Simple L^*-groups of even type with weakly embedded subgroups

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1 Introduction

It has been conjectured that infinite simple groups of finite Morley rank are linear algebraic groups over algebraically closed fields. In this paper we continue to follow a recent approach, which, if successful, would give an affirmative partial answer to this *algebraicity conjecture* in the case of the simple groups of finite Morley rank of *even type*, which would be independent from inductive hypotheses but still use ideas and methods coming from the inductive approach. This is the analysis of *L^*-groups*, initiated in [4] (see Section 2 below for definitions).

The notion of L^*-group is a weakening of the notion of *K^*-group* (a group of finite Morley rank all of whose infinite definable simple proper sections are algebraic groups over algebraically closed fields), and an affirmative answer to the algebraicity conjecture for the simple L^*-groups of even type would in fact be an affirmative answer to the entire conjecture for the simple groups of finite Morley of even type although this is not the case if one uses the stronger notion of K^*-group. The major difficulty that appears when one weakens one's hypotheses from K^* to L^* is that definable connected sections of *degenerate type* may no longer be solvable. We refer the reader to [4] and [5] for a detailed account of these issues.

The notion of *weakly embedded subgroup*, which is expected to yield a characterization of PSL_2 over an algebraically closed field of characteristic 2, has turned out to be a key notion for the classification of simple K^*-groups of even type. The same is expected in the L^*-context since the analysis of simple L^*-groups of even type is modeled on that of simple K^*-groups of even type. As a result, in [4], an initial analysis of weak embedding was made in the L^*-context, and in [5], a first L^*-characterization of PSL_2 in characteristic 2, using the stronger notion of *strongly embedded* subgroup, was obtained under additional hypotheses.

In this paper we will extend the results of [5] to simple L^*-groups of even type with weakly embedded subgroups. The additional hypothesis (the hypothesis (*) in the statement of Theorem 1 below) will be kept. Nevertheless, the results obtained until Section 6 are independent from the hypothesis (*) and are expected to provide the first steps of the analysis in the absence of this hypothesis. The main theorem in this paper is the following:

Theorem 1 *Let G be a simple L^*-group of even type with a weakly embedded subgroup M. Assume*

(∗) *$C_G(A_1, A_2)$ is finite whenever A_1 and A_2 are two distinct conjugates of $\Omega_1^\circ(M)$.*

Then $G \cong \mathrm{PSL}_2(F)$, where F is an algebraically closed field of characteristic 2.

The proof of Theorem 1 can be seen as a reduction to the main theorem of [5]. Indeed, what is finally proven is that under the given assumptions, weak embedding is equivalent to strong embedding. The main ingredient to achieve this is the analysis of the Sylow 2-structure in a simple L^*-group of even type with a weakly but *not* strongly embedded subgroup. In this vein, we will prove Theorems 2 and 3 below. Theorem 1 will be proven only after Theorem 3 completely clarifies the structure of the Sylow° 2-subgroups of the simple group under analysis.

This paper can be seen as an L^*-analogue of [3]. Indeed, the analysis of the Sylow° 2-subgroups is based on a finite Morley rank analogue of a theorem of Landrock and Solomon ([12]), proven in [3] (Fact 3.2 below). The ensuing preliminary analysis follows closely the K^*-case provided one is careful about sections of degenerate type (see for example Fact 3.8).

Similarities become more limited in Section 4 although the reduction has the same final target as in [3]. The proof of Theorem 3 requires a much longer analysis than in the K^*-context because it is much harder to *linearize* nonsolvable groups. This is the most substantial result in the paper, and its proof involves arguments of considerable power in the context of simple L^*-groups, notably *genericity arguments* using *good tori*. On the other hand, the analysis in Subsection 4.1 and the proof of Theorem 2 are substantially simpler than the corresponding section in [3]. Moreover, they are more correct! Some time ago we discovered a gap in the proof of Lemma 7.11 of [3]. The complicated Thompson rank analysis for which this lemma was needed disappeared in the course of eliminating this gap. As a result the proof of Theorem 2 no longer requires an elaborate fusion analysis.

The rest of the paper is more in the spirit of [5] and Section 3 of [11]. At various points however the arguments have their own peculiarities, notably in Sections 5 and 6.

2 Background

Before we go over the necessary background material, it is worth recalling some fundamental facts and definitions.

Fact 2.1 ([7]) *Let G be a group of finite Morley rank. Then the Sylow 2-subgroups of G are conjugate. If S is a Sylow 2-subgroup of G then $S^\circ = B * T$ where B is a definable connected group of bounded exponent, T is divisible abelian, $*$ denotes the central product and $B \cap T$ is finite.*

The *connected component of S*, denoted by S°, is $d(S)^\circ \cap S$. In this definition, $d(S)$ is the *definable closure* of S, the smallest definable subgroup of G containing S. By the descending chain condition on definable subgroups in groups of finite Morley rank, such a smallest subgroup exists; it is the intersection of all definable subgroups of G containing S.

Definition 2.2

1. *A* unipotent *subgroup is a connected definable solvable subgroup of bounded exponent.*
2. *A p-*torus *is a divisible abelian p-group. It is the direct sum of copies of the quasicyclic group $\mathbb{Z}_{p^\infty}$. The* Prüfer p-rank *of a p-torus is the number of these factors.*

   In a group of finite Morley rank a torus *is a definable divisible abelian subgroup. Since it is divisible it is connected. The Prüfer p-rank of a torus is the Prüfer p-rank of its maximal p-torus. By exercise 9 on page 93 of [6], this is finite. A nontrivial p-torus is not definable, but its definable closure is a torus.*
3. *A group of finite Morley rank is of* even type *if the connected component of a Sylow 2-subgroup is unipotent and nontrivial.*

4. *A group of finite Morley rank is of* odd type *if the connected component of a Sylow* 2*-subgroup is a nontrivial* 2*-torus.*

5. *A group of finite Morley rank is of* mixed type *if the connected component of a Sylow* 2*-subgroup is the central product of a nontrivial unipotent subgroup and a nontrivial* 2*-torus.*

6. *A group of finite Morley rank is of* degenerate type *if the connected component of a Sylow 2-subgroup is trivial (that is, the Sylow 2-subgroups are finite).*

7. *A* Sylow° 2-subgroup *of a group G of finite Morley rank is the connected component of a Sylow* 2*-subgroup of G.*

Definition 2.3

1. *An L*-group *is a group of finite Morley rank in which every infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type; in other words, we exclude definable simple sections of mixed type, and we require definable simple sections of even type to be algebraic.*

2. *An L^**-group *is a group of finite Morley rank in which every* proper *infinite definable simple section is either an algebraic group over an algebraically closed field, or of odd or degenerate type.*

2.1 From nilpotent to solvable

Fact 2.4 ([15]) *Let H be a nilpotent group of finite Morley rank. Then $H = D * B$, where D and B are definable characteristic subgroups, with D divisible and B of bounded exponent. Moreover, $D \cap B$ is finite and B is the direct sum of its maximal unipotent p-subgroups.*

In a group G of finite Morley rank, $F(G)$ is the *Fitting subgroup* of G. This is the subgroup of G generated by its normal nilpotent subgroups; it is definable and nilpotent ([14]). The analogous notion for solvable subgroups is the *solvable radical* of G, denoted by $\sigma(G)$. It is the subgroup generated by the normal solvable subgroups of G. In [14], it is shown that $\sigma(G)$ is definable and solvable for G of finite Morley rank.

Fact 2.5 ([13]) *Let G be a connected solvable group of finite Morley rank. Then $G/F^\circ(G)$ is divisible and abelian.*

2.2 Elements of finite order

Fact 2.6 ([6, Exercise 11, p. 93]) *Let G be a group of finite Morley rank and H a normal definable subgroup. If $x \in G$ is such that $\overline{x} \in G/H$ is a p-element, then the coset xH contains a p-element.*

Fact 2.7 $\mathrm{Aut}(\mathbb{Z}_{p^\infty})$ *has no elements of order p.*

Fact 2.8 ([7]) *Let T be a p-torus in a group G of finite Morley rank. Then $|N_G(T) : C_G(T)| < \infty$. Moreover there exists a natural number c such that $|N_G(T) : C_G(T)| < c$ for any p-torus $\leq G$.*

Fact 2.9 ([6, Theorem 9.29];[9]) *The Hall π-subgroups of a connected solvable group of finite Morley rank are connected.*

Fact 2.10 ([3, Proposition 2.43]) *Let $G = H \rtimes Q$ be a group of finite Morley rank where H, Q and the action of Q on H are definable. Let $H_1 \lhd H$ be a solvable Q-invariant definable π-subgroup of bounded exponent in G. Assume that Q is a solvable $\pi^\perp$-subgroup. Then $C_H(Q)H_1/H_1 = C_{H/H_1}(Q)$.*

Fact 2.11 ([6, Proposition 10.2]) *Let G be a group of finite Morley rank, and i, j be two involutions in $I(G)$. Then, either i and j are conjugate by an element of $d(ij)$, or they commute with an involution in $d(ij)$.*

Fact 2.12 ([6, Exercise 13, p. 78]) *Let G be a group of finite Morley rank. Let α be a definable involutive automorphism with finitely many fixed points. Then there is a definable normal subgroup B of G of finite index in G and inverted by α.*

Fact 2.13 ([6, Exercise 14, p. 73]) *Let G be a group of finite Morley rank without involutions. If α is a definable involutive automorphism of G then $G = C_G(\alpha)G^-$, where $G^- = \{g \in G : g^\alpha = g^{-1}$. Moreover, for $c \in C_G(\alpha)$ and $g \in G^-$, the map $(c, g) \longmapsto cg$ is a definable bijection. In particular, G is connected if and only if $C_G(\alpha)$ is connected and G^- is of Morley degree 1.*

Fact 2.14 (Borovik) *Let $G = UX$ be a group of finite Morley rank with $U \lhd G$. Assume that U, X and the action of X on U are definable. Let p be a prime number. Assume also that U is a unipotent p-subgroup of G, X is connected, solvable, and does not contain elements of order p. If the action of X on U is faithful then X is divisible and abelian.*

Fact 2.15 *If G is a nontrivial connected $2^\perp$-group of finite Morley rank then $C_G(x)$ is infinite for every $x \in G$.*

2.3 Weak/strong embedding

Definition 2.16

1. *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* strongly embedded *if $I(M) \neq \emptyset$ and for any $g \in G \setminus M$, $I(M \cap M^g) = \emptyset$.*

2. *Let G be a group of finite Morley rank. A proper definable subgroup M of G is said to be* weakly embedded *if M has infinite Sylow 2-subgroups and for $g \in G \setminus M$, $M \cap M^g$ has finite Sylow 2-subgroups.*

