\varphi_n, k_n)$, consisting of formulas φ_i and natural numbers k_i as in ($*$), such that for any $T \in \mathcal{C}$ the pair (φ, k) in ($*$) can be taken to be one of the (φ_i, k_i).

Indeed, consider the following 1-type $p(S)$ in G^{eq}, where φ varies over all formulas defined over $\emptyset$ and k varies over all natural numbers.

$$S \in \mathcal{C}; \qquad \neg\exists X \in \mathcal{C}_{\varphi,k}\,(S \ll_\varphi X)$$

Observe that the cardinality of this 1-type is at most the cardinality of the language $|\mathcal{L}|$.

By condition ($*$), the type $p(S)$ is not realized in G^{eq}. However, we may take G to be $|\mathcal{L}|$-saturated, and conclude that $p(S)$ is inconsistent. Hence there are finitely many formulas φ_i and natural numbers k_i such that

$$S \in \mathcal{C} \Rightarrow \exists i \leq n\, \exists X \in \mathcal{C}_{\varphi_i,k_i}(S \ll_{\varphi_i} X)$$

This is the desired uniformity.

Let $\mathcal{C}_i = \mathcal{C}_{\varphi_i,k_i}$. Before proceeding, it will be convenient to modify this choice of the $\mathcal{C}_i$. We would like the rank and degree $\mathrm{rk}\,(A), \deg(A)$ for $A \in \mathcal{C}_i$ to be constant; this is achieved by partitioning $\mathcal{C}_i$ into finitely many subsets on which the rank and degree are constant—and the defining formula φ_i is altered accordingly, while n, the number of formulas, increases. Let $r_i = \mathrm{rk}\,(A)$ for $A \in \mathcal{C}_i$ (a constant), and similarly $d_i = \deg(A)$ for $A \in \mathcal{C}_i$. We will write $\ll_i$ for $\ll_{\varphi_i}$.

Now with φ_i, k_i $(1 \leq i \leq n)$ as described, let $\mathcal{C}_i = \mathcal{C}_{\varphi_i,k_i}$ and set

$$\mathcal{C}^i = \{T \in \mathcal{C} : \exists X \in \mathcal{C}_i\,(T \ll_i X)\}$$

Then $\mathcal{C}^i$ is a uniformly definable family, over $\emptyset$.

We now pass to rank computations. We have $\bigcup \mathcal{C}$ generic in G, and $\mathcal{C}$ is the union of the $\mathcal{C}^i$, so for some i the union $\bigcup \mathcal{C}^i$ is generic in G.

Let $\mathcal{C}'_i = \{A \in \mathcal{C}_i : \exists B \in \mathcal{C}^i\, B \ll_i A\}$. To reach a contradiction it suffices to show that $\mathrm{rk}\,\left(\bigcup \mathcal{C}^i\right) < \mathrm{rk}\,(\bigcup \mathcal{C}'_i)$.

For $A \in \mathcal{C}'_i$, let $X_A = \bigcup\{B \in \mathcal{C} : B \ll_i A\}$ and let $Y_A = \bigcup\{A \cap B : B \in \mathcal{C}_i, B \neq A\}$. Note that if $A \neq B$ with $A, B \in \mathcal{C}_i$, then $A \cap B < A$, as $\mathrm{rk}\,(A) = \mathrm{rk}\,(B)$ and $\deg(A) = \deg(B)$. Thus X_A and Y_A are unions of proper subgroups of A, and by the definition of the classes $\mathcal{C}_{\varphi,k}$ only finitely many subgroups are involved. We consider these two sets in more detail.

The subgroups making up X_A have infinite index in A, so their union has rank less than r_i. Furthermore $Z_A = A \setminus Y_A$ has rank r_i by Lemma 1.47

As $Z_A \cap Z_B = \emptyset$ and $\mathrm{rk}\, Z_A = r_i$ for $A \neq B$ in $\mathcal{C}'_i$, we have $\mathrm{rk}\,(\bigcup \mathcal{C}'_i) \geq r_i + \mathrm{rk}\,\mathcal{C}'_i$. On the other hand $\mathrm{rk}\,\left(\bigcup \mathcal{C}^i\right) \leq \mathrm{rk}\,(\bigcup\{X_A : A \in \mathcal{C}'_i\}) < r_i + \mathrm{rk}\,\mathcal{C}'_i$. Thus $\mathrm{rk}\,\left(\bigcup \mathcal{C}^i\right) < \mathrm{rk}\,(\bigcup \mathcal{C}'_i)$, as claimed. □

2 Preliminaries

G is a simple L^*-group of even type with a weakly embedded subgroup M. The main results of [7, 8] apply to G. In this section we describe the general structure of G, and in particular that of M, under the hypothesis $(\neg *)$.

Proposition 5.10 in [6] and Theorem 3 in [8] show that $A = \Omega_1(O_2{}^\circ(M))$ is a definable, connected, elementary abelian group that is central in $O_2{}^\circ(M)$. Moreover Fact 1.19 implies that $M^\circ/O_2{}^\circ(M)$ is of degenerate type. Thus, $M^\circ/C_{M^\circ}(A)$ is of degenerate type. These remarks will be used in the sequel without mention. Furthermore, we have the following.

Fact 2.1 ([8, Fact 3.6 (1)]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup M. If $i \in I(O_2{}^\circ(M))$ then $C_i \leq M$.*

We may assume that one of the A_i mentioned in the hypothesis $(\neg *)$ is A. Let H be a maximal, definable, connected subgroup in $C(A)$ such that $C(H) \not\leq M$. By Fact 1.17, H is of degenerate type. The hypothesis $(\neg *)$ implies $H \neq 1$. Then $C_G(H) < G$ and $C_G{}^\circ(H)$ is an L-group with a weakly embedded subgroup $M \cap C_G{}^\circ(H)$ and Fact 1.24 applies. Thus we conclude that $B(C_G(H)) \cong \mathrm{PSL}_2$ in characteristic two, and $B(C_G(H)) \cap M = A \rtimes T$ where $A \rtimes T$ is a Borel subgroup of $B(C_G(H))$, T being a maximal torus of that Borel subgroup.

We let $L = B(C_G(H))$.

Lemma 2.2 *For $i \in I(A)$, we have $M = C_M(i)T$, as well as $M^\circ = C_{M^\circ}(i)T$, and $C_M(i) \cap T = 1$.*

Proof. Since $A \rtimes T$ is a Borel subgroup of $B(C_G(H))$, the structure of Borel subgroups in PSL_2 over fields of characteristic two implies that T acts on $I(A)$ regularly. In particular this action is transitive. The first part of the statement follows from this transitivity and the fact that $T \leq M^\circ$. That $C_M(i) \cap T = 1$ is a consequence of the regularity of the action. □

Proposition 2.3 *For $i \in I(A)$, we have $C_M{}^\circ(i) = C_M{}^\circ(A) = C_{M^\circ}(i) = C_{M^\circ}(A)$. In particular, $M^\circ = C_{M^\circ}(A) \rtimes T$.*

Proof. We let $\overline{M^\circ} = M^\circ/C_{M^\circ}(A)$. We claim that $C_M{}^\circ(i) = C_{M^\circ}(i)$. Since by Lemma 2.2 $C_M(i) \cap T = 1$, we have $1 = \deg(M^\circ) = \deg(C_{M^\circ}(i))\deg(T) = \deg(C_{M^\circ}(i))$. Hence, $C_{M^\circ}(i) = C_{M^\circ}{}^\circ(i) = C_M{}^\circ(i)$.

Now, suppose towards a contradiction that $C_M{}^\circ(i) > C_M{}^\circ(A)$. Hence $\overline{C_M{}^\circ(i)}$ is an infinite definable connected subgroup of $\overline{M^\circ}$. Note also that $\overline{T} \cong T$ because $C_T(A) = 1$ by the structure of $A \rtimes T$. $\overline{T}$ is contained in a Borel subgroup of $\overline{M^\circ}$. We will show that $\overline{T}$ is indeed a Borel subgroup of $\overline{M^\circ}$. By Fact 1.9, the Borel subgroups of $\overline{M^\circ}$ are divisible abelian. Thus it suffices to show that $C_{\overline{M^\circ}}{}^\circ(\overline{T}) = \overline{T}$. Since $\overline{T}$ acts transitively on $A^\times$, Facts 1.14 and 1.16 imply that $C_{\overline{M^\circ}}{}^\circ(\overline{T})$ is solvable, as there are no simple algebraic groups with abelian Borel subgroups. Hence, $C_{\overline{M^\circ}}{}^\circ(\overline{T})$ is abelian by Fact 1.9. But by Lemma 2.2, $\overline{M^\circ} = \overline{C_{M^\circ}(i)T}$, and the transitive action of $\overline{T}$ on $A^\times$ forces $C_{\overline{M^\circ}}{}^\circ(\overline{T}) \cap \overline{C_{M^\circ}(i)} \leq \overline{C_{M^\circ}(A)} = 1$. As a result $C_{\overline{M^\circ}}{}^\circ(\overline{T}) = \overline{T}$.

Since $\overline{T}$ is a Borel subgroup of $\overline{M^\circ}$ and $\overline{C_{M^\circ}(i)}$ is infinite, Fact 1.43 implies that a conjugate of $\overline{T}$ in $\overline{M^\circ}$ intersects $\overline{C_{M^\circ}(i)}$ nontrivially. This contradicts the action of $\overline{T}$ on A. It follows that $C_M{}^\circ(i) = C_M{}^\circ(A)$.

Since by Lemma 2.2, $M^\circ = C_{M^\circ}(i)T$ and $C_{M^\circ}(i) = C_M{}^\circ(i)$ as remarked above, we also have $M^\circ = C_M{}^\circ(A) \rtimes T$. The equality $C_M{}^\circ(A) = C_{M^\circ}(A)$ then follows from the fact that $C_M{}^\circ(A) \leq C_{M^\circ}(A)$ and the structure of $A \rtimes T$: $C_{M^\circ}(A) \leq C_{M^\circ}(i) = C_M{}^\circ(i) = C_{M^\circ}(A)$. □

Lemma 2.4 *For $i \in I(A)$ we have $C_G(i) = C_M(A)$. In particular $M = C(A) \rtimes T$.*

Proof. By Fact 2.1 we have $C_G(i) \subseteq M$.

Now let $\overline{M} = M/C_M(A)$. Note that by the structure of $A \rtimes T$, $\overline{T} \cong T$. By Proposition 2.3, $M^\circ = C_{M^\circ}(A) \rtimes T$, and thus $\overline{M}^\circ = \overline{T}$. We apply Fact 1.15 with $\overline{M}$ for G, A for A, $\overline{T}$ for G_1 and H. Since $\overline{T}$ acts transitively on $A^\times$, the vector space is 1-dimensional and $\overline{T} = \overline{M}$. Thus $M = C(A) \rtimes T$ and $C_M(i) = C(A)$. The conclusion follows. □

Proposition 2.5 *Let i and j be involutions in G that are conjugate to involutions in A. For $a \in G$, if $a^i = a$ and $a^j = a^{-1}$, then $a^2 = 1$.*

Proof. The proof of this statement follows two distinct paths according to whether M is strongly embedded or weakly but not strongly embedded. The first is Corollary 5.11 in [6] while the second is Proposition 5.7 in [8]. □

Recall that $L = B(C_G(H))$.

Lemma 2.6 ([16, Lemme 4.2]) *Let X be a nonempty subset of H different from $\{1\}$.*

1. $B(C_G(X)) = L$.
2. ${C_G}^\circ(L) = H$.
3. ${N_G}^\circ(X) \leq HL$.
4. ${N_G}^\circ(X) \cap {N_G}^\circ(T) \leq HT$.

Proof. 1. Since $C_G(X) \not\leq M$ and $C_G(X) < G$, ${C_G}^\circ(X)$ is an L-group with ${C_G}^\circ(X) \cap M$ a weakly embedded subgroup. Hence Fact 1.24 applies to yield $B(C_G(X)) \cong \mathrm{PSL}_2$ in characteristic two. Since $A \leq L \leq B(C_G(X))$, the sizes of the two PSL_2 match and we have equality.

2. As $H \leq {C_G}^\circ(L) \leq C(A)$, this follows from the maximality of H.

3. By (1), ${N_G}^\circ(X) \leq {N_G}^\circ(L) = L \cdot {C_G}^\circ(L)$, the last by Fact 1.20, so we conclude by (2).

4. It follows from the previous point that ${N_G}^\circ(X) \cap {N_G}^\circ(T) \leq {N_{HL}}^\circ(T)$, and ${N_{HL}}^\circ(T) = HT$. □

Corollary 2.7 *For any $g \in G \setminus N_G(H)$, $H \cap H^g = 1$.*

Proof. Suppose $g \in G$ is such that $X = H \cap H^g \neq 1$. Then by Lemma 2.6 (1) applied to H and H^g, $B(C_G(H)) = B(C_G(X)) = B(C_G(H^g))$. The conclusion follows from the second part of the same lemma. □

Lemma 2.8 ([16, Lemme 4.4]) $A = {O_2}^\circ(M)$.

Proof. Recall that by definition $A = \Omega_1({O_2}^\circ(M))$. The action of T on A proves that the structure $({O_2}^\circ(M), T)$ is a free Suzuki 2-group. By Fact 1.11, ${O_2}^\circ(M)$ is abelian.

Let $S = {O_2}^\circ(M)$ and suppose towards a contradiction that $S > A$. Then $C_H(S) = 1$. Indeed, if $C_H(S) \neq 1$ then by Lemma 2.6 (1), $B(C_G(C_H(S))) = B(C_G(H))$ and this forces $S = A$, a contradiction.

Let now $S_1/A = \Omega_1(S/A)$. By Remark 1.12, S_1/A is a definable connected group of rank $\mathrm{rk}\,(T)$. We claim that $C_H(S_1/A)$ is finite. If $C_H(S_1/A)$ is infinite then it has infinite Borel subgroups. Let B be one such. By Fact 1.4, $I(B) = \emptyset$, and then Fact 1.5 implies that B centralizes S_1. But Lemma 2.6 implies as in the last paragraph $S_1 = A$, a contradiction. In particular, $C_H(S_1/A) \le Z(H)$

Now we consider the action of $\overline{TH} = TH/C_{TH}(S_1/A)$ on S_1/A. Again by Remark 1.12, $\overline{T} \cong T$ and $\overline{T}$ acts transitively on $(S_1/A)^{\times}$. Moreover, $\overline{T}$ commutes with $\overline{H}$. Thus, the action of $\overline{H}$ on S_1/A can be linearized using Fact 1.15. Since $\overline{H}$ is of degenerate type, Fact 1.16 shows that $\overline{H}$ is solvable. But $\overline{H} \cong H/C_H(S_1/A)$ and it follows that H is solvable. It follows by Fact 1.4 that $I(H) = \emptyset$. Fact 1.5 implies that H centralizes S, a contradiction. □

We combine the foregoing with Fact 1.19 to get the following, bearing in mind that M contains a Sylow 2-subgroup of G.

Corollary 2.9 *A is a Sylow$^\circ$ 2-subgroup of G. Hence any connected definable 2-subgroup of M is contained in A, and any connected definable 2-subgroup of G is elementary abelian.*

3 Genericity of HT

In this section we keep the same hypotheses on G. We also keep the notation introduced in the previous section. We will prove the following statement:

Proposition 3.1 *Let G_1 be a definable connected subgroup of G which contains HT. Then the union of the G_1-conjugates of HT forms a generic subset of G_1.*

Lemma 3.2 $N_{G_1}{}^\circ(HT) = HT$.

Proof. It suffices to prove that $N^\circ(HT) = HT$. By Lemma 1.34 there is a nontrivial definable subgroup Q of H which is continuously characteristic in HT. We then have $N^\circ(HT) \le N^\circ(Q) \le HL$ by Lemma 2.6 (3). But $N_{HL}{}^\circ(HT) = HN_L{}^\circ(T) = HT$. □

Lemma 3.3 *If $g \in G_1 \setminus N_{G_1}(H)$, then $HT \cap (HT)^g$ is a rigid abelian group.*

Proof.

Let $A = (HT) \cap (HT)^g$ with g as specified. There are two natural maps $\pi_T : HT \to T$ and $\pi_{T^g} : (HT)^g \to T^g$ which combine to give a map $(\pi_T, \pi_{T^g}) : A \to T \times T^g$ defined as follows. Since HT is a direct product an element of A is uniquely written as ht and $h_1^g t_1^g$ with $h, h_1 \in H$ and $t, t_1 \in T$. The homomorphism (π_T, π_{T^g}) assigns to such an element the pair (t, t_1^g).

An element $a = ht = h_1^g t_1^g$ is in the kernel of (π_T, π_{T^g}) if and only if $t = t_1 = 1$. Thus $a = h = h_1^g \in H \cap H^g = 1$ by Corollary 2.7. Since the kernel is trivial and the image is contained in a good torus, Lemma 1.26 (1) and (2) imply that the group A is rigid abelian. □

Now we prove the main result of this section.

Proof of Proposition 3.1. We have $HT \leq G_1$ and we claim that $\bigcup_{g \in G_1} (HT)^g$ is generic in G_1.

Let $\mathcal{X}$ be the set

$$\bigcup_{g \in G_1 \setminus N_{G_1}(HT)} (HT) \cap (HT)^g$$

If $\mathcal{X}$ is not generic in HT, then our claim follows by Lemma 3.2 and Fact 1.38. Accordingly we will assume throughout that $\mathcal{X}$ is generic in HT.

Our aim in this case is to find a subgroup $X < HT$ of the form $X = (HT) \cap (HT)^g$ for which we can show that $\bigcup_{g \in G_1} X^g$ is generic in G_1, which will prove our claim.

Writing X_g for $(HT) \cap (HT)^g$, let

$$\mathcal{C} = \{X_g : g \in G_1 \setminus N_{G_1}(HT)\},$$

$$\mathcal{C}_1 = \{X_g : g \in N_{G_1}(H) \setminus N_{G_1}(HT)\},$$

and

$$\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1.$$

We first analyze $\mathcal{C}_1$. For $g \in N_{G_1}(H)$ we have $H \leq X_g \leq HT$ and hence $X_g = H \cdot (T \cap X_g)$. As T is a good torus, the second factor $T \cap X_g$ varies over a finite set and thus $\mathcal{C}_1$ is a finite collection of proper subgroups of HT, and the union $\bigcup \mathcal{C}_1$ is not generic in HT; so $\bigcup \mathcal{C}_2$ must be generic in HT.

Let $\mathcal{C}_3 = \{X^\circ\langle a\rangle : X \in \mathcal{C}_2, a \in X\} \setminus \{T\}$, as in the proof of Theorem 3. Now Lemma 3.3 applies to the groups in $\mathcal{C}_2$, hence also to those in $\mathcal{C}_3$. Moreover since $(\bigcup \mathcal{C}_2) \setminus T \subseteq \bigcup \mathcal{C}_3$, $\bigcup \mathcal{C}_3$ is generic in HT. Thus Theorem 3 applies and gives us $X \in \mathcal{C}_3$ such that X° is a maximal good torus in $C_{HT}(X)$.

Let T and T^g be so that $X^\circ = (HT \cap (HT)^g)^\circ$. Since T is central in HT and $X^\circ \leq HT$, we have $T \leq C_{HT}{}^\circ(X^\circ)$ and thus $T \leq X$. As $T \notin \mathcal{C}_3$, we find $T < X$. Thus $X = T \cdot (X \cap H)$ with $X \cap H \neq 1$. By Lemma 1.31 (3), we have $N^\circ(X) = C^\circ(X)$. Furthermore by Lemma 2.6 (4), $C^\circ(X) = C^\circ(T, X \cap H) \leq HT$. Now T^g is central in $(HT)^g$, and $X^\circ \leq (HT)^g$, so $T^g \leq C^\circ(X) = C_{HT}{}^\circ(X)$. Hence arguing as in the case of T, we find $N^\circ(X) \leq (HT)^g$. So $N^\circ(X) \leq [HT \cap (HT)^g]^\circ \leq X$.

Thus $[N(X) : X] < \infty$. The set $X_0 = \bigcup_{g \in G_1 \setminus N_{G_1}(X)} (X \cap X^g)$ is a finite union of proper subgroups of X by Lemma 1.32, and the complement $X \setminus X_0$ is generic in X by Lemma 1.47. We conclude using Lemma 1.40 that $\bigcup_{g \in G_1} X^g$ is generic in G_1, as desired. □

Corollary 3.4 *HT contains a definable subgroup K such that $N_G{}^\circ(K) \leq K$ and K is generically disjoint from its conjugates in G.*

Proof. This is how the proof went. We have $N_G{}^\circ(HT) = HT$ and if HT is not generically disjoint from its conjugates, we found a rigid abelian subgroup X of HT, a cyclic extension of a good torus, with $N_G{}^\circ(X) \leq X$; and we showed that such groups are generically disjoint from their conjugates. □

Definable groups satisfying $K = N^\circ(K)$ play an important role throughout, so we introduce the following natural terminology, which makes sense in any group.

Definition 3.5 *A subgroup K of a group G is said to be* almost self-normalizing *(in G) if $N(K)/K$ is finite.*

4 L-blocks

We retain the notations and hypotheses of the previous sections regarding the groups G, M, A.

Definition 4.1 *An* SL_2*-block in* G *is a connected definable subgroup of the form* $H \times L$ *where* $L \cong \mathrm{SL}_2(F)$ *for some algebraically closed field* F *of characteristic two, and* $H = C^\circ(L)$ *is nontrivial. A* toral *block is a subgroup of the form* HT *where* H *is part of an* SL_2*-block* HL*, and* T *is a maximal torus of* L*.*

It will be generally understood, when we consider an SL_2-block of the form, e.g., H_1L_1, that L_1 is the component isomorphic to SL_2, and that $H_1 = C^\circ(L_1)$, and similarly for toral blocks. However, if HL is an SL_2-block and we later consider a group HT, with T a torus, it should *not* be assumed that we are *necessarily* taking this to be a toral block; in practice, we will have to consider tori contained in HL which are *not necessarily* contained in L. So it will be convenient to introduce the following terminology as well.

Definition 4.2

1. The SL_2-component *of an* SL_2*-block* HL *is the subgroup* L*.*

2. The toral component *of a toral block* HT *is the torus* T*; here we assume that it is known what the associated* SL_2*-block* HL *is, and the meaning is that* T *is a maximal torus of* L*.*

Remark 4.3 *If* T *is a toral component of a toral block and* $T \leq M$*, then* T *is a complement to* $C_{M^\circ}(A)$ *in* M°*. This holds because* $T \cap C(A) = 1$ *in view of the structure of* SL_2*, and* $\mathrm{rk}\,(M^\circ/C_{M^\circ}(A)) = \mathrm{rk}\,(A) = \mathrm{rk}\,(T)$*.*

We have observed already that there is an SL_2-block HL in G such that $(HL) \cap M = H \cdot AT$, where AT is a Borel subgroup of L, and in particular M contains toral blocks. However we will be interested subseqently in toral blocks HT for which we assume only that $T \leq M$, and then the main question will be the structure of $H \cap M$.

Observe also that if HL is an SL_2-block, then H and L are uniquely determined: L is the only simple normal subgroup with a nontrivial Sylow° 2-subgroup, and L determines H, which is a subgroup of degenerate type. It is not clear whether a similar statement can be made for toral blocks (the ambiguous case arises when H is abelian).

Lemma 4.4

1. *Any two toral blocks* $H_1T_1, H_2T_2 \leq M$ *with* $H_1, H_2 \leq C(A)$ *are conjugate in* M*.*
2. *Any two toral blocks in* G *are* G*-conjugate.*
3. *Any two* SL_2*-components in* G *are* G*-conjugate.*
4. *The tori which occur as components of toral blocks are conjugate in* G*.*

Proof.

1. Suppose that H_1T_1 and H_2T_2 are not conjugate in M. As in the case of Lemma 3.3, we find that every intersection $B_g = (H_1T_1) \cap (H_2T_2)^g$ is rigid abelian (here we can take any $g \in G$): otherwise, one considers the natural map $B_g \to T_1 \times T_2^g$ whose kernel is contained in $H_1 \cap H_2^g$, and if this intersection is nontrivial one finds $L_1 =$

$B(C(H_1 \cap H_2^g)) = L_2^g$, hence $H_1 = H_2^g$ and the tori T_1 and T_2^g are conjugate in L_1, so H_1T_1 and H_2T_2 are conjugate.

Now we know that H_1T_1 and H_2T_2 contain almost self-normalizing subgroups which are generically disjoint from their conjugates, by Corollary 3.4; this holds in G and hence in particular in M. Working in M, we find that H_1T_1 is generically covered by M-conjugates of H_2T_2, by Lemma 1.41.

Thus letting $B_g = (H_1T_1) \cap (H_2T_2)^g$, the family $\{B_g : g \in M\} \setminus \{T_1\}$ is a generic covering of H_1T_1 by rigid abelian groups, of which at least one such, say B, must be maximal in the sense that there is no good torus in $C_{H_1T_1}(B)$ larger than B°. We may suppose that $g = 1$ so that $B = H_1T_1 \cap H_2T_2$.