Fact 2.17 ([10, Theorem 9.2.1]) *Let G be a group of finite Morley rank with a proper definable subgroup M. Then the following are equivalent:*
1. *M is a strongly embedded subgroup.*
2. *$I(M) \neq \emptyset$, $C_G(i) \leq M$ for every $i \in I(M)$, and $N_G(S) \leq M$ for every Sylow 2-subgroup of M.*
3. *$I(M) \neq \emptyset$ and $N_G(S) \leq M$ for every nontrivial 2-subgroup S of M.*

Fact 2.18 ([2]) *Let G be a group of finite Morley rank, M a proper definable subgroup of G. M is weakly embedded if and only if the following hold:*
1. *M has infinite Sylow 2-subgroups.*
2. *For any nontrivial unipotent 2-subgroup U and nontrivial 2-torus T in M, $N_G(U) \leq M$ and $N_G(T) \leq M$.*

2.4 L-groups

Fact 2.19 ([3, Proposition 3.4],[4]) *Let $X \rtimes Y$ be a group of finite Morley rank where X and Y are definable and connected, X is an L-group of even type, and Y is a $2^\perp$-group. Then Y normalizes a Sylow° 2-subgroup of X.*

Fact 2.20 ([6, Theorem 8.4]) *Let $\mathcal{G} = G \rtimes H$ be a group of finite Morley rank where G, H, and the action of H on G are definable, G is an infinite simple algebraic group over an algebraically closed field, and $C_H(G) = 1$. Then, viewing H as a subgroup of $\mathrm{Aut}(G)$, we have $H \leq \mathrm{Inn}(G)\Gamma$ where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G and Γ is the group of graph automorphisms.*

Remark 2.21 We will frequently use the special case of Fact 2.20 with $G = \mathrm{PSL}_2$. Here, as there are no nontrivial graph automorphisms, all definable actions induce inner automorphisms.

Fact 2.22 ([4]) *Let H be a connected L-group of even type with a weakly embedded subgroup M. Then*

$$H \cong L \times D$$

where $L = B(H) \cong \mathrm{SL}_2(F)$, with F algebraically closed of characteristic 2, and $D = C_H(L)$ is a subgroup of degenerate type. $M^\circ \cap L$ is a Borel subgroup of L and $D \leq M$.

2.5 Fields and good tori

As usual, Zil'ber's fundamental result will be an important ingredient in many arguments.

Fact 2.23 (Zil'ber) *Let $G = A \rtimes T$ be a group of finite Morley rank where A, T and the action of T on A are definable. Assume that T and A are abelian, $C_T(A) = 1$ and A is T-minimal. Then $A \cong F_+$, where F is an algebraically closed field and T is isomorphic to a subgroup of $F^\times$. The action of T on A is by scalar multiplication.*

The following theorem proven independently by Newelski and Wagner is indispensable in extending centralization in a torus from torsion elements to the ambient group.

Fact 2.24 ([16, 18]) *Let F be a field of finite Morley rank and T a definable subgroup of the multiplicative group $F^\times$ containing the multiplicative group of an infinite subfield of F. Then $T = F^\times$.*

The notion of good torus was introduced in [5]. In the context of L^*-groups it has already proven to be a fundamental tool. This will continue to be the case in this paper as well.

Definition 2.25 *A definable divisible abelian group T of finite Morley rank is a* good torus *if every definable subgroup of T is the definable closure of its torsion.*

Fact 2.26 ([5])

1. *If T is a good torus and $T_0 \leq T$ is definable and connected, then T_0 is a good torus.*
2. *A finite product of good tori is a good torus.*

Fact 2.27 ([5]) *Let $A \rtimes B$ be a solvable group of finite Morley rank where A, B, and the action of B on A are definable. Assume that A and B are connected and B acts on A faithfully. If A is an elementary abelian p-group for some prime p, and B has no nontrivial p-elements, then B is a good torus.*

2.6 Linearization

As in [4] and [5], the following three results will play an important role in linearizing actions of various groups on elementary abelian 2-groups.

Fact 2.28 ([6, Theorem 9.1]) *Let $A \rtimes G$ be a connected group of finite Morley rank where G is definable, A is abelian and G-minimal, and $C_G(A) = 1$. Assume further that G has a definable infinite abelian normal subgroup H. Then $C_A(G) = 1$, H is central in G, $F = \mathbb{Z}[H]/ann_{\mathbb{Z}[H]}(A)$ is an interpretable algebraically closed field, A is a finite dimensional F-vector space, and the action of G on A is by vector space automorphisms; so $G \leq \mathrm{GL}_n(F)$ via this action, where n is the dimension. Furthermore, $H \leq Z(G) \leq Z(\mathrm{GL}_n(F))$.*

Fact 2.29 ([6, Theorem 9.7]) *Let $A \rtimes G$ be a group of finite Morley rank such that $C_G(A) = 1$. Let $H \lhd G_1 \lhd G$ be definable subgroups with G_1 connected and H infinite abelian. Assume also that A is G_1-minimal. Then $K = \mathbb{Z}[Z(G^\circ)]/ann_{\mathbb{Z}[Z(G^\circ)]}(A)$ is an interpretable algebraically closed field, A is a finite dimensional vector space over K, G acts on A as vector space automorphisms and H acts scalarly. In particular, $G \leq \mathrm{GL}_n(K)$ for some n, $H \leq Z(G)$ and $C_A(G) = 1$.*

Fact 2.30 ([17]) *If F is a field of finite Morley rank of characteristic $p \neq 0$, then every simple definable section of $\mathrm{GL}_n(F)$ is definably isomorphic to an algebraic group over F.*

2.7 Genericity

In the context of L^*-groups, where definable connected sections of degenerate type are not necessarily solvable, genericity of various conjugacy classes replaces various conjugacy theorems known only for solvable groups of finite Morley rank. The following list contains the facts that will be needed in this paper.

Let us also recall an important notion which will be frequently used in the sequel. A *Borel subgroup* of a group of finite Morley rank is a definable connected solvable subgroup which is maximal with respect to these properties. Since infinite groups of finite Morley rank contain infinite definable abelian subgroups, a Borel subgroup is of finite index in its normalizer.

Fact 2.31 ([5]) *Let G be a connected group of finite Morley rank with a conjugacy class of definable divisible abelian subgroups that are of finite index in their normalizers. Assume that any two distinct elements of this family have finite intersection. If B is a subgroup in this family and $x \in N_G(B) \setminus B$, then $C_B(x)$ is finite.*

Fact 2.32 ([5]) *Let G be a connected group of finite Morley rank. Assume that B is a good torus which is of finite index in $N_G(B)$. Then the set $\mathcal{B} = \cup_{g \in G} B^g$ is generic in G.*

Fact 2.33 ([5]) *Let G be a connected group of finite Morley rank. Assume that B is a good torus which is of finite index in $N_G(B)$. Assume also that B_1 is a definable connected subgroup of G such that $\mathcal{B}_1 = \cup_{g \in G} B_1^g$ is a generic subset of G. Then B is conjugate to a subgroup of B_1.*

Fact 2.34 ([5]) *Let $A \rtimes G$ be a group of finite Morley rank where G, A and the action of G on A are definable. Assume that A is connected and elementary abelian of exponent 2, that G is connected of degenerate type, and that G acts faithfully on A. If B is a Borel subgroup of G then $B \cap (\cup_{g \in G \setminus N_G(B)} B^g)$ is not generic in B, and the Borel subgroups of G are conjugate in G.*

3 Preliminary analysis

The proof of Theorem 1 consists mainly of the analysis of the case left open in [5], that is, a simple L^*-group of even type with a weakly embedded subgroup which is *not* strongly embedded. In the sequel, unless otherwise stated, G will denote a simple L^*-group of even type with a weakly embedded subgroup M that is not strongly embedded. Moreover, until the end of Section 5 the hypothesis (*) will *not* be assumed. This more general part of the argument corresponds roughly to sections 3-8 of [3], where simple groups of finite Morley rank of even type with weakly but not strongly embedded subgroups were analyzed under stronger inductive assumptions. From Section 6 on, the assumption (*) will be applied.

By Facts 2.17 and 2.18, M has an *offending involution* α, that is, an involution α such that $C_\alpha \not\leq M$. As was already explained in [4], the starting point of the analysis is the following lemma:

Lemma 3.1 ([3], Lemma 3.1, Proposition 3.2; [4], Lemma 6.1) *Let G be a simple L^*-group of even type with a weakly embedded subgroup M that is not strongly embedded. Then there exists an involution $\alpha \in M$ with the following properties:*

1. *$C_\alpha^\circ \not\leq M$.*

2. $C_\alpha^\circ \cap M$ *is a weakly embedded subgroup of* C_α°.

3. $C_\alpha^\circ = L_\alpha \times D$ *where* $L_\alpha = B(C_\alpha)$, $D = C_{C_\alpha^\circ}(L_\alpha)$, *and* $L_\alpha \cong \mathrm{PSL}_2(K)$ *with* K *algebraically closed of characteristic* 2, *and* D *is a definable connected subgroup of degenerate type.*

4. $C_M^\circ(\alpha) = (L_\alpha \cap M) \times D$ *and* $L_\alpha \cap M = A \rtimes T$ *is a Borel subgroup of* L_α.

5. $\alpha \notin \sigma^\circ(C_\alpha^\circ)$.

Next, corresponding to Proposition 3.5 and Corollary 3.6 in [3], one shows using Fact 2.19 that M has an $(\langle\alpha\rangle \times T)$-invariant Sylow$^\circ$ 2-subgroup S that contains A, and that $C_S^\circ(\alpha) = A$. In this situation the following classification result of Landrock-Solomon type applies:

Fact 3.2 ([3], Theorem 4.1; [12]) *Let* $H = S \rtimes T$ *be a group of finite Morley rank, where* S *is a unipotent* 2*-group of bounded exponent, and* T *is also definable. Assume that* S *has a definable subgroup* A *such that* $A \rtimes T \cong K_+ \rtimes K^\times$ *for some algebraically closed field* K *of characteristic 2, with the multiplicative group acting naturally on the additive group. Assume also that* α *is a definable involutory automorphism of* H *such that* $C_H^\circ(\alpha) = A \rtimes T$. *Under these assumptions* S *is isomorphic to one of the following groups:*

(*i*) *If* S *is abelian then either* S *is homocyclic with* $I(S) = A^\times$, *or* $S = E \oplus E^\alpha$, *where* E *is a* T*-invariant elementary abelian group isomorphic to* K_+. *In the latter case,* $A = \{xx^\alpha : x \in E\}$, *and both* E *and* E^α *are* T*-modules.*

(*ii*) *If* S *is nonabelian then* S *is an algebraic group over* K *whose underlying set is* $K \times K \times K$ *and the group multiplication is as follows:*

For $a_1, b_1, c_1, a_2, b_2, c_2 \in K$,

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + \epsilon\sqrt{a_1a_2} + \sqrt{b_1b_2} + \sqrt{b_1a_2}),$$

where ϵ *is either* 0 *or* 1.