On the other hand, $T_1 \leq Z(H_1T_1)$, so this implies that $T_1 \leq B$ and hence by definition of the family, $B > T_1$. Thus $B \cap H_1 > 1$. But $B \cap H_1 \leq C_{H_2T_2}(A) = H_2$, so $H_1 \cap H_2 > 1$; this implies $L_1 = L_2$ and hence $H_1 = H_2$. Now T_1 and T_2 are conjugate in $N_{L_1}(A) \leq C_M(H_1)$, so finally H_1T_1 is conjugate to H_2T_2 in M.

2. This follows from (1). It suffices to show that for any toral block HT in G, H can be conjugated into $C(A)$. Now HT is associated with an SL_2-block HL with $T \leq L$, and T normalizes a Sylow 2-subgroup of L which can be conjugated to A, and our claim follows.

3. We consider two SL_2-blocks H_1L_1 and H_2L_2 in G. After conjugation, we may suppose that $A \leq L_1, L_2$. As H_i centralizes A, and a torus T_i of L_i normalizes A, we then have $H_iT_i \leq M$ in both cases. We may conjugate the toral block H_2T_2 to H_1T_1 in M, so we suppose $H_1T_1 = H_2T_2$. As $H_i = C_{H_iT_i}(A)$ for $i = 1, 2$, we find $H_1 = H_2$, and hence $L_1 = L_2$.

4. This follows from (3). □

Lemma 4.5 *Let G be a group of finite Morley rank, T a good torus in G, and K a definable subgroup with $T \leq Z(K)$. Then $N_G{}^\circ(K) \leq C_G{}^\circ(T)$. In particular, if $K = C_G{}^\circ(T)$ then $N_G{}^\circ(K) = K$.*

Proof. We have $T \leq Z(K)$ and $Z(K)$ contains a unique maximal good torus $\hat{T}$. Then $N^\circ(K) \leq N^\circ(\hat{T}) = C^\circ(\hat{T}) \leq C^\circ(T)$. □

Lemma 4.6 *Let $T \leq M$ be a torus which is a complement to $C^\circ(A)$ in M°. Then $C_M(A, T)$ is infinite.*

Proof. We may pass to an $\aleph_0$-saturated elementary extension.

The torus T is a good torus. Let $K = C_M{}^\circ(T)$. Then $K = C_M{}^\circ(A, T) \cdot T$. Our claim is that $K > T$.

We have $N_M{}^\circ(K) = K$ by Lemma 4.5. On the other hand T, being a good torus, is generically disjoint from its conjugates in M (Lemma 1.32). Assuming toward a contradiction that $K = T$, it is then an almost self-normalizing group which is generically disjoint from its conjugates in M.

We know that there is a toral block H_1T_1 contained in M, with $H_1 \leq C(A)$, and the union of its conjugates in M is generic there by Proposition 3.1. Thus by Lemma 1.41, the group T is generically covered by conjugates of H_1T_1. However a good torus which is generically covered by a family of uniformly definable groups is contained in one of them, so after conjugation, without loss of generality we have $T \leq H_1T_1$.

We then have $T_1 \leq K$ and by our assumption $K = T$, so we have $T_1 \leq T$; but then $T_1 = T$ as each is a complement to $C^\circ(A)$ in M°, and hence $H_1 \leq K$, which is finally a contradiction. $\square$

Lemma 4.7 *Let $T \leq M$ be a torus which is a complement to $C^\circ(A)$ in M°, and suppose that T is inverted by an involution w conjugate to an involution of A. Then $C(T, w)$ is infinite.*

Proof. Let $\hat{T} = C^\circ(T)$. Then the involution w acts on $\hat{T}$. Suppose toward a contradiction that $C_{\hat{T}}(w)$ is finite. Then w inverts $\hat{T}$ by Fact 1.22.

We claim that a Sylow 2-subgroup P of $\hat{T}$ is trivial. The group P is connected (Fact 1.4), and in view of Corollary 2.9 it is elementary abelian, hence contained in $C_{\hat{T}}(w)$, and hence is finite and therefore trivial.

By Proposition 2.5 with $j = w$, the group $(\hat{T} \cap C(A))$ is an elementary abelian 2-group, so $\hat{T} \cap C(A) = 1$ and hence $C_M{}^\circ(T) = T$, which contradicts Lemma 4.6. $\square$

The next result will ultimately furnish the main connection between the structure of the set of involutions in G, and properties of SL_2-blocks.

Proposition 4.8 *Let w be an involution in $G \setminus M$, conjugate to an involution in A, and such that the set $X = \{g \in M : g^w = g^{-1}\}$ has rank at least $\mathrm{rk}\,(A)$. Then there is an SL_2-block HL such that $L \cap M$ contains a maximal torus of L inverted by w.*

Proof. Let $K = d(\langle X \rangle)$. Then $K \leq M$ is w-invariant.

Suppose first that K is abelian. Then $K = X$ is inverted by w. We claim that a Sylow 2-subgroup P of K° is trivial. The group P is connected by Fact 1.4, and in view of Corollary 2.9 is contained in A. But as w inverts P this means $P = 1$.

Now by Proposition 2.5, $K^\circ \cap C(A)$ is an elementary abelian 2-group, hence trivial by the above. Hence by rank considerations, in view of Lemma 2.4, K° is a complement to $C(A)$ in M. In particular K° is a complement to $C^\circ(A)$ in M° as well, in view of Lemma 2.2. By Lemma 4.7 we have $C_G{}^\circ(K^\circ, w) > 1$. Let $H_1 = C_G{}^\circ(K^\circ, w)$.

Let A_w be a conjugate of A containing w. By Proposition 2.3 we have $A_w \leq C(H_1)$. Furthermore as w inverts K°, the group K° does not normalize A_w (otherwise, w would centralize K°). So $B(C(H_1)) \cong \mathrm{SL}_2(F)$ for some algebraically closed field F of characteristic two, by Fact 1.24, and H_1 is part of an SL_2-block $\hat{H}_1 L$ containing K°, with $w \in L$. Now w inverts K°, centralizes $\hat{H}_1$, and normalizes L, while $K^\circ \leq \hat{H}_1 L$ contains no involutions, so we have $K^\circ \leq L$. As $\mathrm{rk}\,(K^\circ) \geq \mathrm{rk}\,(A)$, K° must be a maximal torus of L, so $L \cap M$ contains a maximal torus of L.

So now suppose that K is nonabelian. By Lemma 2.4, $K' \leq C(A)$, and similarly $K' \leq C(A^w)$. So $B(C(K')) = L$ is a group of type $\mathrm{SL}_2(F)$ with F an algebraically closed field of characteristic two. Let T be a maximal torus of L normalizing A and A^w. As the involution w normalizes K', it acts also on L, like an element of L. As w swaps A and A^w, the action of w on T is by inversion. Furthermore $T \leq N(A) \leq M$, as desired. $\square$

We introduce some notation for the relevant class of involutions.

Definition 4.9 $I_1 = \{i \in I(G) : i$ *is conjugate to an element of* $A\}$.

$I^* = \{w \in I_1 : w$ *inverts a subset of* M *of rank at least* $\mathrm{rk}\,(A)\}$

The class I^* is defined relative to a specific choice of M, not indicated explicitly in the notation, and this must be borne in mind. On the other hand I_1 is a single conjugacy class in G, and as G is connected, I_1 has Morley degree 1.

Lemma 4.10 *$I_1 \cap M = I(A)$.*

Proof. Let $i \in M$ be an involution which lies in a conjugate A_i of A. As A commutes with i (Lemma 2.4), it follows from Lemma 2.4 that $A_i \leq C_M{}^\circ(A)$. As $M/O_2{}^\circ(M)$ is of degenerate type, it follows that $A_i \leq \Omega_1(O_2{}^\circ(M)) = A$, and thus $A_i = A$, and $i \in A$. $\square$

Corollary 4.11 *Let L be a definable subgroup of G of the form SL_2. Then $I_1 \cap N(L) \subseteq L$.*

Proof. Let $w \in I_1 \cap N(L)$. Then w induces an inner automorphism on L of order at most 2 and hence w centralizes a conjugate A_1 of A with $A_1 \leq L$. Then by Lemma 4.10 we have $w \in A_1$. $\square$

Lemma 4.12 *I^* is generic in I_1.*

Proof. Let $g = \mathrm{rk}\,(G)$, $m = \mathrm{rk}\,(M)$, and $c = \mathrm{rk}\,(C(A))$. For $i \in A$ we have $C(i) = C(A)$ by Lemma 2.4, and thus $\mathrm{rk}\,(I_1) = g - c$.

Now by Lemma 2.2 $m - c = a$. Thus $\mathrm{rk}\,(I_1) = g - c = (g - m) + a$. For $w \in I_1 \setminus I^*$, we have $\mathrm{rk}\,(I_1 \cap wM) < a$, so $\mathrm{rk}\,(I_1 \setminus I^*) < \mathrm{rk}\,(G/M) + a = g - c = \mathrm{rk}\,(I_1)$. So $I_1 \setminus I^*$ is nongeneric in I_1, and I^* is generic. $\square$

Corollary 4.13 *A generic involution in I_1 inverts a maximal torus T of some SL_2-block for which $T \leq M$.*

This shows that in some sense the situation is reminiscent of SL_2; in a sense, we aim to prove eventually that we are in SL_2, though under our hypothesis $(\neg *)$ this produces a contradiction.

5 Toral block types

We have considered the behavior of involutions with respect to M, and the associated tori. We must now consider the types of toral blocks that arise when we take into account the relationship with M (fixed). Accordingly we make the following definition.

Definition 5.1 *Let HT be a toral block, associated with the SL_2-block HL, with $T \leq L$, and suppose that $T \leq M$.*

> *HT is* type I *if $C_H(A) \neq 1$.*
>
> *HT is* type II *if $C_H(A) = 1$ and $(H \cap M)^\circ > 1$.*
>
> *HT is* type III *if $H \cap M$ is finite.*

Lemma 5.2

1. *If HT is a toral block of type I, then $[H, A] = 1$. In particular, $H \leq M$.*
2. *If H_1T_1 and H_2T_2 are toral blocks of type I, then H_1T_1 is conjugate to H_2T_2 by an element of M which carries T_1 to T_2.*

Proof.
1. By Lemma 2.6(2), $B(C(C_H(A))) = B(C(H))$. As $A \leq B(C(C_H(A)))$, we have $[H, A] = 1$. So $H \leq C(A) \leq M$.
2. By part (1) and Lemma 4.4 (1), the toral blocks in question are conjugate in M, so we may suppose $H_1T_1 = H_2T_2$. Then $H_i = C_A(H_iT_i)$, so $H_1 = H_2$. Hence with $L_i = B(C(H_i))$ we have $L_1 = L_2$ and then $T_1 = T_2$. □

Having defined *toral block* types, we now make a slightly more subtle definition, correspondingly, for *toral* types.

Definition 5.3 *Let $T \leq M$ be a torus which occurs as the toral component of some toral block. Then T is said to be of type I, II, or III, respectively, if it occurs as the toral component of a toral block of the same type.*

From Lemma 5.2 (1) it follows that the types of toral blocks are mutually exclusive, assuming one knows the pair (H, T), and not simply their product. However this is by no means true of the types of tori which occur as toral components of toral blocks. This is an important and even useful point, as our main goal at the outset is to show that type I and type II tori behave reasonably well, and that there are no other types, in the quite limited sense that a type III torus must also be of type I or II (this leads to stronger statements subsequently).

Definition 5.4 *$\mathcal{T}$ is the family of tori T contained in M such that T is the toral component of some toral block.*

Observe that M acts by conjugation on $\mathcal{T}$, preserving types.

Proposition 5.5 *The family $\mathcal{T}$ divides into finitely many conjugacy classes under the action of M.*

Proof. As this statement is unaffected by passage to an elementary extension, we may suppose the group G is $\aleph_0$-saturated. The family $\mathcal{T}$ is uniformly definable in view of Lemma 4.4 (4). So Fact 1.37 applies. □

In the long run, we aim to prove that all tori in $\mathcal{T}$ are of type I, or rather, in sharper form, that no tori in $\mathcal{T}$ are of types II or III. We begin with a result about Type I tori.

Proposition 5.6 *Let HT be a toral block of type I. Then $C_M{}^\circ(T) = HT$.*

Proof. Set $\hat{H} = C_M{}^\circ(A, T)$. Then $\hat{H} \geq H$ and we claim $\hat{H} = H$.

Let HL be the SL_2-block associated with HT, and $w \in L$ an involution inverting T. Set $\Gamma = \{[w, x] : x \in C^\circ(T)\}$. We show first

$$\mathrm{rk}\,(C^\circ(T)) = \mathrm{rk}\,(\Gamma) + \mathrm{rk}\,(H) \tag{1}$$

We have the natural map from $C^\circ(T)$ to Γ induced by commutation with w, and for $x, y \in C^\circ(T)$ we have

$$[w,x] = [w,y] \iff w^{xy^{-1}} = w \iff xy^{-1} \in C(A_w)$$

where A_w is the conjugate of A containing w, by Lemma 2.4. On the other hand, $xy^{-1} \in C^\circ(T)$ and $\langle T, A_w\rangle = L$. So $[w,x] = [w,y] \iff xy^{-1} \in C(L)$, and as $C^\circ(L) = H$ this implies that $\mathrm{rk}\,(C^\circ(T)) = \mathrm{rk}\,(H) + \mathrm{rk}\,(\Gamma)$. So (1) holds.

Next we claim

$$\text{(2)} \qquad \mathrm{rk}\,(\hat{H}\cdot\Gamma) = \mathrm{rk}\,(\hat{H}) + \mathrm{rk}\,(\Gamma)$$

For this we consider the natural map $\hat{H}\times\Gamma \to \hat{H}\cdot\Gamma$ defined by $(\hat{h},\gamma) \mapsto \hat{h}\gamma$. It suffices to show that this is a bijection. So we need to show that for $\hat{h}\gamma_1 = \gamma_2$, with $\hat{h} \in \hat{H}$ and $\gamma_1, \gamma_2 \in \Gamma$, we have $\hat{h} = 1$. Suppose toward a contradiction that $\hat{h} \neq 1$.

Now as w inverts every element of Γ, we have $\hat{h}^w\gamma_1^{-1} = (\hat{h}\gamma_1)^w = \gamma_2^w = \gamma_2^{-1} = \gamma_1^{-1}\hat{h}^{-1}$, and thus $\hat{h}^{w\gamma_1^{-1}} = \hat{h}^{-1}$.

Now $w\gamma_1^{-1}$ is conjugate to w by the following calculation: taking $\gamma_1 = [w,x]$, we have $w\gamma_1^{-1} = wx^{-1}wxw = w^{xw}$. Thus $w\gamma_1^{-1} \in I_1$.

Now by Proposition 2.5, the element $\hat{h}$ must be an involution commuting with $w\gamma_1^{-1}$.

Now $\hat{h}w\gamma_1^{-1} = \hat{h}\gamma_1 w = \gamma_2 w = w\gamma_2^{-1}$ is another conjugate of w, as above. As $\hat{h}w\gamma_1^{-1}$ and $w\gamma_1^{-1}$ are commuting involutions in I_1, it follows that they lie in the same conjugate of A and hence their product $\hat{h}$ is also in I_1, hence in $I_1 \cap C(A) = A$, by Lemma 4.10. But $\hat{h} \in C(T)$, so this is a contradiction.

Accordingly, $\hat{h} = 1$, and the map in question is bijective.

Now we combine points (1) and (2). We have $\hat{H}\cdot\Gamma \subseteq C(T)$ and hence

$$\mathrm{rk}\,(\hat{H}) + \mathrm{rk}\,(\Gamma) \le \mathrm{rk}\,(\Gamma) + \mathrm{rk}\,(H)$$

or in other words $\mathrm{rk}\,(\hat{H}) \le \mathrm{rk}\,(H)$. As $H \le \hat{H}$ we find $H = \hat{H}$ and the result is proved.
□

Corollary 5.7 *Let HT be a toral block of type I. Then any maximal good torus of HT is a maximal good torus of M.*

Lemma 5.8 *Let HT be a type I toral block contained in M, and HL the corresponding* SL_2*-block, with T a maximal torus of L. Then for $w \in L$ inverting T, $(M\cap M^w)^\circ = HT$.*

Proof. The group $M \cap M^w$ normalizes $\langle A, A^w\rangle = L$, and the claim follows. □

6 Type III Tori

Rather than analyzing type III toral blocks directly, we show that every torus in $\mathcal{T}$ is of type I or II, which generally allows us to restrict our attention to those types.

We will make use of the following.

Lemma 6.1 *Let G be an $\aleph_0$-saturated group of finite Morley rank, T a good torus in G, and $K = C_G{}^\circ(T)$. Suppose that K is generically covered by a family of intersections $\{K \cap K^g\}$ with g varying over a definable set, with each of these intersections a rigid abelian group.*

Then K contains a good torus which is almost self-normalizing in G.

Proof. By Theorem 3 there is at least one such intersection $B = K \cap K^g$ such that B is maximal rigid in the sense that B° is the unique maximal good torus in $C_K(B)$.

As T is central in K, we then find that $T \le B$. Thus $C^\circ(B^\circ) \le C^\circ(T) = K$. As $B \le K^g$, we have $T^g \le C^\circ(B)$ and thus $T^g \le C_K(B)$; again by maximality, we have $T^g \le B$. Thus $C^\circ(B^\circ) \le C^\circ(T^g) = K^g$. Accordingly $C^\circ(B^\circ) \le K \cap K^g = B$, and thus $N^\circ(B^\circ) = C^\circ(B^\circ) = B^\circ$. □

Proposition 6.2 *Let $T \le M$ be the toral component of a toral block HT. Then T is of type I or II.*

We do not assert here that the toral block HT will itself be of type I or II.

Proof. We may suppose after passing to an elementary extension that M is $\aleph_0$-saturated.

Let H_1T_1 be a type I toral block, and T^* a maximal good torus of H_1T_1. Then by Corollary 5.7, T^* is a maximal good torus of M, and hence after conjugation, invoking Fact 1.35, we may suppose $T \le T^*$.

Thus we have a toral block HT with toral component $T \le M$, and another toral block H_1T_1 of type I, with $T \le H_1T_1$. Here $H_1 \le C(A)$ and T, T_1 are complements to $C(A)$ in M, so in particular $H_1T = H_1T_1$.

We consider $R = C_{H_1T_1}{}^\circ(T)$. If $T_1 \le T$ then $T = T_1$, so T is of type I, as desired. Suppose therefore that

$$T_1 \not\le T$$

and in particular $R > T$.

We claim

(1) $$R = C_G{}^\circ(TT_1) \text{ is almost self-normalizing in } G$$

Notice that $TT_1 > T$ so TT_1 meets H_1 nontrivially, and $C(TT_1) \le C(H_1 \cap (TT_1)) \le N(H_1)$, in view of Lemma 2.6 (1,2). Thus $C^\circ(TT_1) \le [N^\circ(H_1) \cap C(T_1)]^\circ = H_1T_1$, by Lemma 2.6 (4), and $C^\circ(TT_1) \le C_{H_1T_1}{}^\circ(T) = R$. This proves that $R = C_G{}^\circ(TT_1)$, and as TT_1 is a good torus, it then follows that R is almost self-normalizing (Lemma 4.5). So (1) holds.

For the remainder of the argument, we view R primarily as a subgroup of $C(T)$. We claim that

(2) R contains an almost self-normalizing connected subgroup of $C(T)$ which is generically disjoint from its conjugates in $C(T)$.

One possibility is that R itself is generically disjoint from its conjugates in $C(T)$, in which case (2) holds. Now assume the contrary: $R \cap \bigcup_{g \in C(T) \setminus N(R)} R^g$ is generic in R.

Now $R \cap R^g \le (H_1T_1) \cap (H_1T_1)^g$, and by Lemma 3.3, this is a rigid abelian group unless $g \in N(H_1)$. On the other hand, taking $L_1 = B(C(H_1))$, we have $N_{C(T)}{}^\circ(H_1) \le C_{H_1L_1}{}^\circ(T)$ and as T normalizes H_1 and L_1 we have $C_{H_1L_1}{}^\circ(T) = C_{H_1}{}^\circ(T)C_{L_1}{}^\circ(T) \le C_{H_1}{}^\circ(T)T_1 = R$. Thus $N_{C(T)}{}^\circ(H_1) \le R$, and there are only finitely many conjugates of R of the form R^g with $g \in N_{C(T)}(H_1)$. These may accordingly be discarded, and our conclusion in this case is that R is generically covered by a definable family of groups $R \cap R^g$ which are all rigid abelian. In this case as we work in an $\aleph_0$-saturated model, Lemma 6.1 applies, as R is the connected component of the centralizer of the good torus TT_1, and the conclusion is that R contains a good torus B which is almost self-normalizing in $C(T)$. But then B is also generically disjoint from its conjugates in $C(T)$ (Lemma 1.32), and (2) is proved.

We now fix a connected subgroup $K \leq R$, almost self-normalizing and generically disjoint from its conjugates in $C(T)$. Then automatically $T \leq K$, and as T is not almost self-normalizing, in fact $T < K$. Now by Proposition 3.1 and Lemma 1.41, we find that R is generically covered by conjugates of HT in $C(T)$.

We claim now that we may suppose that all the intersections $R \cap (HT)^g$ with $g \in C(T)$ are rigid abelian groups. We have an inclusion $R \leq H_1T_1$ so there is a natural homomorphism $R \cap (HT)^g \to T_1 \times T$ induced by the projections associated with H_1T_1 and $(HT)^g = H^gT$. The kernel of the combined map is contained in $H^g \cap H_1$: if this is nontrivial, one finds that $H^g = H_1$. In this case, since H^gT is a toral block with toral component T, we again have T of type I, as desired. Leaving that case aside, we conclude that the map $R \cap (HT)^g \to T_1 \times T$ is an embedding, and hence the intersection is a rigid abelian group.

But in this case we have a covering of R, generically, by rigid abelian groups $R \cap (HT)^g$, and hence by Theorem 3, and saturation, at least one such, say $B = R \cap (HT)^g$, must be maximal in R in the sense that B° is a maximal good torus of $C_R(B)$. However this implies that $T_1 \leq B$ and thus $TT_1 \leq (HT)^g \cap M$. It follows that $[(HT)^g \cap M]^\circ > T$, and we have a toral component H^gT of type I or II containing T, as claimed. □

7 The rank of I_1

We have the formula $\mathrm{rk}\,(I_1) = g - c$ where $g = \mathrm{rk}\,(G)$ and $c = \mathrm{rk}\,(C(A))$, in view of Lemma 2.4. We wish to make a second computation of the rank using the fact that $\mathrm{rk}\,(I^*) = \mathrm{rk}\,(I_1)$ (Lemma 4.12).

Definition 7.1 *For $T \in \mathcal{T}$ let I_T be the set of involutions in I_1 which act by inversion on T. For $\mathcal{T}_0 \subseteq \mathcal{T}$ let $I_{\mathcal{T}_0}$ be $\bigcup_{T \in \mathcal{T}_0} I_T$.*

Lemma 7.2 *If $T_1, T_2 \in \mathcal{T}$ are distinct, then I_{T_1} and I_{T_2} are disjoint.*

Proof.

Suppose $w \in I_{T_1} \cap I_{T_2}$. Then $T_1, T_2 \leq M \cap M^w$ are inverted by w, and are complements to $C(A)$ in M. If $K = (M \cap M^w)^\circ$ is abelian, then T_1T_2 is inverted by w, and this forces $T_1 = T_2$ by Lemma 2.4 and Proposition 2.5, a contradiction. So K is nonabelian.

Then $B(C(K')) = L$ is a group of type SL_2 containing A and A^w. So $K \leq N^\circ(L) = HL$ with $H = C^\circ(L)$. Furthermore $K \leq (HL \cap M \cap M^w)^\circ = HT$ for some maximal torus T of L, inverted by w (which belongs to L by Corollary 4.11). So T is central in K, and TT_1 is a torus inverted by w. Furthermore by Proposition 2.5 $TT_1 \cap C(A) = 1$, and hence $T_1 \leq T$; as the ranks are equal, we have $T_1 = T$. Similarly $T_2 = T$ and we have a contradiction. □

Note that by Corollary 4.13 $I_{\mathcal{T}}$ is generic in I_1. By Proposition 5.5 $\mathcal{T}$ is a finite union of conjugacy classes with respect to the action of M, and hence for at least one of these classes $\mathcal{T}_0 \subseteq \mathcal{T}$, the set $I_{\mathcal{T}_0}$ is generic in I_1; by Lemma 7.2 this holds for exactly one such class, as the Morley degree of I_1 is 1.

Definition 7.3 *A conjugacy class $\mathcal{T}_0 \subseteq \mathcal{T}$, with respect to the action of M, is a* generic *class if $I_{\mathcal{T}_0}$ is generic in I_1.*

So we know now that there is a unique generic conjugacy class in $\mathcal{T}$. We use this class to estimate the rank of G, and eventually we show that the generic class consists of type I tori.

Lemma 7.4 *For $T \in \mathcal{T}$, the rank of I_T is independent of T and is at most* $\mathrm{rk}\,(C(T)) - \mathrm{rk}\,(H)$ *where HT is a toral block for which T is the toral component (relative to some* SL_2*-block). Furthermore,* $\mathrm{rk}\,(C(T))$ *and* $\mathrm{rk}\,(H)$ *are constant, independent of the choice of T and the choice of the associated toral block.*

Proof. It follows from Lemma 4.4 (4) that any two $T \in \mathcal{T}$ are conjugate in G. As I_T is determined by T, its rank is independent of the choice of T.