In this case α *acts by* $(a, b, c)^\alpha = (a, a+b, a+b+c+\sqrt{ab})$ *and* $[\alpha, S] = \{(0, b, c) : b, c \in K\}$.

In particular, if S *is nonabelian then* S *has exponent 4.*

We state for future reference several facts that were obtained while proving Fact 3.2. The notation will be the same as in Lemma 3.1 and Fact 3.2.

Fact 3.3 ([3, Proposition 4.5]) *Let* S *and* T *be as in the statement of Fact 3.2. Then for every* $t \in T^\times$, $C_S(t) = 1$.

Fact 3.4 ([3, Corollary 4.6]) *Let* S *and* T *be as in the statement of Fact 3.2. If* X *is a definable normal* T*-invariant subgroup of* S, *then for every* $t \in T^\times$, $C_{S/X}(t) = 1$.

Fact 3.5 ([3, Corollary 4.7]) *Let* S, T *and* α *be as in the statement of Fact 3.2. Any definable normal* T*-invariant subgroup of* S *is connected. In particular,* $C_S(\alpha)$ *is connected and thus* $C_S(\alpha) = A$.

S_1 will denote the largest, proper, normal, definable, $(\langle\alpha\rangle \times T)$-invariant subgroup of S. Note that S_1 is also connected by Fact 3.5.

Fact 3.6 ([3, Proposition 4.13]) $S_1 = [\alpha, S] = \{[\alpha, x] : x \in S\}$ *is an abelian group inverted by* α.

The following fact can be extracted from the analysis of the nonabelian case in Fact 3.2.

Fact 3.7 *For* S *nonabelian,* $\mathrm{rk}\,(S) = 3\mathrm{rk}\,(A)$.

As was explained in Section 6 of [4], one can adapt arguments from [3] and prove the following using Fact 3.2.

Fact 3.8 ([4, Theorem 1]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup M. Then $M/O_2^\circ(M)$ is of degenerate type.*

We will refer to this structural property of M, as well as of M°, as *weak solvability*. The weak solvability of M° has the following structural consequences:

Fact 3.9 ([3, Proposition 5.6]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup. If $i \in I(\sigma^\circ(M))$ then $C_i \leq M$.*

Fact 3.10 ([3, Proposition 5.7]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup. Then for $g \in G \setminus M$, we have $M^g \cap I(\sigma^\circ(M)) = \emptyset$.*

Fact 3.11 ([3, Corollary 5.8]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup. Then $I(G) = I_1 \sqcup I_2$, where I_1 is the set of involutions in G conjugate to an involution in $\sigma^\circ(M)$, and I_2 is the set of involutions conjugate to an involution in $M \setminus \sigma^\circ(M)$.*

Fact 3.12 ([3, Corollary 5.9]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup. Then M controls fusion in $I(\sigma^\circ(M))$.*

4 Sylow 2-structure

In this section, G will continue to denote a simple L^*-group of even type with a weakly but not strongly embedded subgroup M. We will continue to use the notation introduced in the previous section.

We will show (Theorem 3 below) that $S = O_2^\circ(M)$ is a homocyclic group with $I(S) = A^\times$. We will proceed in two steps. First, we will prove that the Sylow° 2-subgroup of M, namely S, is abelian (Theorem 2). This first step will leave us with the two possibilities in case (i) of Fact 3.2. In the second step we will reduce the structure of S to that of a homocyclic group such that $I(S) = A^\times$. These two steps correspond to the two subsections below.

4.1 Reduction to the abelian case

In this subsection all the arguments will be made under the following assumption:

(†) *The Sylow° 2-subgroups of G are* not *abelian.*

Fact 3.2 (ii) gives a detailed description of the two isomorphism types for the Sylow° 2-subgroups under the assumption (†). Using this we will analyze the fusion of involutions in G. This analysis constitutes the largest part of this subsection. We will keep the same notation as above, and recall that I_1 is the set of involutions of G conjugate to an involution in $\sigma^\circ(M)$, and I_2 is the set of involutions conjugate to an involution in $M \setminus \sigma^\circ(M)$.

Lemma 4.1 *If $s \in S$ is such that αs is an involution, then α is conjugate to αs by an element of S.*

Proof. By Fact 3.6, we have $S_1 = \{[\alpha, x] : x \in S\}$. Therefore it suffices to show that $s \in S_1$. As αs is an involution, we have $s^\alpha = s^{-1}$. By Fact 3.2 (ii), we have $(a,b,c)^\alpha(a,b,c) = (0, a, \epsilon a + \sqrt{ab})$. Thus (a,b,c) is inverted by α if and only if $a = 0$, and we conclude that the only elements of S inverted by α are those of S_1. □

In the rest of this subsection R will denote a Sylow 2-subgroup of M which contains α. Note that $R^\circ = S$.

Lemma 4.2 *Let $\beta \in I(M)$ be such that $C_\beta^\circ \not\leq M$, and $L_\beta = B(C_\beta)$. If $L_\alpha \leq L_\beta$, then $L_\alpha = L_\beta$*

Proof. Lemma 3.1 applies to β as well. Hence $C_\beta^\circ = L_\beta \times C_{C_\beta^\circ}(L_\beta)$, where $L_\beta \cong \mathrm{PSL}_2(K_\beta)$ with K_β algebraically closed of characteristic 2. Moreover, $L_\beta \cap M = A_\beta \times T_\beta$ and $A_\beta \leq S$. By Fact 3.7, $\mathrm{rk}\,(S) = 3\mathrm{rk}\,(A_\beta)$ as well as $\mathrm{rk}\,(S) = 3\mathrm{rk}\,(A)$. This implies that $\mathrm{rk}\,(L_\alpha) = 3\mathrm{rk}\,(A) = 3\mathrm{rk}\,(A_\beta) = \mathrm{rk}\,(L_\beta)$. Hence if $L_\alpha \leq L_\beta$, then $L_\alpha = L_\beta$. □

Lemma 4.3 $A \lhd M$.

Proof. The group law in Fact 3.2 (ii) shows that $A = Z(S)$. As S is the only Sylow° 2-subgroup of M (Fact 3.8), the conclusion follows. □

Lemma 4.4 ([3, Lemma 7.3]) *Let $C/S = C_{R/S}(\alpha)$. Then $C = SC_C(L_\alpha)$.*

Proof. It suffices to show that $C \leq SC_C(L_\alpha)$. Let $x \in C$. Then $[\alpha, x] \in S$. We define $s = [\alpha, x]$. Thus, $\alpha^x = \alpha s \in I(S)$. By Lemma 4.1, there exists $s_1 \in S$ such that $\alpha^x = \alpha^{s_1}$. It follows that $xs_1^{-1} \in C_\alpha$ and normalizes L_α. By Remark 2.21, there exists $i \in I(L_\alpha)$ such that $xs_1^{-1}i \in C(L_\alpha)$. Since by Lemma 4.3 xs_1^{-1} normalizes A, we have $i \in A$. As $S \leq C$, $xs_1^{-1}i \in C_C(L_\alpha)$. It follows that $x \in SC_C(L_\alpha)$. □

Corollary 4.5 ([3, Lemma 7.3]) $R = S \rtimes C_R(L_\alpha)$.

Proof. We let $R_1/S = \Omega_1(Z(R/S))$. $R_1 > S$ because $R > S$. Let $x \in R_1 \setminus S$. By Lemma 4.4, there exist $s \in S$ and $\beta \in C_R(L_\alpha)$ such that $x = s\beta$. Since $x^2 \in S$, $\beta^2 \in C_S(L_\alpha)$. β is an involution because $C_S(L_\alpha) = 1$ by Fact 3.9. By Lemma 4.2, we have $L_\alpha = L_\beta$. We then apply Lemma 4.4 to $C/S = C_{R/S}(\beta)$. By the choice of x, $C = R$. Hence, $R = SC_C(L_\beta) = SC_C(L_\alpha) = SC_R(L_\alpha)$. As $C_S(L_\alpha) = 1$, we in fact have $R = S \rtimes C_R(L_\alpha)$. □

Lemma 4.6 ([3, Lemma 7.4]) $I(C_R(L_\alpha)) = \{\alpha\}$.

Proof. Let $\beta \in I(C_R(L_\alpha))$. We may assume that either α or β is in $Z(C_R(L_\alpha))$. Then, as α and β commute and $[\beta, T] \leq [\beta, L_\alpha] = 1$, the definable connected subgroup $[\beta, S]$ is $(\langle\alpha\rangle \times T)$-invariant. Therefore, $[\beta, S] \leq S_1$. A symmetric argument using Fact 3.6 shows that $[\beta, S] = S_1$. The same fact also implies that β inverts S_1. If $\alpha \neq \beta$ then $\alpha\beta$ is an involution that centralizes S_1. But $\alpha\beta$ centralizes α and the preceding argument for β can be applied to $\alpha\beta$ and yields that $\alpha\beta$ inverts S_1. This contradicts the structure of S_1 as described in Fact 3.2 (ii). Hence $\alpha = \beta$. □

Corollary 4.7 ([3, Lemma 7.4]) *The involutions in $R \setminus \sigma^\circ(M)$ are conjugated by S. In particular, $I(M \setminus \sigma^\circ(M))$ is a single conjugacy class.*

Proof. Let $\beta \in I(R \setminus \sigma^\circ(M))$. Then by Corollary 4.5, $\beta = s\beta_1$ for some $s \in S$ and $\beta_1 \in C_R(L_\alpha)$. It follows using Fact 3.9 as in the proof of Corollary 4.5 that β_1 is an involution. Then by Lemma 4.6, $\beta_1 = \alpha$. Now the first part of the statement follows from Lemma 4.1. The second part follows from the first and the conjugacy of Sylow 2-subgroups. □

The rudimentary fusion information provided by the following lemma suffices for the rest of the argument although it is possible to obtain more.