Now fix an SL_2-block HL with T a maximal torus of L. For $w \in I_T$ we have $I_T = \{w' \in I_1 : w' = wx \text{ with } x \in C(T)\}$. In other words, $I_T = I_1 \cap wC(T)$. For the upper bound on $\mathrm{rk}\,(I_T)$, we aim to show that I_T meets each coset of H in $wC(T)$ in at most one element, which is sufficient.

Suppose toward a contradiction that $w, w' \in I_T$ and $w = w'h$ with $h \in H^\times$. Then w inverts $h \in C(A)$. As $w \in I_1$, it follows from Proposition 2.5 that h is an involution, so that w and w' commute. If A_w is the conjugate of A containing w, and M_w the corresponding conjugate of M, then $w' \in C(A_w) \le M_w$. By Lemma 4.10 we have $w' \in A_w$ and hence $h = ww' \in A_w$. Then $B(C(h)) = A_w$, and this is a contradiction. This gives the estimate $\mathrm{rk}\,(I_T) \le \mathrm{rk}\,(C(T)) - \mathrm{rk}\,(H)$.

As the tori T in $\mathcal{T}$ are conjugate in G, and the toral blocks HT are also conjugate, it follows that $\mathrm{rk}\,(T)$, $\mathrm{rk}\,(C(T))$, and $\mathrm{rk}\,(HT)$ are independent of T; so $\mathrm{rk}\,(H) = \mathrm{rk}\,(HT) - \mathrm{rk}\,(T)$ is also constant (and for that matter, the various H are also conjugate). $\square$

Definition 7.5 *Let $c_1 = \mathrm{rk}\,(C(T))$ for $T \in \mathcal{T}$ and $h = \mathrm{rk}\,(H)$ where HT is a toral block. As remarked above, this is well defined.*

Now we record the corresponding rank formula for conjugacy classes in $\mathcal{T}$. We recall that m, g, c are $\mathrm{rk}\,(M)$, $\mathrm{rk}\,(G)$, and $\mathrm{rk}\,(C(A))$ respectively.

Lemma 7.6 *For $\mathcal{T}_0$ a conjugacy class in $\mathcal{T}$ with respect to the action of M, and $T \in \mathcal{T}_0$ we have*

$$\mathrm{rk}\,(I_{\mathcal{T}_0}) = c - \mathrm{rk}\,(C(A,T)) + \mathrm{rk}\,(I_T) \le c - \mathrm{rk}\,(C(A,T)) + c_1 - h$$

Proof.

We have shown that the rank $\mathrm{rk}\,(I_T)$ is constant, and as $I_{\mathcal{T}_0}$ is the disjoint union of the I_T $(T \in \mathcal{T}_0)$ by Lemma 7.2, we find $\mathrm{rk}\,(I_{\mathcal{T}_0}) = \mathrm{rk}\,(\mathcal{T}_0) + \mathrm{rk}\,(I_T)$, where $\mathcal{T}$ is viewed as a definable subset of G^{eq}.

Now for $T \in \mathcal{T}_0$ we have $\mathrm{rk}\,(\mathcal{T}_0) = \mathrm{rk}\,(M/N_M(T)) = m - \mathrm{rk}\,(N_M{}^\circ(T))$ and $N_M{}^\circ(T) = C_M{}^\circ(T) = C_M{}^\circ(A,T) \times T$, so $\mathrm{rk}\,(\mathcal{T}_0) = m - t - \mathrm{rk}\,(C(A,T))$ and as $m - t = c$ our first equality holds. For the second we use our preceding estimate of $\mathrm{rk}\,(I_T)$. $\square$

Corollary 7.7 *Let $\mathcal{T}_0$ be a generic conjugacy class in $\mathcal{T}$ with respect to the action of M, and let $T \in \mathcal{T}_0$. Then*

1. $g \le 2c - \mathrm{rk}\,(C(A,T)) + c_1 - h$

2. $\mathrm{rk}\,(C(A,T)) \le h$

Proof. For the first point we just use the fact that $g = c + \mathrm{rk}\,(I_1) = c + \mathrm{rk}\,(I_{\mathcal{T}_0})$, together with our final estimate above.

For the second point, our formula for $\mathrm{rk}\,(I_{\mathcal{T}_0})$ shows that this rank is independent of the choice of $\mathcal{T}_0$, except for the term $\mathrm{rk}\,(C(A,T))$, which must be *minimized.* However in the case of a type I torus we know by Proposition 5.6 that this rank is h, so the minimum is at most h. This proves the claim. □

8 $C^\circ(A)C(T)C^\circ(A)$

We now seek a lower bound for $g = \mathrm{rk}\,(G)$ by computing the rank of $C^\circ(A)C(T)C^\circ(A)$ in those cases for which $C(T)$ is not contained in M; while ultimately it turns out that this does not occur, at present it is certainly a strong possibility a priori, and in addition it might also depend on the specific choice of T (for example, its type).

Lemma 8.1 *Suppose $T \in \mathcal{T}$, $h_1, h_2 \in C^\circ(A)$, and $c_1, c_2 \in C(T) \setminus M$, with*

$$h_1c_1C^\circ(A) = h_2c_2C^\circ(A)$$

Then $h_1 \in h_2[C^\circ(A) \cap C(T)]$ and $c_1 \in [C^\circ(A) \cap C(T)]c_2[C^\circ(A) \cap C(T)]$

Proof. We write $h_1c_1 = h_2c_2v$ with $v \in C^\circ(A)$. With $u = h_2^{-1}h_1$, we have $u \in C^\circ(A)$ as well, and

$$v = c_2^{-1}uc_1; u, v \in C^\circ(A), c_1, c_2 \in C(T)$$

We aim to show that $v \in C(T)$, so that $u \in C(T)$, $h_1 = h_2u \in h_2[C^\circ(A) \cap C(T)]$, and $c_1 = u^{-1}c_2v \in [C^\circ(A) \cap C(T)]c_2[C^\circ(A) \cap C(T)]$.

Consider the group $X = [T, v]$ (generated by the set of commutators). Suppose toward a contradiction that $X > 1$.

We have $X \le C^\circ(A)$ by Proposition 2.3. Furthermore $X = [T, c_2^{-1}uc_1] = [T, uc_1] = [T, u]^{c_1} \le C_{M^\circ}(A^{c_1})$. Now as $c_1 \notin M$ and $X > 1$ we find $B(C(X)) = \langle A, A^{c_1}\rangle \cong \mathrm{SL}_2(F)$ for some algebraically closed field F of characteristic two.

Let $L_1 = B(C(X))$ and $H_1 = C^\circ(L_1)$. Then $H_1 \le [M \cap M^{c_1}]^\circ$. Let T_1 be the maximal torus of L_1 contained in $M \cap M^{c_1}$, and $w_1 \in L_1$ an involution inverting T_1.

Now T normalizes X and hence acts on L_1, so $T \le H_1L_1$. Furthermore $T \le M \cap M^{c_1}$ since $c_1 \in C(T)$, and $M^{c_1} = M^{w_1}$, so $T \le ((H_1L_1) \cap M \cap M^{w_1})^\circ = H_1T_1$.

Now $A^{c_1} = A^{w_1}$ so $c_1w_1 \in N(A) = M$. Write $c_1 = xw_1$ with $x \in M$. For $t \in T$ we have $t = t^{xw_1}$, so $t^x = t^{w_1}$. Writing $t = ht_1$ with $h \in H_1$ and $t_1 \in T_1$, we find $h^xt_1^x = ht_1^{-1}$; reading this in $\overline{M} = M/C(A)$ we get $\bar{t}_1 = \bar{t}_1^{-1}$ and this forces $t_1 = 1$.

From this we conclude that $T \le H_1$, but this contradicts Proposition 2.5. Hence $X = 1$, as claimed. □

Lemma 8.2 *Suppose that $T \in \mathcal{T}$. For $h \in C^\circ(A)$ and $c \in C(T) \setminus M$, the rank of the set of pairs (h', c') for which $h' \in C^\circ(A)$, $c' \in C(T) \setminus M$, and $hcC^\circ(A) = h'c'C^\circ(A)$ is $2\mathrm{rk}\,(C^\circ(A) \cap C(T))$.*

Proof. By the preceding lemma, a necessary condition for $hcC^\circ(A) = h'c'C^\circ(A)$ is $h' = hu_1$, $c' = u_2cu_3$ with $u_1, u_2, u_3 \in C^\circ(A) \cap C(T)$, and if $u_1u_2 = 1$ then this is clearly sufficient. It remains to prove that necessarily $u_1u_2 = 1$.

We may suppose therefore that we have $cC^\circ(A) = ucC^\circ(A)$ with $c \in C(T) \setminus M$ and $u \in C^\circ(A) \cap C(T)$, and we aim to show $u = 1$. Since $cC^\circ(A) = ucC^\circ(A)$, we have $u^c \in C^\circ(A)$. Thus $u \in C(A, A^{c^{-1}})$.

Suppose $u \neq 1$, and let $L = B(C(u))$. As $c \notin M$, we have $A \neq A^{c^{-1}}$, and L is of type SL_2.

There is an involution $w \in L$ such that $A^w = A^{c^{-1}}$ and thus $wc \in N(A) = M$. Now w inverts $T_1 = N_L(A) \cap N_L(A^w)$ and $T \leq N(A) \cap N(A^w) \leq N(L)$, so $T \leq T_1C(L)$.

We have $[w, T] = [wc, T] \leq C(A)$ (Lemma 2.4); as $T \leq T_1C(L)$ with w inverting T_1 and centralizing $C(L)$, we find $[w, T] \leq T_1 \cap C(A) = 1$, and hence $T \leq C(L) \leq C(A)$. As $T \in \mathcal{T}$, this contradicts Proposition 2.5. $\square$

Lemma 8.3 *Suppose that $T \in \mathcal{T}$ and that $C(T)$ is not contained in M. Then*

$$\mathrm{rk}\,(C^\circ(A)[C(T) \setminus M]C^\circ(A)) = c_1 + 2c - 2\mathrm{rk}\,(C(A, T))$$

and, in particular,

$$g \geq c_1 + 2c - 2\mathrm{rk}\,(C(A, T))$$

Proof. Let $X = C(T) \setminus M$. Then $\mathrm{rk}\,(X) = c_1$ by assumption. We claim $\mathrm{rk}\,(C^\circ(A)XC^\circ(A)) = c_1 + 2c - 2\mathrm{rk}\,(C(A, T))$.

Define an equivalence relation $\sim$ on $C^\circ(A) \times X$ by $(h_1, c_1) \sim (h_2, c_2)$ iff $h_1c_1C^\circ(A) = h_2c_2C^\circ(A)$. Then $\mathrm{rk}\,(C^\circ(A)XC^\circ(A)) = \mathrm{rk}\,((C^\circ(A) \times X)/\sim) + c$. So to get our formula, it suffices to check that the equivalence classes for $\sim$ in $C^\circ(A) \times X$ have rank $2\mathrm{rk}\,(C(A, T))$.

By Lemma 8.2, the correct value is $2\mathrm{rk}\,(C^\circ(A) \cap C(T))$. Since $C^\circ(A) \cap C(T)$ has finite index in $C(A, T)$, the ranks are the same. $\square$

Actually, we need to improve the foregoing inequality by making it strict. This requires some further analysis.

Lemma 8.4 *Suppose $T \in \mathcal{T}$ and $C(T) \not\leq M$. Then $I_1 \cap C^\circ(A)[C(T) \setminus M]C^\circ(A) = \emptyset$.*

Proof. Suppose $h_1, h_2 \in C^\circ(A)$, $c \in C(T) \setminus M$, and $h_1ch_2 \in I_1$. After conjugating by h_1 we find $ch \in I_1$, with $h = h_2h_1 \in C^\circ(A)$.

Let $i = ch$. Then $[i, T] = [h, T] \leq C^\circ(A)$.

Now for $t \in T^\times$ we have $[i, t] \neq 1$ since $i \in I_1$ and $T \in \mathcal{T}$, using Proposition 2.5. But $[i, t]$ is inverted by i and $i \in I_1$, so again by Proposition 2.5 the element $[i, t]$ must be an involution. But $[i, t] = ii^t$ so i and i^t are commuting involutions in I_1. In this case, by Lemma 2.4 they lie in a single conjugate of A, which must also contain $[i, t]$. But i is outside M, and $[i, t]$ is inside M, a contradiction. $\square$

Lemma 8.5 *Let $w \in I^*$. Then* $\mathrm{rk}\,(\{x \in M \cap M^w : x = ww' \text{ for some } w' \in I_1\}) = rk(A)$.

Proof. By Proposition 4.8 there is a torus T in $M \cap M^w$ inverted by w with $\mathrm{rk}\,(T) = \mathrm{rk}\,(A)$. Let $X_1 = \{x \in M \cap M^w : x^w = x^{-1}\}$, and let $X = \{x \in X_1 : \exists w' \in I_1\, x = ww'\}$. Then $T \subseteq X \subseteq X_1$.

If $M \cap M^w$ is abelian then X_1 is a group and $T = X_1{}^\circ$, in which case our claim is clear.

Assume $M \cap M^w$ is nonabelian, and let K be its commutator subgroup. Then $L = B(C(K)) = \langle A, A^w \rangle$ is a group of type SL_2 and $M \cap M^w$ acts on it, so we have $M \cap M^w \le RL$ with $R = C(L)$.

Now $w \in L$ by Corollary 4.11. Since $X_1 \subseteq RL$ and $w \in L$, it follows that the elements of X_1 are of the form rt with $r \in R$, $r^2 = 1$, and $t \in T_1$, where $T_1 \le L$ is the maximal torus inverted by w, normalizing A.

Suppose such an element rt belongs to X, that is $rt = ww'$ with $w' \in I_1$. Now $t = ww''$ with $w'' \in I(L)$, so $r = (w'w'')^w$ and conjugating by w, $r = w'w''$. In particular w', w'' are commuting involutions in I_1, hence lie in the same conjugate of A, which therefore contains r. If $r \neq 1$ then $r \in I_1 \cap C(A) = A^\times$, a contradiction. So $r = 1$ and $rt \in T_1$. □

Lemma 8.6 *$I_1 M^\circ$ is generic in G.*

Proof. We claim that the map $I_1 \times M^\circ \to G$ given by multiplication is generically surjective. The rank of $I_1 \times M^\circ$ is $\mathrm{rk}\,(I_1) + m = g - c + m = g + a$ where a is the rank of A, which is also the rank of a complement T to $C(A)$ in M.

It suffices to check that the fibers have rank at most a. But if $i, j \in I_1$, $x, y \in M^\circ$ and $ix = jy$, then $ji = yx^{-1}$ is an element of M° inverted by i, and the rank of the set

$$\{j \in I_1 : ji \in M^\circ\}$$

is at most a if $i \in I^*$, by the preceding lemma, and is strictly less than a otherwise. □

Proposition 8.7 *Suppose that $T \in \mathcal{T}$ and $C(T)$ is not contained in M. Then $g > c_1 + 2c - 2\mathrm{rk}\,(C(A,T))$.*

Proof. By Lemma 8.3 we have $g \ge c_1 + 2c - 2\mathrm{rk}\,(C(A,T))$ already.

If we have equality, then by that same lemma the set $C^\circ(A)[C(T) \setminus M]C^\circ(A)$ is generic in G. On the other hand $I_1 M^\circ$ is also generic in G. We claim that these two sets are disjoint, and hence the former is not generic, so the inequality is strict.

Now $C^\circ(A)[C(T) \setminus M]C^\circ(A) = C^\circ(A)[C(T) \setminus M]M^\circ$ since $M^\circ = TC^\circ(A)$ and $T \le C(T)$. Since

$$(C^\circ(A)[C(T) \setminus M]C^\circ(A)) \cap I_1 = \emptyset$$

none of the left cosets of M° making up this set meets I_1. On the other hand, every left coset of M° in $I_1 M^\circ$ meets I_1, so the two sets are disjoint. □

9 $C(T)$ and $\mathrm{rk}\,(G)$

Now we can combine the upper and lower bounds for g derived in the previous sections to show the following.

Proposition 9.1

1. For a torus T in the generic conjugacy class $\mathcal{T}_0$ under the action of M, we have $C(T) \le M$.

2. $g = m + c - h$.

Proof.

1. Fix $T \in \mathcal{T}_0$. Suppose that $C(T)$ is not contained in M. In this case, the estimates of the last two sections apply, namely Proposition 8.7 and Corollary 7.7, giving

$$c_1 + 2c - 2\mathrm{rk}\,(C(A,T)) < g \le 2c - \mathrm{rk}\,(C(A,T)) + c_1 - h$$

Cancelling common terms we find

$$-\mathrm{rk}\,(C(A,T)) < -h$$

or $h < \mathrm{rk}\,(C(A,T))$. This contradicts Corollary 7.7 and proves the first point.

2. In particular we find $c_1 = \mathrm{rk}\,(C_M(T)) = \mathrm{rk}\,(A) + \mathrm{rk}\,(C(A,T))$ and hence our estimate in Corollary 7.7 becomes

$$g \le 2c + \mathrm{rk}\,(A) - h = m + c - h$$

On the other hand, if we choose $w \in I_1$ associated with a type I toral block $HT \le M$, then $\mathrm{rk}\,(MwM) = 2m - \mathrm{rk}\,(M \cap M^w)$. By Lemma 5.8, $(M \cap M^w)^\circ = HT$.

So we have now

$$g \ge \mathrm{rk}\,(MwM) = 2m - \mathrm{rk}\,(HT) = 2m - h - \mathrm{rk}\,(T) = m + c - h$$

and so we have determined g:

$$g = m + c - h$$

□

10 2-Transitivity, Connectivity, Strong Embedding

In this section we study the action of G on the cosets of M in G. We use the notations $g = \mathrm{rk}\,(G)$, $m = \mathrm{rk}\,(M)$, $c = \mathrm{rk}\,(C(A))$, $h = \mathrm{rk}\,(H)$, $a = \mathrm{rk}\,(A)$ as before, though in what follows one should bear in mind rather that $a = \mathrm{rk}\,(T)$.

Lemma 10.1 *Let $g \in G \setminus M$. Then $\mathrm{rk}\,(M \cap M^g) \ge h + a$.*

Proof. We know $g = m+c-h$ and by considering MgM we find $g \ge 2m - \mathrm{rk}\,(M \cap M^g)$, so $\mathrm{rk}\,(M \cap M^g) \ge m - c + h = h + a$. □

Lemma 10.2 *Let $g \in G \setminus M$. Then $\mathrm{rk}\,(M \cap M^g) = h + a$.*

Proof. We need to prove the upper bound $\mathrm{rk}\,(M \cap M^g) \le h + a$. If the intersection has rank a then this is clear, so we assume $\mathrm{rk}\,(M \cap M^g) > a$. Let $R = (M \cap M^g)^\circ$. Let $K_1 = C_R(A)$ and $K_2 = C_R(A^g)$. As $M = C(A)T$ with $\mathrm{rk}\,(T) = a$, we have K_1 infinite, and similarly K_2 is infinite.

Suppose first that

(1) $$(K_1 \cap K_2)^\circ = 1$$

As $M = C(A)T$ and $M^g = C(A^g)T^g$ we have a natural map $R \to T \times T^g$ with kernel $K_1 \cap K_2$, so R is a good torus by Lemma 1.26 (4).

We claim that R is almost self-normalizing. We have

$$N^\circ(R) = C^\circ(R) \leq N(B(C(K_1))) \cap N(B(C(K_2)))$$

If $B(C(K_1)) = A$ then $N^\circ(R) \leq N(A) = M$. Suppose $L_1 = B(C(K_1))$ is of type SL_2. If R centralizes L_1, then $\mathrm{rk}\,(R) \leq h$ and we are done. So suppose R acts nontrivially on L_1. Then R acts like part of a maximal torus $T_1 \leq L_1$, normalizing A, and thus $N^\circ(R)$ also acts on L_1 like part of T_1. So $N^\circ(R) \leq T_1 \cdot C(L_1)$. But of course $C(L_1) \leq N(A) = M$, so again $N^\circ(R) \leq M$. Thus in all cases $N^\circ(R) \leq M$. Similarly, $N^\circ(R) \leq M^g$ and thus $N^\circ(R) = R$.

As R is a good torus, it follows that the union of the conjugates of R in M° is generic in M°.

Taking a type I toral block $HT \leq M^\circ$, it follows from Proposition 3.1 and Lemma 1.41 that R is generically covered by the conjugates of HT in M°, and hence we may suppose without loss of generality that $R \leq HT$. But then $\mathrm{rk}\,(R) \leq h + a$, as claimed. That proves the result in Case (1).

Now suppose

$$(K_1 \cap K_2)^\circ > 1 \tag{2}$$

Then $L = B(C(K_1 \cap K_2)) = \langle A, A^g \rangle$ is of type SL_2 and $H = C^\circ(L) \leq M$. Now $M^g = M^w$ for some $w \in L$, and by Lemma 5.8 we have $(M \cap M^g)^\circ = HT$ with T the torus inverted by w. So $\mathrm{rk}\,(R) = h + a$ in this case as well. □

Proposition 10.3 *The action of G on G/M is doubly transitive.*

Proof. For $g \in G \setminus M$ we consider the double coset MgM. We find $\mathrm{rk}\,(MgM) = 2m - \mathrm{rk}\,(M \cap M^g) = 2m - h - a = m + c - h = \mathrm{rk}\,(G)$. Thus each such double coset is generic in G, and there can be only one. □

Corollary 10.4

1. *For any two distinct conjugates A_1, A_2 of A, the group $\langle A_1, A_2 \rangle$ is a group of type SL_2 over an algebraically closed field of characteristic two, and is a factor of an SL_2-block.*

2. *For any conjugate M_1 of M distinct from M, the group $(M \cap M_1)^\circ$ is a toral block of type I.*

Lemma 10.5 *Let HT be a toral block with $HT \leq M$. Then HT is of type I, and hence $[H, A] = 1$.*

Proof. We take an SL_2-block HL with T a maximal torus of L, inverted by the involution $w \in L$. By double transitivity, $L_1 = \langle A, A^w \rangle$ is also a group of type SL_2, part of an SL_2-block H_1L_1. As L_1 is normalized by w, by Corollary 4.11, we have $w \in L_1$. So w inverts the maximal torus T_1 of L_1 which normalizes A and A^w. By Lemma 7.2 we have $T = T_1$.

As $w \in L \cap L_1$, it follows that the conjugate A_w of A containing w is also contained in both L and L_1, hence $L = \langle A_w, T \rangle = L_1$, and $H = H_1$. Thus HT is of type I. □

Corollary 10.6 *The generic class of tori in $\mathcal{T}$ with respect to the action of M consists of the type I tori, and these tori are not of types II or III.*

Proof. If T belongs to the generic class $\mathcal{T}_0 \subseteq \mathcal{T}$ then we know that $C(T) \leq M$. So any toral block containing T is contained in M and is therefore of type I exclusively (Lemma 10.5).

Conversely, the type I tori are conjugate under the action of M (Lemma 5.2 (2)), so they are all in the generic class. $\square$

Lemma 10.7 *There are no tori in $\mathcal{T}$ of type II or III.*

Proof. We know that a type III torus is of type I or II. We also know that there is a single conjugacy class of type I tori in M, and such tori are not of types II or III. So we only have to eliminate type II tori.

Let T be a type II torus inverted by the involution $w \in L$ where HL is an SL_2-block for which T is a maximal torus of L. By double transitivity we have $(M \cap M^w)^\circ = H_1T_1$ for some toral block of type I. Let H_1L_1 be the associated SL_2-block.

As H_1 centralizes A and A^w, we have $L_1 = \langle A, A^w \rangle$, so w normalizes L_1. By Corollary 4.11, $w \in L_1$. But $L_1 \cap H_1T_1 = T_1$ and hence w normalizes T_1. Now $C_{T_1}(w) = 1$, so w inverts T and T_1, and this forces $T = T_1$ (Lemma 7.2). So in this case T is of type I, and hence not of type II (Corollary 10.6). $\square$

Corollary 10.8 *The family $\mathcal{T}$ consists of a single conjugacy class under the action of M.*

Proof. By Lemma 5.2 (2). $\square$

Our next goal is the connectivity of M. The following is a consequence of Lemma 2.4.

Lemma 10.9 *For $T \in \mathcal{T}$ we have $N_M(T) = C_M(T)$.*

Proposition 10.10 *M is connected.*

Proof. Take a toral block $HT \leq M$ and an involution $w \in C(H)$ inverting T. Suppose toward a contradiction that $x \in M \setminus M^\circ$.

Now

$$\mathrm{rk}\,(C^\circ(A)wM^\circ) = c + m - \mathrm{rk}\,(C^\circ(A) \cap (M^\circ)^w)$$

and $[C^\circ(A) \cap (M^\circ)^w]^\circ = [C^\circ(A) \cap (HT)]^\circ = H$, so $\mathrm{rk}\,(C^\circ(A)wM^\circ) = c + m - h = g$ and $C^\circ(A)wM^\circ$ is generic in G.