Lemma 4.8 ([3, Lemma 7.6]) *The involutions in A are not conjugate to those in $S \setminus A$.*

Proof. By Corollary 3.12, M controls fusion in $I(\sigma^\circ(M))$. But $A \lhd M$ by Lemma 4.3. □

Now we can prove the main result of this subsection:

Theorem 2 ([3, Theorem 7.8]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup that is not strongly embedded. Then the Sylow° 2-subgroups of G are abelian.*

Proof. Let $x \in I(S \setminus S_1)$, and let $y \in I(G \setminus M)$ be conjugate to $I(A)$. By Lemma 4.8, x and y are not conjugate. Therefore by Fact 2.11, there exists an involution i that commutes with both x and y. By Fact 3.9, $C_x \leq M$ and $C_y \leq M^g$ where M^g is the conjugate of M containing y. Fact 3.10 implies that $i \notin \sigma^\circ(M)$. In particular, $i \notin S$, and by Corollary 4.7, i is conjugate to α in M. By Fact 3.5, $C_S(\alpha) = A$. As i and α are conjugate in M and $A \lhd M$, $C_S(i) = A$. But $x \in S \setminus A$ and commutes with i. This contradiction finishes the proof of the theorem. □

4.2 Reduction to the Abelian case with $I(S) = A^\times$

In this subsection we will continue to use the same notation as in the last one. The following theorem will complete the Sylow 2-analysis:

Theorem 3 ([3, Theorem 8.1]) *Let G be a simple L^*-group of even type with a weakly embedded subgroup M that is not strongly embedded. Then the only Sylow° 2-subgroup S of M is homocyclic with $I(S) = A^\times$.*

All arguments in this subsection will be carried out under the following assumption which corresponds to the subcase of Fact 3.2 (i) with $I(S \setminus A) \neq \emptyset$:

(††) *$S = E \oplus E^\alpha$, where both E and E^α are definable T-invariant elementary abelian subgroups of S, and $A = \{xx^\alpha : x \in E\}$. Moreover, T acts naturally on A, E, E^α.*

Lemma 4.9 *The elements of $S^\times$ are M°-conjugate.*

Proof. Suppose first towards a contradiction that $S^\times$ has at least two distinct conjugacy classes. Let $u \in I(A)$, and v be an involution not conjugate to u, which does not commute with u and is conjugate to $I(S \setminus A)$. One can find such an involution v thanks to the contradictory assumption that $S^\times$ has at least two conjugacy classes of involutions and Fact 3.9. Evidently, $v \notin M$. By Fact 2.11, there exists a third involution w that commutes with u and v. By Fact 3.9, $w \in M$. As $v \notin M$, $C_w \not\leq M$. We may assume that $\alpha = w$.

Since v is not conjugate to u, Lemma 3.1 (3) implies that $v \in C_\alpha \setminus L_\alpha$. The involution v acts on L_α as an inner automorphism by Remark 2.21. Hence, there exists $i \in I(L_\alpha)$ such that vi centralizes L_α. Since v and i commute, vi is an involution. Since $vi \in C(L_\alpha)$, $vi \in M$ by Fact 3.9. As v is conjugate to an involution in S, Fact 3.10 implies that $v \in \sigma^\circ(M^g)$, where M^g is distinct from M. As i commutes with v, $i \in M^g$ by Fact 3.9. Moreover, $i \in \sigma^\circ(M^g)$ by Fact 3.10, because, being an element of L_α, it is conjugate to the involutions in $A \leq \sigma^\circ(M)$. Therefore, $vi \in I(\sigma^\circ(M^g))$. But $C_{vi} \not\leq M^g$, a contradiction to Fact 3.9.

The conclusion of the last paragraph and Fact 3.12 show that the elements of $S^\times$ are M-conjugate. We will now strengthen this to conjugacy under the action of M°. For $i \in I(S)$, we have $\mathrm{rk}\,(i^{M^\circ}) = \mathrm{rk}\,(M^\circ) - \mathrm{rk}\,(C_{M^\circ}(i)) = \mathrm{rk}\,(M) - \mathrm{rk}\,(C_M(i)) = \mathrm{rk}\,(i^M)$. But the last paragraph shows that $\mathrm{rk}\,(i^M) = \mathrm{rk}\,(S)$. As S is connected and M° is a group, it follows that $S^\times = i^{M^\circ}$. This finishes the argument. □

Let w be an involution in L_α that inverts T. We will keep this notation until the end of this section.

Lemma 4.10 *If A_1 is a unipotent 2-subgroup of S such that $A_1 > A$, then $C_G(A_1, A^w) = 1$.*

Proof. Let $x \in C_G(A_1, A^w)$. Suppose $x \neq 1$. Then by Fact 2.22, $B(C_x)$ is a copy of PSL_2 in characteristic 2. As $A_1 > A$, $B(C_x) > L_\alpha$ because $L_\alpha = \langle A, A^w \rangle$. A comparison of a maximal torus of L_α with that of $B(C_x)$ containing it and Fact 2.24 yield a contradiction. □

Proposition 4.11 $C^\circ(L_\alpha) = 1$.

Proof. Let $X = C^\circ(L_\alpha)$. Note that by Facts 3.8 and 3.9, X is of degenerate type. Suppose towards a contradiction that $X \neq 1$.

We will now argue that $K = T \cup \{0\}$ carries a field structure that turns S into a 2-dimensional vector space over this field. By Fact 3.3, T embeds into $\mathrm{End}(S)$. So we will see T a subgroup of the multiplicative group of $\mathrm{End}(S)$, and we will consider the subring R of $\mathrm{End}(S)$ generated by T. As E is T-invariant, E is R-invariant. So there is a natural restriction map $\rho : R \longrightarrow \mathrm{End}(E)$. Furthermore, $\rho[R] = \rho[T \cup \{0\}] \cong K$ acting on E identified with K_+. It suffices therefore to check that ρ is an injection.

Now, α commutes with T and hence with R, so if $r \in R$ annihilates E then r annihilates $EE^\alpha = S$, and $r = 0$. Thus ρ is injective and $R = T \cup \{0\}$ is a field which we call K. It follows from (††) and the action of T on E that S is a 2-dimensional K-vector space.

Since X commutes with T, the action of X on S is K-linear. Lemma 4.10 implies that $C_X(S) = 1$ because $C_X(S) \le C(S, A^w) = 1$. As a result, Fact 2.30 implies that X is solvable. Since X is solvable of degenerate type, Fact 2.9 implies that $I(X) = \emptyset$. Since $C_X(S) = 1$, Fact 2.14 shows that X is divisible and abelian. It follows that XT is abelian and does not have involutions.

Next we define $X_1 = C_{XT}(S/A)$. As S/A and A are isomorphic T-modules by (††), the assumption that $X \neq 1$ and the fact that XT is abelian imply that $X_1 \neq 1$. We have $S/A = C_{S/A}(X_1) = C_S(X_1)A/A$ by Fact 2.10. Thus, $S = C_S(X_1)A$. But $C_S(\alpha) = A$ by Fact 3.5 and $C_S(X_1)$ is $(\langle\alpha\rangle \times T)$-invariant. It follows that $C_S(X_1) \ge A$, and we have $X_1 \le C_{XT}(S)$.

We will now show that $X_1 = 1$, the final contradiction which will prove the proposition. Since X_1 centralizes X, X_1 normalizes $B(C(X)) = L_\alpha$. But X_1 has no involutions and centralizes A. This, together with Remark 2.21, implies that X_1 centralizes L_α. Lemma 4.10 implies that $X_1 = 1$. □

Proposition 4.12 *w inverts $C^\circ(T)$.*

Proof. It suffices to show that $C^\circ(w, T) = 1$ (Fact 2.12). Let $X = C^\circ(w, T)$. Note that X is normalized by α, and the structure of PSL_2 forces $X \cap L_\alpha = 1$. Since $C^\circ(L_\alpha) = 1$ by Proposition 4.11, $C_X(\alpha)$ is finite (Remark 2.21 and Lemma 3.1) and by Fact 2.12 α inverts X. In particular, X is abelian. We suppose towards a contradiction that $X \neq 1$.

By Fact 3.9, $X \le M_w$, where M_w is the conjugate of M containing w. Let S_w denote the Sylow° 2-subgroup of M_w. We have $\alpha \in M_w$ and $L_\alpha \cap M_w = A_w \rtimes T_w$ by Lemma 3.1 (4). By the structure of PSL_2, $T \not\le M_w$, indeed $T \cap M_w = 1$. As a result, $C(X) \not\le M_w$.

We will prove that if X_0 is a nontrivial definable connected subgroup of X then $C_{S_w}(X_0)$ is finite. If this is not the case then since $C(X_0) \ge C(X)$, and thus $C(X_0) \not\le M_w$, Lemma 2.22 implies that $B(C(X_0)) \cong \mathrm{PSL}_2$ in characteristic 2. As α inverts X, α normalizes X_0. As a result $C_{S_w}(X_0)$ is normalized by α and has infinite intersection with $C_{S_w}(\alpha)$. By Fact 3.5, $C_{S_w}(\alpha) = A_w$. Since X_0 centralizes also T, it centralizes $\langle T, C_{A_w}(X_0)\rangle$. This last subgroup is nothing but L_α. But $C^\circ(L_\alpha) = 1$ by Proposition 4.11, and we reach a contradiction which proves that $C_{S_w}(X_0)$ is finite.

Next we consider the quotient $\overline{M_w^\circ} = M_w^\circ / C_{M_w^\circ}(S_w)$. By the preceding paragraph, $C_X(S_w)$ is finite. Hence $\overline{X}$ is isogenous to X. Both $\overline{X}$ and $\overline{T}_w$ are contained in Borel subgroups of $\overline{M_w^\circ}$. By Fact 2.34, an $\overline{M_w^\circ}$-conjugate $\overline{X_1}$ of $\overline{X}$ is in the same Borel of $\overline{M_w^\circ}$ as $\overline{T_w}$. There exists $w_1 \in C_{S_w}(\overline{X_1})^\times$ and it follows from Fact 3.3 that $\overline{T_w} \cap \overline{X_1} = 1$. Since $\overline{T_w}$ and $\overline{X_1}$ are in the same Borel, $[\overline{X_1}, \overline{T_w}] = 1$ (Fact 2.14). Therefore, for any $\overline{t} \in \overline{T_w}$, $[\overline{X_1}, w^{\overline{t}}] = 1$. Since $C_{S_w}(X)$ is finite, so is $C_{S_w}(\overline{X})$, and the same applies to $C_{S_w}(\overline{X_1})$. Therefore we reach a contradiction. □

Proposition 4.13 *$C_M^\circ(T) = T$. Thus, $N_M^\circ(T) = T$.*

Proof. Let $Y = C_M^\circ(T)$. By Proposition 4.12, w inverts Y. Hence every subgroup of Y is w-invariant. As a result $C_Y(S) = C_Y(S, S^w)$, and by Lemma 4.10, $C_Y(S) = 1$.

By Fact 3.3, $Y \cap S = 1$. It follows that Y is of degenerate type (Fact 3.8). Since Y is connected and abelian, we conclude using Fact 2.9 that $I(Y) = \emptyset$. The group $S \rtimes Y$ is connected and solvable. Since $C_Y(S) = 1$, Fact 2.14 implies that Y is divisible abelian. The faithful action of Y on S and Fact 2.27 imply that Y is a good torus.

Now, suppose towards a contradiction that $Y > T$. Since Y is a good torus and $Y > T$, Y contains torsion elements that are not in T. Moreover, T is a full torus in characteristic 2. It follows that the Prüfer p-rank of Y is at least 2 for some prime p. Since $C_Y(S) = 1$, Fact 2.23 implies that the action of Y on S cannot be irreducible. Let $\tilde{A}$ be a Y-minimal subgroup of S. Then $C_Y(\tilde{A})$ is infinite by Fact 2.23 and our conclusion about the Prüfer p-rank of Y. Let $\tilde{Y} = C_Y^\circ(\tilde{A})$. As every subgroup of Y is w-invariant, $\tilde{Y}$ is w-invariant. Thus $\langle \tilde{A}, \tilde{A}^w\rangle \le C^\circ(\tilde{Y})$.

By Facts 2.22 and 3.10, $\tilde{L} = B(C(\tilde{Y})) \cong \mathrm{PSL}_2$ in characteristic 2. By Remark 2.21, w acts on $\tilde{L}$ as an element of $\tilde{L}$, and $\tilde{L}$ has a Sylow 2-subgroup A_2 that is centralized by w. By Fact 3.9, $A_2 \leq M_w$, where M_w is the conjugate of M containing w. Let S_w denote the unique Sylow° 2-subgroup of M_w. Clearly A_w and A_2 are subgroups of S_w. Since $\tilde{Y}$ centralizes A_2, $\tilde{Y} \leq M_w$ by Fact 2.18. Since $w \in A_w$ and $A_w \leq S_w \lhd M_w$, $[w, \tilde{Y}] \leq \tilde{Y} \cap S_w = 1$. But w inverts $\tilde{Y}$ and $I(Y) = \emptyset$, a contradiction. We conclude that $Y = T$.

The second conclusion follows from Facts 2.8 and 2.24. □

Corollary 4.14 *The set $\cup_{g \in M^\circ} T^g$ is generic in M°.*

Proof. The conclusion follows from Proposition 4.13, Fact 2.32 and the fact that T is a good torus. □

Let $\overline{M^\circ} = M^\circ / C_{M^\circ}(S)$. The $\overline{}$-notation will be used to denote the quotients by $C_{M^\circ}(S)$.

Proposition 4.15 *$\overline{T}$ is a Borel subgroup of $\overline{M^\circ}$.*

Proof. By Corollary 4.14, the set $\tau = \cup_{g \in M^\circ} T^g$ is generic in M°. Hence, so is its image $\overline{\tau}$ in $\overline{M^\circ}$. Let $\overline{B}$ be a Borel subgroup of $\overline{M^\circ}$. $\overline{B}$ is a good torus by Fact 2.27, and since it is a Borel subgroup of $\overline{M^\circ}$, it is of finite index in $N_{\overline{M^\circ}}(\overline{B})$. By Fact 2.33, $\overline{B}$ is conjugate to a subgroup of $\overline{T}$. Since $\overline{B}$ is a Borel subgroup of $\overline{M^\circ}$, we conclude that $\overline{T}$ and $\overline{B}$ are conjugate in $\overline{M^\circ}$. □

Proposition 4.16 *For $i \in I(S)$, $C_{\overline{M}^\circ}(i)$ is finite.*

Proof. If $C_{\overline{M^\circ}}(i)$ is infinite for some $i \in I(S)$, then Proposition 4.15 shows that a Borel subgroup of $C_{\overline{M^\circ}}(i)$ can be conjugated to a subgroup of $\overline{T}$. This contradicts Fact 3.3. Hence, $C_{\overline{M^\circ}}(i)$ is finite. □

Corollary 4.17 $\mathrm{rk}\,(\overline{M^\circ}) = 2\mathrm{rk}\,(T)$.

Proof. The assumption on the structure of S implies that $\mathrm{rk}\,(S) = 2\mathrm{rk}\,(T)$. Since $S = I(S) \cup \{1\}$, Proposition 4.16 and the transitive action of M° (hence that of $\overline{M^\circ}$) on $I(S)$ (Lemma 4.9) yield the conclusion. □

Proposition 4.18 *For $\overline{x} \notin N_{\overline{M^\circ}}(\overline{T})$, $\overline{T} \cap \overline{T}^{\overline{x}}$ is finite.*

Proof. We suppose towards a contradiction that $\overline{T_0} = (\overline{T} \cap \overline{T}^{\overline{x}})^\circ \neq 1$ for some $\overline{x} \notin N_{\overline{M^\circ}}(\overline{T})$. We consider $\overline{X} = C^\circ_{\overline{M^\circ}}(\overline{T_0})$. This is a nonsolvable group by Proposition 4.15. Let $B \leq S$ be an $\overline{X}$-minimal subgroup of S. As $\overline{T_0} \neq 1$ and $\overline{X}/C_{\overline{X}}(B)$ is of degenerate type, Facts 2.28 and 2.30 imply that $\overline{X}/C_{\overline{X}}(B)$ is a solvable group. But by Proposition 4.16, $C_{\overline{X}}(B)$ is finite. Hence $\overline{X}$ is solvable, a contradiction. □

Proposition 4.19 $I(\overline{M^\circ}) = \emptyset$.

Proof. Suppose towards a contradiction that $\overline{w} \in I(\overline{M^\circ})$. If $C_{\overline{M}}(\overline{w})$ is finite, then by Fact 2.12, $\overline{w}$ inverts $\overline{M^\circ}$. It follows using Fact 2.9 that $\overline{M^\circ}$ has infinite Sylow 2-subgroups. This contradicts that $\overline{M^\circ}$ is of degenerate type. Thus $C_{\overline{M}}(\overline{w})$ is infinite.

The conclusion of the last paragraph shows that $C_{\overline{M}}(\overline{w})$ has infinite Borel subgroups. These Borel subgroups are contained in Borel subgroups of $\overline{M}$. It follows from Proposition 4.18 that $\overline{w}$ normalizes a Borel subgroup of $\overline{M^\circ}$ of which it centralizes an infinite subgroup. This contradicts Fact 2.31. □

Corollary 4.20 $N_{\overline{M^\circ}}(\overline{T}) = \overline{T}$

Proof. As $\overline{T}$ is a full torus of dimension 1 in characteristic 2, Facts 2.7, 2.31 and 4.15 imply that $N_{\overline{M^\circ}}(\overline{T})/\overline{T}$ is a 2-group. But by Proposition 4.19, $I(\overline{M^\circ}) = \emptyset$. The conclusion follows from Fact 2.6. □

Proposition 4.21 *$\overline{M^\circ}$ acts doubly transitively on the cosets of $\overline{T}$ in $\overline{M}_0$.*

Proof. Let $\overline{g} \in \overline{M^\circ} \setminus \overline{T}$. By Proposition 4.18 and Corollary 4.20, $\overline{T} \cap \overline{T}^{\overline{g}}$ is finite. It follows that the mapping

$$\begin{array}{ccc} \overline{T} \times \overline{T} & \longrightarrow & \overline{T}\overline{g}\overline{T} \\ (\overline{t_1}, \overline{t_2}) & \longmapsto & (\overline{t_1}\overline{g}\overline{t_2}) \end{array}$$

has finite fibers. Therefore $\text{rk}\,(\overline{T}\overline{g}\overline{T}) = 2\text{rk}\,(T)$. But by Corollary 4.17, $\text{rk}\,(\overline{M^\circ}) = 2\text{rk}\,(\overline{T})$. We conclude that $\overline{M^\circ} = \overline{T} \sqcup \overline{T}\overline{g}\overline{T}$ for any $\overline{g} \in \overline{M^\circ} \setminus \overline{T}$. □

Proof of Theorem 3. By Proposition 4.19, $I(\overline{M^\circ}) = \emptyset$. On the other hand, in the doubly transitive action of $\overline{M^\circ}$ on the cosets of $\overline{T}$ proven in Proposition 4.21 the element of $\overline{M^\circ}$ that swaps two distinct points yields an involution in $\overline{M^\circ}$. This contradiction shows that (††) cannot hold. □

5 Structure of M

In this section we will prove several corollaries of Theorem 3 that will provide useful tools for the proof of Theorem 1. These corollaries will help clarify the structure of M. We start by reviewing the notation that we will keep from the last section. G will denote a simple L^*-group of even type with M a weakly but not strongly embedded subgroup. S will be the Sylow° 2-subgroup of M. Let α be a fixed offending involution in M and $L_\alpha = B(C_\alpha)$. $L_\alpha \cap M$, which is a Borel subgroup of L_α, is of the form $A \rtimes T$. By Theorem 3, we know that S is homocyclic and $\Omega_1^\circ(M) = I(S) \cup \{1\} = A$. We will continue to denote by w the Weyl group element in L_α that inverts T.

In understanding the structure of M, the Borel subgroup $L_\alpha \cap M$ of L_α and w are important ingredients. Indeed the particular interaction among w, T and A and the field structure involved in $A \rtimes T$ have very strong consequences on G, as we will see. It will turn out that a large class of involutions share similar properties. Thus we introduce some notation and terminology that will help us not to limit our discussion to w and $A \rtimes T$. In the following definition, i can be taken to be a fixed involution in $I(A)$.

Definition 5.1

1. $T(u) = \{x \in M^\circ : x^u = x^{-1}\}$.
2. $X_1 = \{u \in i^G \setminus M : \text{rk}\,(T(u)) < \text{rk}\,(A)\}$.
3. $X_2 = \{u \in i^G \setminus M : \text{rk}\,(T(u)) \geq \text{rk}\,(A)\}$.
4. $XL_2 = \{u \in i^G \setminus M : u$ inverts a subgroup $T[u]$ of M° such that $A \rtimes T[u] \cong K_+ \rtimes K^\times\}$

We will occasionally refer to involutions in X_i (resp. in XL_2) as X_i-involutions (resp. XL_2-involutions). More important is that $XL_2 \subseteq X_2$, and that XL_2, hence X_2, is not empty, as $w \in XL_2$. Evidently, we may take $T[w] = T$, and we will prefer T as notation to $T[w]$ whenever we concentrate on the particular involution w.

Our first target is to show that $C_M(i) = C_M(A)$ for any $i \in I(A)$. This will be done by first showing the presence of a semidirect product structure on M° in terms of $C_{M^\circ}(A)$ and any $T[u]$, for any $u \in XL_2$.