More generally, the same holds with w replaced by any $g \in G \setminus M$, since $C^\circ(A)gM^\circ = C^\circ(A)M^{g^{-1}}g$ and the rank of this set is clearly determined by the pair $(M, M^{g^{-1}})$, as $C^\circ(A) = C^\circ(\Omega_1(O_2{}^\circ(M)))$.

We conclude that both $C^\circ(A)wM^\circ$ and $C^\circ(A)wxM^\circ$ are generic in G, and hence that $C^\circ(A)wM^\circ = C^\circ(A)wxM^\circ$. So we have an equation $cwM^\circ = wxM^\circ$ with $c \in C^\circ(A)$. So $c^w \in xM^\circ \subseteq M$ and $c \in C^\circ(A) \cap M^w \leq N(A) \cap N(A^w)$. Let $L = \langle A, A^w \rangle$. So $c \in N(L)$ and $c \in C^\circ(A)$. Then c acts on L like an element of A while normalizing A^w, forcing $c \in C(L)$, and in particular c commutes with w. The equation reduces to $wM^\circ = wxM^\circ$ and shows $x \in M^\circ$. $\square$

Corollary 10.11 *$C^\circ(A) = C(A) = C(i)$ for $i \in I(A)$.*

Proof. We know $C(A) = C(i)$ by Lemma 2.4.

We also have $M = C(A)T$ and $M = M^\circ = C^\circ(A)T$ and as $C(A) \cap T = 1$ our claim follows. □

Lemma 10.12 *Let $T \in \mathcal{T}$, HL a corresponding SL_2-block with T a maximal torus in L, and $w \in I(L)$ inverting T. Let $\hat{H} = C(L)$. Then*

1. *$M \cap M^w = \hat{H}T$.*
2. *$C(A) \cap M^w = C(L)$.*

Proof. For the first point, consider $R = C_{M\cap M^w}(A)$. Then $M \cap M^w = RT$ and $\hat{H} \leq R$, so the claim is that R centralizes L. Now $R^\circ = H$ so R normalizes H and hence also L. As R centralizes A, R acts on L like a subgroup of A. So $R \leq A\hat{H}$ and $R = \hat{H} \cdot (A \cap R)$. But $A \cap R \leq A \cap M^w = 1$, so $R = \hat{H}$. This proves (1).

Now $M \cap M^w = TC(L)$, so $C(A) \cap M^w = C(A) \cap TC(L) = C(L)$. □

Lemma 10.13 *Let $T \in \mathcal{T}$. Then for $t \in T^\times$ we have $C(t) \leq M$.*

Proof. Let HL be the associated SL_2-block and fix $w \in L$ inverting T. We have $G \setminus M = C(A)wM = C(A)wC(A)T$ and we claim that nothing in this set commutes with t.

Assuming the contrary, we get an element $c_1wc_2 \in C(t)$ with $c_1, c_2 \in C(A)$. So $(c_1wc_2) = (c_1wc_2)^t = c_1^t w^t c_2^t$. Now $w^t = wt^2$. So we find $xwt^2 = wy$ for appropriate x, y in $C(A)$, or $x^w = yt^{-2} \in C(A)^w \cap M \leq M \cap M^w = C(L)T$ by the Lemma 10.12. So $x^w \in C_{C(L)T}(A^w) = C(L)$.

As $y \in C(A)$ and $x^w = yt^{-2}$ this implies $t^{-2} \in C(A)$ and hence $t = 1$. This contradiction proves the claim. □

Proposition 10.14 *M is strongly embedded in G.*

Proof. Supposing the contrary, we have an *offending involution* $i \in M$, that is an involution i whose centralizer is not contained in M. If $C^\circ(i) \leq M$ then it follows easily by a Frattini argument that $C(i) \leq M$; so we have $C^\circ(i) \not\leq M$.

Then by Fact 1.24, we have $C^\circ(i) = \hat{H}_i \times L_i$ with L_i of type SL_2, and $\hat{H}_i \leq C^\circ(L_i)$ of degenerate type; $\hat{H}_i \leq M$ and $M \cap L_i$ is a Borel subgroup of L_i. Possibly $\hat{H}_i = 1$. Set $H_i = C^\circ(L_i)$, which is nontrivial in view of Proposition 4.8.

Now let us replace i by one of its conjugates outside M. Then $C^\circ(i)$ has the same structure as above, and we will again write $C^\circ(i) = \hat{H}_i L_i$, and $H_i = C^\circ(L_i)$.

On the other hand, by Proposition 10.3 we also have $(M \cap M^i)^\circ = HT$ with $T = (M \cap M^i) \cap L$ for some SL_2-block HL; specifically, $L = \langle A, A^i \rangle$. Here i acts on L, with nontrivial action, and hence also acts on H. As i acts as an involution on L, it centralizes some Sylow 2-subgroup A_0 of L. Therefore $A_0 \leq L_i \cap L$.

Now set $K = C_H{}^\circ(i) \leq H_i L_i$, and suppose $K \neq 1$. Then K centralizes L, hence centralizes A_0. As K acts on L_i centralizing A_0 and is of degenerate type, K centralizes L_i by Fact 1.20. Thus $K \leq H \cap H_i$ and $L = L_i$, which contradicts the action of i, trivial in one case and nontrivial in the other. We conclude, therefore, that $C_H{}^\circ(i) = 1$ and i inverts H, which must then be abelian.

Now i acts on L like an involution $w \in L$, and in particular i commutes with w. Let $j = iw$, another involution. Then j centralizes L and inverts H. Furthermore $j \in N(A) = M$. We will study the coset jHT. As HT is abelian and j inverts H, $j \notin HT$.

As M is connected, the conjugates of HT in M are generic in M by Proposition 3.1. As $N^\circ(HT) = HT$, it follows by Fact 1.44 that the set

$$X = \{x \in jHT : \text{For some } g \in M \setminus N_M(HT),\ x \in (\langle j\rangle HT)^g\}$$

is generic in the coset jHT, as otherwise the conjugates of jHT provide a second generic subset of M which is generically disjoint from the union of the conjugates of HT.

For $a \in jHT$, if we write $a = jht$ with $h \in H$, $t \in T$ we find $a^2 = (ht)^j(ht) = t^2 \in T$. Therefore for $a \in X$, if $g \in M \setminus N(HT)$ and $a \in (\langle j\rangle HT)^g$, we find $a^2 \in T \cap (HT)^g$.

There is a coset jhT of T with $h \in H$ for which $X \cap jhT$ is generic in jhT. In other words, the set $T_0 = \{t \in T : jht \in X\}$ is generic in T, and for $t \in T_0$ we have $t^2 \in T \cap (HT)^g$ for some $g \in M \setminus N_M(HT)$. So the set of all such intersections $T \cap (HT)^g$ generically covers T, and as T is a good torus we find one such $g \in M \setminus N_M(HT)$ for which $T \leq (HT)^g$. As HT is abelian, we have $(HT)^g \leq C^\circ(T) = HT$ by Proposition 5.6 and Lemma 10.7. Hence $g \in N(HT)$, a contradiction. □

Corollary 10.15 *$I(G) = I_1$ and H contains no involutions*

Proof. All involutions in G are conjugate by strong embedding. On the other hand $I_1 \cap M = A$, so H contains no involutions. □

11 The final chapter

In this section we study elements of order 3 in wM, in order to reach a contradiction, proving Theorem 2. Here we fix $w \in I(G) \setminus M$ and write $(M \cap M^w)^\circ = HT$ with HT a toral block, corresponding to an SL_2-block HL with $w \in L$.

11.1 Taking stock

We first rework some of our earlier results to put them in the form we will require here.

Lemma 11.1 *For any $w \in I(G) \setminus M$, setting $L = \langle A, A^w\rangle$, we have the following.*

1. L is of type SL_2 and $N_L(A) \cap N_L(A^w)$ is a torus inverted by w, contained in $M \cap M^w$, and $w \in L$.

2. $G = M \sqcup C(A)wC(A)T$; the representation is unique in the following sense:

3. If $c_1wc_2t_1 = c_1'wc_2't_1'$ with $c_1, c_2, c_1', c_2' \in C(A)$ and $t_1, t_1' \in T$, then $t_1 = t_1'$ and for some $x \in C(L)$ we have $c_1 = c_1'x$, $c_2 = x^{-1}c_2'$.

Proof. 1. The first claim follows by double transitivity, Proposition 10.3. The rest then follows, using Corollary 4.11 and Corollary 10.15 for the final point.

2. Again, this is double transitivity, together with Lemma 2.4.

3. The equation can be written in the form

$$x^w = c_2' t c_2^{-1}$$

with $x = (c_1')^{-1} c_1$ and $t = t_1' t_1^{-1}$. Hence $x \in C(A) \cap M^w = C(L)$ by Lemma 10.12. As $w \in L$ our equation becomes $x = c_2' t c_2^{-1}$ and hence $t \in C(A)$. As $t \in T$ we find $t = 1$ and $t_1 = t_1'$, $x = c_2' c_2^{-1}$, and everything has been checked. $\square$

Lemma 11.2 *For $T \in \mathcal{T}$ and $t \in T^\times$ we have $C(t) = C(T) \leq M$.*

Proof. By Lemma 10.13 we have $C(t) \leq M$. Taking $w \in I(G)$ which inverts T we have $C(t) \subseteq M^w$. Now $C(t) = T \cdot C(t, A)$ and $C(t, A) \leq C(A) \cap M^w = C(L)$ by Lemma 10.12. As $C(L) \leq C(T)$ we find $C(t) \leq C(T)$. $\square$

Lemma 11.3 *Let T be the toral component of a toral block, and suppose $T \cap M \neq 1$. Then $T \leq M$.*

Proof. Let $T_0 = T \cap M$. Let $w \in I(G)$ invert T. As w inverts T_0, we have $w \notin M$. Thus $L = \langle A, A^w \rangle$ is a group of type SL_2 and T_0 acts on L normalizing A and A^w. Hence T_0 normalizes the torus $T_1 = N_L(A) \cap N_L(A^w)$, and by Lemma 10.9, it also centralizes T_1. So $T_1 \leq C(T_0) = C(T)$ by Lemma 11.2. Thus $T \leq C(T_1) \leq M$ by Lemma 10.13. $\square$

The next result corresponds to Lemma 4.5 of [16], which in the K^* context was formulated as follows: $O(F(M)) > 1$. This was a cornerstone of the analysis as given in [16], which our analysis follows in parallel, allowing for the difference in perspective and notation resulting from the removal of the theory of solvable groups from the discussion, of which the present instance provides a good example.

Lemma 11.4 *$C(A) > AH$*

Proof. Suppose $C(A) = A \times H$. Then as $H = O(AH)$ is characteristic in AH we have $M \leq N(H)$. As $w \in C(H)$ we find $G = M \cup MwM \subseteq N(H)$, $H \triangleleft G$. Hence $H = 1$, contradicting the hypothesis $(\neg *)$. $\square$

Lemma 11.5 *Suppose H_1T_1 and H_2T_2 are toral blocks, $t_1 \in T_1^\times$, $t_2 \in T_2^\times$, and t_1, t_2 commute. Then $T_1 = T_2$ and $H_1 = H_2$.*

Proof.

As $C(t_1) = C(T_1)$ and $C(t_2) = C(T_2)$, the tori T_1 and T_2 commute.

We may suppose that $H_1T_1 \leq M$. Then T_1 is a type I toral block by Lemma 10.7, associated with an SL_2-block H_1L_1. If $w \in L_1$ is an involution inverting T_1, then $C^\circ(T_1) = H_1T_1 = (M \cap M^w)^\circ$ by Proposition 5.6 and Lemma 5.8. So $T_2 \leq M \cap M^w$ and thus $C(T_2) \leq M \cap M^w$. On the other hand T_2 belongs to a toral block H_2T_2 and just as in the case of T_1, we have $C^\circ(T_2) = H_2T_2$. So we have $H_2T_2 \leq H_1T_1$ and as the ranks are equal, we have $H_1T_1 = H_2T_2$.

As both T_1 and T_2 are type I, we have $H_1 = C_{H_1T_1}(A) = H_2$. So the associated SL_2 components $L_1 = B(C(H_1))$ and $L_2 = B(C(H_2))$ also coincide. So T_1, T_2 are commuting maximal tori in SL_2, hence coincide. $\square$

11.2 Elements of order 3

We now begin the study of elements of order 3 in G, and more particularly, those which are conjugate to an element of the SL_2-component of an SL_2-block.

The following is fundamental, and will be used without explicit mention.

Lemma 11.6 *Let $x = wct$ with $c \in C(A)$ and $t \in T$. Then $x^3 = 1$ if and only if $c^w = c^{-t}wc^{-1}t$.*

Proof. Expanding and bearing in mind $wt = t^{-1}w$, we find

$$(wct)^3 = c^w t^{-1} cwc^t = c^w (c^{-t}wc^{-1}t)^{-1}$$

□

Corollary 11.7 *If $c \in C(A)$, $t_1, t_2 \in T$, and wct_1, wct_2 are of order 3, then $t_1 = t_2$.*

Proof. The representation of c^w is unique (Lemma 11.1). □

Definition 11.8

1. *An element $t \in G$ is* toral *if t lies in a toral component T of some toral block.*
2. *$X_3 = \{c \in C(A) : \exists t \in T$ [$\mathrm{ord}\,(wct) = 3$ and wct is toral]$\}$*

Lemma 11.9 *X_3 is invariant under conjugation by HT.*

Proof. Suppose wct is a toral element of order 3. For $h_1 \in H$ and $t_1 \in T$ we may compute

$$(wct)^{h_1 t_1^{-1}} = w^{t_1^{-1}} c^{h_1 t_1^{-1}} t = wt_1^{-2} c^{h_1 t_1^{-1}} t = wc^{h_1 t_1} t_1^{-2} t$$

and from this we see that $c^{h_1 t_1} \in X_3$. □

Lemma 11.10 $X_3 \cap AC(L) = A^\times$.

Proof.

Working inside SL_2 one can see that X_3 meets $A^\times$; since $A^\times$ is a single conjugacy class under the action of T, by Lemma 11.9 we have $A^\times \subseteq X_3$.

Now suppose $ax \in X_3$ with $a \in A$, $x \in C(L)$. Then we have $t \in T$ with $waxt$ a toral element of order 3, and thus

$$(ax)^w = (ax)^{-t} w (ax)^{-1} t = (a^{-t} wat) x^{-2}$$

Comparing this with $(ax)^w = a^w x$ yields $x^3 \in L$. As $x \in C(L)$ this gives $x^3 = 1$ and hence from the foregoing equations, $a^w = a^{-t}wat$. Hence wat is also a toral element of order 3. As wat and $waxt$ commute, the associated torus is the same in both cases, by Lemma 11.5. As $wat \in L$ this implies $waxt \in L$, forcing $x = 1$, as claimed. Then evidently $a \neq 1$. □

Lemma 11.11 $\mathrm{rk}\,(X_3) = c - h$

Proof. Fix $T \in \mathcal{T}$ and $t_0 \in T$ of order 3. Note that the toral elements of order three make up the conjugacy class t_0^G, and there is a function $wM \cap t_0^G \leftrightarrow X_3$ defined by $wct \mapsto c$ for $c \in C(A)$, $t \in T$ (when $wct \in t_0^G$). This is a bijection since c determines t by Corollary 11.7.

Now $\mathrm{rk}\,(t_0^G) = g - \mathrm{rk}\,(C(t_0))$ and $C(t_0) = C(T) = C_M(T)$ so in view of Propositions 9.1 and 5.6, we have $\mathrm{rk}\,(t_0^G) = (m + c - h) - (a + h) = 2(c - h)$.

On the other hand $\mathcal{T}$ is a single conjugacy class in M, so $M \cap t_0^G = \{t_0, t_0^{-1}\}^M$, taking into account Lemma 11.3, and $rk(t_0^M) = m - a - h = c - h$. Hence t_0^G lies generically outside M, and by double transitivity it is evenly distributed over the cosets gM for $g \in G \setminus M$. Then writing $r = \mathrm{rk}\,(gM \cap t_0^G)$ with r constant, we find

$$2(c - h) = \mathrm{rk}\,(t_0^G) = r + \mathrm{rk}\,(G/M) = r + g - m = r + c - h$$

and hence $r = c - h$, as claimed. □

Definition 11.12 $X_3' = X_3 \setminus A$.

Lemma 11.13 X_3' *is generic in* X_3.

Proof. We know that $\mathrm{rk}\,(X_3) = c - h$, and $c - h > a$ by Lemma 11.4. □

Lemma 11.14 $\mathrm{rk}\,(X_3'C(L)) = \mathrm{rk}\,(C(A))$ *and the natural map* $X_3' \times C(L) \leftrightarrow X_3'C(L)$ *is a bijection.*

Proof. Since $\mathrm{rk}\,(X_3' \times C(L)) = \mathrm{rk}\,(C(A))$, the second claim suffices. Suppose therefore that $c \in X_3'$, $h \in C(L)$, and $ch \in X_3'$. We must show that $h = 1$.

Now we have

$$c^w = c^{-t_1} w c^{-1} t_1$$

and then

$$(ch)^w = (ch)^{-t_2} w (ch)^{-1} t_2 = h^{-1} c^{-t_2} h^{-1} w c^{-1} t_2; \tag{1}$$

$$(ch)^w = c^w h = c^{-t_1} w c^{-1} h t_1 \tag{2}$$

Comparing these representations, we have $t_1 = t_2$ and hence $h^{-1} c^{-t_1} h^{-1} w c^{-1} = c^{-t_1} w c^{-1} h$, so that $(c^{t_1} c^{-t_1 h} h^{-2})^w = c^{-1} h c \in C(A) \cap C(A^w) = C(L)$.

If $h \neq 1$ then as $h^c \in C(L)$ it follows easily that $c \in N(L)$: $L^c = B(C(h))^c = B(C(h^c)) = L$. Hence $c \in C(A) \cap N(L) = AC(L)$. Thus $c \in X_3' \cap AC(L) = \emptyset$ by Lemma 11.10. □

Corollary 11.15 $H = C(L)$

Proof. By the first part of Lemma 11.13 and Corollary 10.11, the set $X_3'C(L)$ has Morley degree 1, and hence by the second part of Lemma 11.13 the set $X_3' \times C(L)$ also has Morley degree 1. Hence $C(L)$ has Morley degree 1, and so $C(L) = C^\circ(L) = H$. □

Now at long last we can derive a contradiction from our initial assumption $(\neg *)$.

Proof of Theorem 2. We are going to calculate a fairly complicated element of G in two distinct ways to reach a contradiction.

We fix the usual notation HL, HT with $HT \le M$ and $w \in I(L)$ inverting T. We choose a base point $a_0 \in A$ satisfying $a_0^w = a_0 w a_0$ (this takes place inside SL_2).

We know that $X_3' H$ is generic in $C(A)$ by Lemma 11.14 and Corollary 11.15. If we consider the intersections $cAH \cap X_3'H$ with $c \in C(A)$, it follows that for a generic set of $c \in C(A)$, the intersection of cAH with $X_3'H$ is generic in cAH. Equivalently, fixing our base point $a_0 \in A$, the set of $c \in C(A)$ satisfying

$$(\bullet) \qquad \{t \in T : ca_0^t \in X_3'H\} \text{ is generic in } T$$

is generic in $C(A)$. As $X_3'H$ is generic in $C(A)$, the set of $c \in X_3'H$ satisfying the same condition $(\bullet)$ is generic in $C(A)$. As this set is closed under multiplication by H on the right, and as the multiplication map $X_3' \times H \to X_3'H$ is bijective, the set X_3'' of $c \in X_3'$ satisfying the condition $(\bullet)$ is generic in X_3. Now X_3 is closed under inversion, as can be seen by inverting both sides of the equation $c^w = c^{-t} w c^{-1} t$. So the following set is generic in X_3'.

$$\{c \in X_3' : c, c^{-1} \in X_3''\}$$

Fix such an element $c \in X_3''$. Then the following sets are generic in T.

$$\{t \in T : ca_0^t \in X_3'H\}, \quad \{t \in T : c^{-1} a_0^{t^{-1}} \in X_3'H\}$$

Hence also their intersection

$$T_0(c) = \{t \in T : ca_0^t \in X_3'H; c^{-1}a_0^{t^{-1}} \in X_3'H\}$$

is generic in T.

Set $T_0 = T_0(c)$. As $c \in X_3'$ we have $c^w = c^{-t_1} w c^{-1} t_1$ for some $t_1 \in T$. Choose $t_0 \in T_0 \cap t_1^{-1} T_0$. Set $a = a_0^{t_0}$ and $t = t_0^{-2}$.

After these preparations, we can begin the calculation of $(ca)^w$ in two ways. We have $a_0^{t_0^{-1}} = a^t$ and hence

$$\begin{array}{rcl} a^w & = & a_0^{wt_0^{-1}} = a^t w a t \\ c^w & = & c^{-t_1} w c^{-1} t_1 \\ ca & \in & X_3'H \\ c^{-1} a^{tt_1^{-1}} & = & c^{-1} a_0^{(t_0 t_1)^{-1}} \in X_3'H \end{array}$$

Therefore we have expansions of the following form, for suitable $h_2, h_3 \in H$ and $t_2, t_3 \in T$.

$$\begin{array}{rcl} (cah_2)^w & = & (cah_2)^{-t_2} w (cah_2)^{-1} t_2 \\ & = & h_2^{-1} a^{t_2} c^{-t_2} w h_2^{-1} a c^{-1} t_2; \\ (c^{-1} a^{tt_1^{-1}} h_3)^w & = & (c^{-1} a^{tt_1^{-1}} h_3)^{-t_3} w (c^{-1} a^{tt_1^{-1}} h_3)^{-1} t_3 \\ & = & h_3^{-1} a^{tt_1^{-1} t_3} c^{t_3} w h_3^{-1} a^{tt_1^{-1}} c t_3 \end{array}$$

We now compute $(ca)^w$ again:

$$\begin{aligned}
(ca)^w &= c^w a^w = c^{-t_1} w c^{-1} t_1 a^t wat \\
&= c^{-t_1}\left[c^{-1}a^{tt_1^{-1}}\right]^w t_1^{-1}at \\
&= c^{-t_1}[h_3^{-1}a^{tt_1^{-1}t_3}c^{t_3}wh_3^{-1}a^{tt_1^{-1}}ct_3h_3^{-1}]a^{t_1}t_1^{-1}t \\
&= [a^{tt_1^{-1}t_3}c^{-t_1}h_3^{-1}c^{t_3}]w[a^{tt_1^{-1}+t_1t_3^{-1}}h_3^{-1}ch_3^{-1}]t_1^{-1}tt_3
\end{aligned}$$

Comparing this with

$$(ca)^w = [a^{t_2}h_2^{-1}c^{-t_2}]w[ah_2^{-1}c^{-1}h_2^{-1}]t_2$$

we find (Lemma 11.1)

$$\begin{aligned}
&(3) & t_1^{-1}tt_3 &= t_2 \\
&(4) & a^{tt_1^{-1}t_3}c^{-t_1}h_3^{-1}c^{t_3} &= a^{t_2}h_2^{-1}c^{-t_2}\cdot h, \qquad \text{some } h\in H \\
&(5) & a^{tt_1^{-1}+t_1t_3^{-1}}h_3^{-1}ch_3^{-1} &= h^{-1}\cdot ah_2^{-1}c^{-1}h_2^{-1}
\end{aligned}$$

Now put (5) in the form

$$\begin{aligned}
a^{t^*} &= h^{-1}h_2^{-1}c^{-1}h_2^{-1}h_3c^{-1}h_3 \\
&= h'c^{-1}h''c^{-1}
\end{aligned}$$

with $h', h'' \in H$ and $t^* = tt_1^{-1} + t_1t_3^{-1} + 1$ in $T \cup \{0\}$, where at the end we conjugate by h_3^{-1} before collecting terms. We can recast this further as

$$a^{t^*} = h^*(h''c^{-1})^2$$

with $h^* = h'h''^{-1}$.

As the element $h''c^{-1}$ centralizes a^{t^*}, it also centralizes h^*. If $h^* \neq 1$ then $h''c^{-1} \in N_{C(A)}(H) = AC(L)$ and hence $c \in AC(L)$, a contradiction to Lemma 11.10. We conclude that $h^* = 1$ and hence $a^{t^*} = (h''^{-1}c^{-1})^2$, or after inversion:

$$a^{t^*} = (ch'')^2$$

Now if $t^* = 0$ then $(ch'')^2 = 1$ and hence by strong embedding $ch'' \in A$, again contradicting Lemma 11.10. So $t^* \in T$, and $a^{t^*} \in A^\times$.

Let S be a Sylow 2-subgroup of $C(A)$, and $\Phi(S)$ its Frattini subgroup, the subgroup generated by the commutator subgroup S' together with all squares of elements in S. As S/A is finite and A is central in S, it follows easily that $\Phi(S)$ is finite, and in particular definable. By a Frattini argument, $M = N(C(A)) = C(A)N_M(\Phi(S) \cap S^\circ) = N_M(\Phi(S) \cap A)$. So M normalizes the finite group $\Phi(S) \cap A$ and hence centralizes it, which forces $\Phi(S) \cap A = 1$. However as seen above, $a^{t^*} \in \Phi(S)^\times$ if $ch \in S$.

This contradiction proves Theorem 2. □

Now Theorem 1 follows from Theorem 2.

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