Lemma 5.2 *For $i \in I(A)$ and $u \in XL_2$, we have $M = C_M(i)T[u]$, as well as $M^\circ = C_{M^\circ}(i)T[u]$, and $C_M(i) \cap T[u] = 1$.*

Proof. The assumption on the structure of $A \rtimes T[u]$ implies that $T[u]$ acts on $I(A)$ regularly. In particular this action is transitive. The first part of the statement follows from this transitivity and the fact that $T[u] \leq M^\circ$. That $C_M(i) \cap T[u] = 1$ is a consequence of the regularity of the action. □

Proposition 5.3 ([3, Lemma 9.3]) *For $i \in I(A)$ and $u \in XL_2$, we have $C^\circ_M(i) = C^\circ_M(A) = C_{M^\circ}(i) = C_{M^\circ}(A)$. In particular, $M^\circ = C_{M^\circ}(A) \rtimes T[u]$.*

Proof. We let $\overline{M^\circ} = M^\circ/C_{M^\circ}(A)$. By Lemma 5.2, $C^\circ_M(i) = C_{M^\circ}(i)$. Now, suppose towards a contradiction that $C^\circ_M(i) > C^\circ_M(A)$. Hence $\overline{C^\circ_M(i)}$ is an infinite definable connected subgroup of $\overline{M^\circ}$. Note also that $\overline{T[u]} \cong T[u]$ because $C_{T[u]}(A) = 1$ by the structure of $A \rtimes T[u]$. $\overline{T[u]}$ is contained in a Borel subgroup of $\overline{M^\circ}$. We will show that $\overline{T[u]}$ is indeed a Borel subgroup of $\overline{M^\circ}$. By Fact 2.14, the Borel subgroups of $\overline{M^\circ}$ are divisible abelian. Thus it suffices to show that $C^\circ_{\overline{M^\circ}}(\overline{T[u]}) = \overline{T[u]}$. Since $\overline{T[u]}$ acts transitively on $A^\times$, Facts 2.28 and 2.30 imply that $C^\circ_{\overline{M^\circ}}(\overline{T[u]})$ is solvable. Hence, $C^\circ_{\overline{M^\circ}}(\overline{T[u]})$ is abelian by Fact 2.14. But by Lemma 5.2, $\overline{M^\circ} = \overline{C_{M^\circ}(i)T[u]}$, and the transitive action of $\overline{T[u]}$ on $A^\times$ forces $C^\circ_{\overline{M^\circ}}(\overline{T[u]}) \cap \overline{C_{M^\circ}(i)} \leq \overline{C_{M^\circ}(A)} = 1$. As a result $C^\circ_{\overline{M^\circ}}(\overline{T[u]}) = \overline{T[u]}$.

Since $\overline{T[u]}$ is a Borel subgroup of $\overline{M^\circ}$ and $\overline{C_{M^\circ}(i)}$ is infinite, Fact 2.34 implies that a conjugate of $\overline{T[u]}$ in $\overline{M^\circ}$ intersects $\overline{C_{M^\circ}(i)}$ nontrivially. This contradicts the action of $\overline{T[u]}$ on A. It follows that $C^\circ_M(i) = C^\circ_M(A)$.

Since by Lemma 5.2, $M^\circ = C_{M^\circ}(i)T[u]$ and $C_{M^\circ}(i) = C^\circ_M(i)$ as remarked above, we also have $M^\circ = C^\circ_M(A) \rtimes T[u]$. The equality $C^\circ_M(A) = C_{M^\circ}(A)$ then follows from the fact that $C^\circ_M(A) \leq C_{M^\circ}(A)$ and by the structure of $A \rtimes T[u]$. □

Corollary 5.4 ([3, Lemma 9.3]) *$C_M(i) = C_M(A)$ for $i \in I(A)$.*

Proof. Let $\overline{M} = M/C_M(A)$. Note that by the structure of $A \rtimes T$, $\overline{T} \cong T$. By Proposition 5.3, $M^\circ = C_{M^\circ}(A) \rtimes T$, and thus $\overline{M}^\circ = \overline{T}$. We apply Fact 2.29 with $\overline{M}$ for G, A for A, $\overline{T}$ for G_1 and H. Since $\overline{T}$ acts transitively on $A^\times$, the vector space is 1-dimensional and $\overline{T} = \overline{M}$. The conclusion follows. □

Before proving the last conclusion of this section we recall some group theoretic notions:

Definition 5.5 *Let G be a group.*

1. *For $x \in G$, $C^*_G(x) = \{g \in G : x^g = x$ or $x^{-1}\}$.*
2. *An element of G is said to be* strongly real *if it is the product of two involutions.*

Proposition 5.6 ([3, Lemma 9.17]) *Let i and j be involutions in G that are conjugate to involutions in A. For $a \in G$, if $a^i = a$ and $a^j = a^{-1}$, then $a^2 = 1$.*

Proof. We may assume that $i \in A$. By Fact 3.9, we have $a \in M$. Since $[a, i] = 1$, by Corollary 5.4, $[a, A] = 1$. Suppose towards a contradiction that $a^2 \neq 1$. Then $j \in C^*_G(a) \setminus C_G(a)$, and i and j are not conjugate in $C^*_G(a)$. By Fact 2.11, there exists an involution $k \in C^*_G(a)$ that commutes with both i and j. Fact 3.9 implies that $k \in M$.

We will consider two cases. The first case is the one in which $k \in I(\sigma^\circ(M))$. By Theorem 3, $k \in A$. Since $[j, k] = 1$, Fact 3.9 implies that $j \in M$. Since j is conjugate to an involution in A, Fact 3.10 implies that $j \in A$. Since $[a, A] = 1$, we conclude $a^2 = 1$, a contradiction.

We now consider the case in which $k \in I(M \setminus \sigma^\circ(M))$. By Corollary 5.4, we have $[k, A] = 1$. Since j is conjugate to involutions in A, there exists $h \in G$ such that $j \in A^h$. By Fact 3.9, $k \in M^h$, and by Corollary 5.4, $[k, A^h] = 1$. Now, $j \notin A$ because $[a, A] = 1$, $a^j = a^{-1}$ and it is assumed that $a^2 \neq 1$. By Fact 3.10, $j \notin M$. Hence k is an offending involution with $C_k^\circ = B(C_k) \times C_{C_k^\circ}(B(C_k))$ and $B(C_k) \cong \mathrm{PSL}_2$ in characteristic 2 by Fact 2.22. We have $B(C_k) = \langle A, A^h \rangle$. We also have $B(C_k) = \langle A, A^j \rangle$ since $A^j \neq A$. Since a centralizes both A and A^j, a centralizes $B(C_k)$. Therefore a centralizes A^h. In particular $a^j = a$, a contradiction. □

6 Large intersections

In this section we start the proof of Theorem 1. We assume the hypothesis (*) of Theorem 1. We will keep the same notation as in the previous sections. The notation introduced in Definition 5.1 will be frequently used.

The main theme in this section is the structure of $(M \cap M^u)^\circ$ for an arbitrary $u \in X_2$. Such an intersection will be denoted R_u and we will show that u inverts R_u. A consequence of this is that $X_2 = XL_2$, as we will see in Corollary 6.13. Along the way, we will obtain a crucial piece of information, namely that offending involutions in M are in $M \setminus M^\circ$.

We start with the following consequence of Proposition 5.3 although it will be used only later (Proposition 6.10). It shows the strong effects of the structural description given by Proposition 5.3.

Lemma 6.1 *Let $u \in X_2$. Then R_u is abelian.*

Proof. We consider the definable connected subgroup $[R_u, R_u]$. Since by Proposition 5.3 $M^\circ/C_{M^\circ}(A)$ is abelian, we have $[R_u, R_u] \leq [M^\circ, M^\circ] \leq C^\circ(A)$. Since u normalizes R_u, u normalizes $[R_u, R_u]$. It follows that $[R_u, R_u] \leq C^\circ(A)^u$ as well. Since $A^u \neq A$, the hypothesis (*) implies that $[R_u, R_u]$ is finite and thus trivial. □

The next set of lemmas were corollaries in a section in [5]. In that section the main result was an analogue of the inversion statement that we are about to start proving. However some of the conclusions that became available only after that analogue (notably Corollary 6.10 of that article provided that this numbering survives until its publication)) are already known in this new context (Proposition 5.3 and Corollary 5.4) although this knowledge is for the time being restricted to the XL_2-involutions.

Lemma 6.2 ([11, Lemme 3.2]) *Let $u \in XL_2$. For any $t \in T[u]^\times$, u inverts $C^\circ(t)$.*

Proof. By Fact 2.12, it suffices to prove that $X = C^\circ(t, u) = 1$. Suppose $X \neq 1$. Then X is infinite, and by Fact 3.9, $X \leq M_u$ where M_u is the conjugate of M containing u. By Proposition 5.3 and Fact 3.9, $C^\circ(u) = C^\circ(A_u)$ where A_u is the conjugate of A which contains u. As $u \in XL_2$, by the definition of XL_2, $u \notin M$. Thus $A \neq A_u$ and $M \neq M_u$ (Fact 3.10). The assumption (*) implies that $C(X) \leq M_u$ as otherwise one could find distinct conjugates of A_u in $C(X)$ using elements in $C(X) \setminus M_u$. Hence $t \in M_u$. Since u inverts t, we have $[t, u] \in A_u \cap T[u]$. But $I(T[u]) = \emptyset$ since $T[u]$ is isomorphic to the multiplicative group of a field in characteristic 2. Therefore, $[t, u] = 1$ and this forces $t^2 = 1$, a contradiction. □

Lemma 6.3 *Let $u \in XL_2$. For any nontrivial subgroup $X \leq T[u]$, $C_M^\circ(X) = T[u]$. In particular, $C(A, T[u])$ is finite.*

Proof. By Proposition 5.3, $C_M^\circ(X) = C_{M^\circ}(AX) \rtimes T[u]$. As $C_M^\circ(X)$ is connected, it follows that $C_{M^\circ}(AX)$ is connected. Thus if $T[u] < C_M^\circ(X)$, then $C_M^\circ(AX)$ is nontrivial. By Lemma 6.2, u inverts $C_M^\circ(AX)$, and thus by Proposition 5.6, $C_M^\circ(AX)$ is an infinite connected elementary abelian 2-group which is centralized by u. But then Fact 2.18 implies that $u \in M$, a contradiction. □

Corollary 6.4 *Let $u \in XL_2$. Then Two distinct M°-conjugates of $T[u]$ have trivial intersection.*

Proof. If $x \in M^\circ$ is such that $T[u] \cap T[u]^x \neq 1$ then by Lemma 6.3, $T[u] = C_M^\circ(T[u] \cap T[u]^x) = T[u]^x$. □

Lemma 6.5 *Let $u \in XL_2$. Then $N_{M^\circ}(T[u]) = T[u]$. In particular, $C_{M^\circ}(A, T[u]) = 1$.*

Proof. We first prove that $N_M^\circ(T[u]) = T[u]$. $N_M^\circ(T[u])$ centralizes the torsion subgroup of $T[u]$ by Fact 2.8. But, by the definition of the XL_2-involutions, $T[u]$ is definably isomorphic to the full multiplicative group of an algebraically closed field in characteristic 2. Then Fact 2.24 implies that $N_M^\circ(T[u])$ centralizes $T[u]$. But by Lemma 6.3, $C_M^\circ(T[u]) = T[u]$.

By Proposition 5.3, $N_{M^\circ}(T[u]) = (C_{M^\circ}(A) \cap N_{M^\circ}(T[u]))T[u]$. Let $X = (C_{M^\circ}(A) \cap N_{M^\circ}(T[u]))$. Then the semidirect product structure of M° implies that X centralizes $T[u]$. The last paragraph shows that $T[u]$ is of finite index in its normalizer in M°. Moreover, by Corollary 6.4, two distinct M°-conjugates of $T[u]$ have trivial intersection. Since $T[u]$ is divisible abelian, Fact 2.31 can be applied to M° and $T[u]$. Since $C_{T[u]}(X)$ is infinite (namely $T[u]$), we conclude that $X = 1$. □

Corollary 6.6 *Let $u \in XL_2$. If X is any nontrivial subgroup of $T[u]$, then $C_{M^\circ}(X) = T[u]$.*

Proof. Corollary 6.4 implies that $C_{M^\circ}(X)$ normalizes $T[u]$. The conclusion follows from Lemma 6.5. □

The following is the aforementioned crucial information about the place in M of its offending involutions.

Corollary 6.7 *Offending involutions in M are in $M \setminus M^\circ$.*

Proof. Suppose α is an offending involution in M. Let L_α and w be as at the beginning of Section 5. As we noted there, $w \in XL_2$ and $T = T[w]$. Since α centralizes $A \rtimes T$, the conclusion follows from Lemma 6.5. □

We recall that for any $u \in X_2$, $R_u = (M \cap M^u)^\circ$. The following is another easy but important conclusion of the preceding discussion.

Corollary 6.8 *Let $u \in X_2$. Then $I(M^\circ \cap M^{\circ u}) = \emptyset$. In particular, $I(R_u) = \emptyset$.*

Proof. Suppose towards a contradiction that $\beta \in I(M^\circ \cap M^{\circ u})$. Then $\beta \in C(A, A^u)$ by Corollary 5.4. In particular β is an offending involution. But $M^\circ \cap M^{\circ u} \leq M^\circ$ and Corollary 6.7 forces $\beta \in M \setminus M^\circ$, a contradiction. Since $R_u \leq M^\circ \cap M^{\circ u}$, the second statement follows as well. □

Corollary 6.9 *Let $u \in X_2$. Then $\mathrm{rk}\,(R_u^-) \geq \mathrm{rk}\,(A)$, where $R_u^- = \{x \in R_u : x^u = x^{-1}\}$. In particular R_u^- is infinite, and thus contains nontrivial elements.*

Proof. Let $\tilde{R_u} = M^\circ \cap M^{\circ u}$. By Corollary 6.8, $I(\tilde{R_u}) = \emptyset$. By Fact 2.13, $\tilde{R_u} = C_{\tilde{R_u}}(u)\tilde{R_u}^-$ and $\mathrm{rk}\,(\tilde{R_u}) = \mathrm{rk}\,(C_{\tilde{R_u}}(u)) + \mathrm{rk}\,(\tilde{R_u}^-)$. Since $R_u = \tilde{R_u}^\circ$, it follows that $\mathrm{rk}\,(R_u) = \mathrm{rk}\,(C_{R_u}(u)) + \mathrm{rk}\,(\tilde{R_u}^-)$. Thus $\mathrm{rk}\,(R_u^-) = \mathrm{rk}\,(\tilde{R_u}^-)$ by Fact 2.13 applied to R_u and the preceding rank equality. □

Proposition 6.10 *If $u \in X_2$, then u inverts $R_u = (M \cap M^u)^\circ$.*

Proof. Let u and R_u be as in the statement. By Corollary 6.8, $I(R_u) = \emptyset$. Thus, by Fact 2.13, $R_u = C_{R_u}(u)R_u^-$ where $R_u^- = \{x \in R_u : x^u = x^{-1}\}$. Note that, since R_u is abelian by Lemma 6.1, R_u^- is in fact a group. We let $X_u = C_{R_u}(u)$. By Fact 2.13, X_u is a definable connected subgroup.

Since u is conjugate to involutions in A and X_u centralizes u, Corollary 5.4 implies that $X_u \leq C^\circ(A_u)$, where A_u is the conjugate of A containing u. As a result, if $X_u \neq 1$ then it is an infinite group and by the hypothesis (*), $C(X_u) \leq M_u$ where M_u is the conjugate of M containing u. It follows from this and Lemma 6.1 that $R_u \leq M_u$. In particular, $R_u^- \subseteq M_u$. But then $[R_u^-, u] \leq R_u^- \cap A_u$. This is contradictory because u inverts R_u^-, $I(R_u) = \emptyset$ by Corollary 6.8 and R_u^- contains nontrivial elements by Corollary 6.9. □

Corollary 6.11 *Let $u \in X_2$. Then R_u acts semiregularly on A.*

Proof. By Proposition 6.10, u inverts R_u. By Corollary 6.8, $I(R_u) = \emptyset$. Since u is conjugate to the involutions in $I(A)$, we conclude using Proposition 5.6. □

Corollary 6.12 *Let $u \in X_2$. Then* $\mathrm{rk}\,(T(u)) = \mathrm{rk}\,(A) = \mathrm{rk}\,(R_u)$.

Proof. Since u is an X_2-involution, we have $\mathrm{rk}\,(T(u)) \geq \mathrm{rk}\,(A)$. Proposition 6.10 implies that $R_u \subseteq T(u)$. Since $T(u) \subseteq M \cap M^u$, we conclude that $\mathrm{rk}\,(T(u)) = \mathrm{rk}\,(R_u)$. It follows from this equality and the fact that $\mathrm{rk}\,(T(u)) \geq \mathrm{rk}\,(A)$ that $\mathrm{rk}\,(R_u) \geq \mathrm{rk}\,(A)$. If this inequality were strict $C_{R_u}(i)$ would be nontrivial for any $i \in I(A)$ but this contradicts Corollary 6.11. □

Corollary 6.13 *Let $u \in X_2$. Then $A \rtimes R_u \cong K_+ \rtimes K^\times$ where K is an algebraically closed field of characteristic 2. Equivalently, $X_2 = XL_2$. Moreover, $R_u = T[u]$.*

Proof. This follows from Corollaries 6.11, 6.12 and Fact 2.23. □

7 Rank of G

In this section we will compute the rank of the simple group G we are analyzing. We will keep the same notation as in the previous sections. In particular, $i \in I(A)$ is fixed.

The X_2-involutions will play an important role in this section. In Corollary 6.13 we concluded that these involutions belong to the class XL_2 of Definition 5.1. Corollary 6.13 will be used without mention in the sequel and thanks to this corollary we will be able to use various results which were proven in the previous section only for the XL_2-involutions.

We start with a warning. The fact that u is an X_2-type involution does not imply that $T(u) = (M \cap M^u)^\circ$, as it is not clear whether $T(u)$ is a group. Indeed, if $T(u)$ is a group then it is abelian, and we have $R_u \leq T(u) \leq C_{M^\circ}(R_u) = R_u$ by Proposition 6.10 and Corollary 6.6. In any case, we have $(M \cap M^u)^\circ \subseteq T(u) \subseteq M \cap M^u$, and in particular the group $R_u = (M \cap M^u)^\circ$ is in a weak sense the "connected component" of $T(u)$. Since the conjugacy of the $(M \cap M^u)^\circ$ for various $u \in X_2$ is an important theme in this section, we will occasionally use $T^\circ(u)$ to denote $(M \cap M^u)^\circ$. It is worth noting that for $u \in X_2$, $T^\circ(u)$ is the same group as the one denoted by $T[u]$ prior to Corollary 6.13.

Proposition 7.1 $\mathrm{rk}\,(i^G) = \mathrm{rk}\,(X_2)$.

Proof. The proof consists of showing that $\mathrm{rk}\,(X_1) < \mathrm{rk}\,(i^G)$. An equivalence relation $\sim$ is defined on X_1 as follows: for $u_1, u_2 \in X_1$, $u_1 \sim u_2$ if and only if $u_1 M^\circ = u_2 M^\circ$. The second condition is equivalent to $u_2 u_1 \in T(u_1)$. Now $\mathrm{rk}\,(X_1) \leq \mathrm{rk}\,(X_1/{\sim}) + m$ where m is the maximal fiber rank for the quotient map $X_1 \longrightarrow X_1/{\sim}$. By the definition of $\sim$ and X_1, $m < \mathrm{rk}\,(A)$. Moreover, the mapping from $X_1/{\sim}$ into G/M° which assigns to each equivalence class $u/{\sim}$ the coset uM° is an injection, by the definition of $\sim$. Hence, $\mathrm{rk}\,(X_1) < \mathrm{rk}\,(G) - \mathrm{rk}\,(M) + \mathrm{rk}\,(A) = \mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(i)) - \mathrm{rk}\,(T) + \mathrm{rk}\,(A)$, using Proposition 5.3. But $\mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(i)) - \mathrm{rk}\,(T) + \mathrm{rk}\,(A) = \mathrm{rk}\,(i^G)$ since $\mathrm{rk}\,(T) = \mathrm{rk}\,(A)$. □

Lemma 7.2 *If w_1 and w_2 are two X_2-involutions such that $T^\circ(w_1) \neq T^\circ(w_2)$, then $T^\circ(w_1) \cap T^\circ(w_2) = 1$.*

Proof. If $T^\circ(w_1) \cap T^\circ(w_2) \neq 1$ then Lemma 6.3 implies that $T^\circ(w_1) = C^\circ_M(T^\circ(w_1) \cap T^\circ(w_2)) = T^\circ(w_2)$. □

Proposition 7.3 *If $w_1 \in X_2$, then $T^\circ(w)$ and $T^\circ(w_1)$ are $C^\circ(A)$-conjugate.*

Proof. By Lemma 7.2 and Lemma 6.5, $\cup_{x\in M^\circ} T^\circ(w)^x$ is generic in M°. If w_1 is another X_2-involution, then the connectedness of M° implies that for some $x \in M^\circ$, $T^\circ(w)^x \cap T^\circ(w_1) \neq 1$. Then Lemma 7.2 implies that $T^\circ(w)^x = T^\circ(w_1)$. The $C^\circ(A)$-conjugacy follows from the structure of M° as described by Proposition 5.3. □

It is worth noting that Proposition 7.3 eliminates any difference between T and $T^\circ(u)$ where u is an arbitrary X_2-involution.

Proposition 7.4 $\mathrm{rk}\,(G) = \mathrm{rk}\,(C(T)) + 2\mathrm{rk}\,(C(A))$.

Proof. The standard line of argument (introduced in [8] and also used in [1, 11]) to reach such a conclusion consists of defining a suitable mapping from X_2 into $w^{C(T)C^\circ(A)}$. We have the necessary tools, notably Lemma 6.5, to reproduce the same analysis.

By Proposition 7.3, for any X_2-involution w_1, there exists $f \in C^\circ(A)$ such that $T^f = T^\circ(w_1)$. It follows that $w_1^{f^{-1}}$ inverts T and thus $w_1^{f^{-1}} w$ centralizes T. Note also that by Lemma 6.5 f is unique.

Hence we can define the following definable map:

$$\begin{array}{rccc} \Phi & : & X_2 & \longrightarrow & w^{C(T)C^\circ(A)} \\ & & w_1 & \longmapsto & w^{w_1^{f^{-1}} w f} \end{array}$$

We show that Φ has finite fibers. If $w^{w_1^{f^{-1}} wf} = w^{w_1'^{f'^{-1}} wf'}$, then since this element inverts both T^f and $T^{f'}$, we have $T^f = T^{f'}$. Then Lemma 6.5 implies that $f = f'$. It follows that $w^{fw_1} = w^{fw_1'}$ and $(w_1 w_1')^{f^{-1}} \in C(T, w)$. But $C(T, w)$ is a finite group by Lemma 6.2. This proves the finiteness of the fibers.

The conclusion of the last paragraph implies that $\mathrm{rk}\,(X_2) \leq \mathrm{rk}\,(w^{C(T)C^\circ(A)})$. Since $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(i^G)$ by Proposition 7.1, we have $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(w^{C(T)C^\circ(A)})$

Next we show that $\mathrm{rk}\,(w^{C(T)C^\circ(A)}) = \mathrm{rk}\,(C(T)C^\circ(A))$. We define the following definable map:

$$\begin{array}{rccc} \Psi & : & C(T)C^\circ(A) & \longrightarrow & w^{C(T)C^\circ(A)} \\ & & cf & \longmapsto & w^{cf} \end{array}$$

Its fibers are finite because if $w^{cf} = w^{c'f'}$ then both w^{cf} and $w^{c'f'}$ invert $T^f = T^{f'}$. Then it follows from Lemma 6.5 that $f = f'$, thus $c'c^{-1} \in C(T, w)$, and this last group is finite by Lemma 6.2. Since Ψ is clearly surjective we have $\mathrm{rk}\,(C(T)C^\circ(A)) = \mathrm{rk}\,(w^{C(T)C^\circ(A)})$. The rank computations using Φ now yield $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(C(T)C^\circ(A))$.

Since $C(T) \cap C^\circ(A) = 1$ by Lemma 6.5, it follows from Proposition 7.1 that $\mathrm{rk}\,(C(T)) + \mathrm{rk}\,(C^\circ(A)) = \mathrm{rk}\,(X_2) = \mathrm{rk}\,(i^G) = \mathrm{rk}\,(G) - \mathrm{rk}\,(C_G(i))$. Using Proposition 5.3, we have $\mathrm{rk}\,(G) = \mathrm{rk}\,(C(T)) + 2\mathrm{rk}\,(C(A))$. □

8 Centralizers of tori

We keep the same notation as before.

Lemma 8.1 $\mathrm{rk}\,(X_2 M^\circ) = \mathrm{rk}\,(G)$.

Proof. The following equivalence relation is defined on X_2: $w_1 \sim w_2$ if and only if $w_1 M^\circ = w_2 M^\circ$ (if and only if $w_2 w_1 \in T(w_1)$). As $\mathrm{rk}\,(T(w_1)) = \mathrm{rk}\,(T)$ by Corollary 6.12, we conclude that $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(X_2/\!\sim) + \mathrm{rk}\,(A)$. Since $\mathrm{rk}\,(X_2) = \mathrm{rk}\,(i^G)$ by Proposition 7.1, it follows using Proposition 5.3 that

$$\begin{aligned} \mathrm{rk}\,(G) &= \mathrm{rk}\,(C_G(A)) + \mathrm{rk}\,(X_2/\!\sim) + \mathrm{rk}\,(A) \\ &= \mathrm{rk}\,(M) - \mathrm{rk}\,(T) + \mathrm{rk}\,(X_2/\!\sim) + \mathrm{rk}\,(A) \\ &= \mathrm{rk}\,(M) + \mathrm{rk}\,(X_2/\!\sim) \\ &= \mathrm{rk}\,(X_2 M^\circ) \end{aligned}$$

□

Lemma 8.2 *If $c \in C^\circ(T) \setminus M$ then $i^G \cap fcM^\circ = \emptyset$ for any $f \in C^\circ(A)$.*

Proof. Suppose towards a contradiction that $fcb \in i^G \cap fcM^\circ$ with $b \in M^\circ$ and f, c as in the statement of the lemma. Using Proposition 5.3 we may assume that $b \in C^\circ(A)$. After conjugating fcb by f we conclude that $cu \in i^G \cap cM^\circ$ with $u = bf \in C^\circ(A)$.

If $t \in T$ then $(cu)^t = cu^t$ and $[u,t] = (cu)^{-1}(cu)^t \in T(cu) \cap C^\circ(A)$. Let $x = [u,t]$. By Proposition 5.6, $x^2 = 1$. If $x \neq 1$ then x is an offending involution as $cu \in C(x) \setminus M$. However $x \in M^\circ$ and by Corollary 6.7 offending involutions in M are in $M \setminus M^\circ$. Therefore $x = 1$ and u centralizes t. Since t is an arbitrary element in T, we conclude that $u \in C_{M^\circ}(A,T)$. This last group is trivial by Lemma 6.5. It follows that $c \in i^G \cap C(T)$. T has no involutions and is inverted by w and thus Proposition 5.6 yields a contradiction. □

Lemma 8.3 ([11, Lemme 4.25]) *If for $f_1,\ f_2 \in C^\circ(A)$, $c_1,\ c_2 \in C^\circ(T) \setminus M$, we have $f_1c_1M^\circ = f_2c_2M^\circ$, then $f_1 = f_2$ and $c_1T = c_2T$.*

Proof. Suppose $f_1c_1 = f_2c_2v$ for some $v \in M^\circ$. We may assume $v \in C^\circ(A)$ by Proposition 5.3 and $c_1 = uc_2v$ where $u = f_1^{-1}f_2$.

We claim that $X = [v,T] = 1$. X is a definable connected subgroup contained in $M \cap M^{c_1}$ as $T^v = T^{c_2v} = T^{u^{-1}c_1} \leq M^{c_1}$ and $T^v \leq M^v = M$. As T normalizes X, XT is a group. In fact it is definable and connected. Note also that $X \leq M^{\circ\prime} \leq C^\circ(A)$ by Proposition 5.3. Thus $C_X(A^{c_1})$ is finite by the assumption $(*)$ of Theorem 1 and the fact that $A^{c_1} \neq A$. Since T is inverted by w, it acts freely on A^{c_1} (Proposition 5.6). Let $K = C^\circ_{XT}(A^{c_1})$. By Proposition 5.3 applied to M^{c_1} and the connectedness of XT, we have $XT = KT$. Then $[T,K] \leq (XT)' \leq X \cap K$ as T is abelian and both X and K are normal in XT. Since $C_X(A^{c_1})$ is finite and $[T,K]$ is connected, we conclude that $[T,K] = 1$. Then $K \leq C^\circ(T)$. The group $C^\circ(T)$ is inverted by w (Lemma 6.2) and does not have infinite 2-subgroups. Proposition 5.6 implies that K is an elementary abelian 2-subgroup. Since K is connected by definition, we conclude that $K = 1$. Therefore we have $XT = T$, and $X \leq T$. Since T acts freely on A, $X = 1$.

The last paragraph shows that $v \in C_{M^\circ}(T)$. It follows that $u \in C_{M^\circ}(T,A)$. The conclusion follows using Lemma 6.5. □

Proposition 8.4 *$C^\circ(T) = T$.*

Proof. It suffices to prove that $C^\circ(T) \leq M$. Suppose not and let $Y = \bigcup\{fcM^\circ : f \in C^\circ(A),\ c \in C^\circ(T) \setminus M\}$. By Lemma 8.3, the fact that $C_{M^\circ}(T) = T$ (Corollary 6.6) and Proposition 7.4, $\mathrm{rk}\,(Y) = \mathrm{rk}\,(C(A)) + \mathrm{rk}\,(C(T)) - \mathrm{rk}\,(T) + \mathrm{rk}\,(M) = \mathrm{rk}\,(C(T)) + 2\mathrm{rk}\,(C(A)) = \mathrm{rk}\,(G)$. Since by Lemma 8.1 X_2M° is also generic in G, Y and X_2M° share a coset of M°. This contradicts Lemma 8.2. □

Corollary 8.5 $\mathrm{rk}\,(G) = \mathrm{rk}\,(T) + 2\mathrm{rk}\,(C(A))$.

9 Proof of Theorem 1

In [5], at this point we started to prove that the group G is a Zassenhaus group. In the current situation this will not be necessary because we will reduce the proof of Theorem 1 to the main theorem of [5], namely the strongly embedded analogue of Theorem 1.

Proof of Theorem 1. Let G be a simple L^*-group of even type with a weakly embedded subgroup M. Suppose also that G satisfies the hypothesis (*). If M is strongly embedded then we are done by [5].

If M is not strongly embedded then Facts 2.17 and 2.18 show the existence of an offending involution α in M. The structure of the centralizer of α is given by Lemma 3.1. Structural analysis starting from this lemma and building upon it yields among other things Corollary 6.7. In particular, under the hypothesis that M is weakly but not strongly embedded, M is not connected. We will now prove that this yields a contradiction.

We first show that $C^\circ(A) \cap M^w = 1$ where A and w are as in the preceding sections. Let $x \in C^\circ(A) \cap M^w$. Since by Fact 2.18 $C_G(A) \leq M$, $x \in M^\circ$. By Corollary 6.13, $(M \cap M^w)^\circ = T$ where T is as in the preceding sections. In particular, x, which is in $M \cap M^w$, normalizes T. Thus $x \in N_{M^\circ}(T)$, and it follows from Lemma 6.5 that $x \in C_T(A)$. This last subgroup is trivial.

Now we analyze the following mapping, analogues of which were used for various double transitivity arguments in [5]:

$$\begin{array}{cccc} \theta & : & C^\circ(A) \times M & \longrightarrow & G \\ & & (f, m) & \longmapsto & fwm \end{array}$$

The last paragraph shows that this mapping is injective. As a result $\mathrm{rk}\,(\theta(C^\circ(A) \times M)) = \mathrm{rk}\,(C(A)) + \mathrm{rk}\,(M) = 2\mathrm{rk}\,(C(A)) + \mathrm{rk}\,(T)$. By Corollary 8.5, this is exactly $\mathrm{rk}\,(G)$. Hence $\deg(\theta(C^\circ(A) \times M)) = 1$ since G is connected. Since θ is injective, $\deg(C^\circ(A) \times M) = 1$. Hence $\deg(M) = 1$. But $\alpha \in M \setminus M^\circ$ by Corollary 6.7, a contradiction. □

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