# Theories categorical in power $n+2$

## *On the occasion of Roland Fraïssé's $80^{th}$ birthday*

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**Abstract**

We find all theories in $n$-variable logic which are not categorical in power $n+2$.

*Key words:* Finite variable logic, finite model theory, permutation group, computer science, descriptive complexity, pebble game, categoricity.

## 1 Introduction

Let $L^n$ be the sublogic of ordinary first order logic which is obtained by restricting to formulas involving at most $n$ distinct variables, which may be used repeatedly. This logic is well known in the context of computer science and is well suited to an analysis in terms of "le va-et-vient de Fraïssé", also known as Ehrenfeucht-Fraïssé (or pebble) games. Poizat showed in [Po] that any complete $L^n$ theory is categorical in power $n+1$, in the sense that it has at most one model of cardinality $n+1$, up to isomorphism (a result he attributes to Jurie [Ju], as a refinement of Krasner's generalized Galois theory); and furthermore this model, if it exists, is homogeneous: any partial $L^n$-elementary map extends to an automorphism.

In the same paper Poizat exhibited a complete $L^n$-theory $A(n)$ (called $T_2(n)$ in [Po]) with the following properties:

- $A(n)$ has at least two nonisomorphic models in each power greater than $n+2$.

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- $A(n)$ has two nonisomorphic models in power $n+2$ if $n$ is congruent to 0 or 1 modulo 4, or if $n=2$.
- $A(n)$ is categorical in power $n+2$ if $n$ is congruent to 2 or 3 modulo 4, and $n \neq 2$.

This leaves open the question of the existence of another $L^n$ theory which is not categorical in power $n+2$, for $n$ congruent to 2 or 3 modulo 4 and $n > 2$ [Po, pp. 643 and 657]. We show that the congruence condition on $n$ is essential:

**Proposition 1**
*For $n$ congruent to 2 or 3 modulo 4, with $n > 2$, any complete $L^n$ theory is categorical in power $n+2$.*

We emphasize that we consider a theory categorical in a given cardinality if it has *at most* one model of the specified order, that is we do not concern ourselves in the least with the existence of models of specific orders, an interesting issue in its own right.

The reason Proposition 1 holds is very simple: Poizat's example is the only one possible. That is, we prove:

**Proposition 2**
*Let $T$ be a complete $L^n$ theory which is not categorical in power $n+2$, and suppose that $n > 2$. Then $T = A(n)$.*

Thus Proposition 1 is a consequence of the analysis given by Poizat in the case of $A(n)$, which will be reviewed in §2.

We note that the case $n = 3$ of Proposition 2 was treated in [Po].

A group theoretical twist may be given to the question as follows:

**Problem 3**
*Define $h(n)$ as the least $k$ for which there is some complete $L^n$-theory which has two nonisomorphic homogeneous models of cardinality $k$. Compute (or estimate) $h(n)$.*

The following are immediate, taking into account Poizat's work:

1. $h(2) = 4$;
2. For $n > 2$, we have $n + 2 < h(n) \leq 2n + 1$.

For $n = 2$ one has the complete bipartite graph $K_{2,2}$ and its complement as an example. To see that $h(n) > n+2$ for $n > 2$, by Proposition 2 it suffices to consider Poizat's theory $A(n)$; he shows that this theory has no homogeneous model of order greater than $n+1$ [Po, p. 650]. For the upper bound one can use the theory of a unary predicate picking out a subset of size $n$ or $n+1$ in a model of cardinality $2n+1$; but one might reasonably modify the problem, allowing only theories with only one 1-type over the empty set, in the sense of $L^n$, or even restricting to primitive theories, in the sense defined immediately below.

For the proof of Proposition 2, the idea is a reduction to the *primitive* case. Since we deal with structures which are not homogeneous, we cannot use the usual group theoretic notion of primitivity, but instead use the $L^n$-analog:

**Definition 4**
*A structure is* $L^n$**-primitive** *if it carries no nontrivial equivalence relation which is* $L^n$*-definable.*

Here $L^n$-definability refers to definability without parameters. We will sometimes shorten "$L^n$-definability" to "definability", particularly in the latter part of the proof where specific definable relations are under consideration.

For $L^n$-homogeneous structures, primitivity as defined above means that the automorphism group of the structure is primitive in the usual sense of permutation groups. Our notion of primitivity is considerably weaker than the usual one in general, but it will be sufficient for the proof of Proposition 2.

## 2 The theories $A(n)$ and $S(n)$

We present some material from [Po] as background. This material will not be needed subsequently, apart from the following terminological conventions, based on terminology which is standard in the context of binary relations.

**Definition 5**
*Let $R(x_1,\ldots,x_n)$ be a relation on a set $S$.*

*1. $[S]^n$ is the set of (*ordered*) $n$-tuples from $S$ with distinct entries.*

*2. $R$ is* irreflexive *if $R \subseteq [S]^n$.*

*3. If $R$ is irreflexive, we denote by $-R$ its complement in the Boolean algebra of irreflexive $n$-place relations on $S$.*

*4. The symmetric group* $\mathrm{Sym}(n)$ *acts naturally on the $n$-place relations on $S$ by permutation of variables; we write $R^\sigma$ for the image of $R$ under $\sigma \in \mathrm{Sym}(n)$.*

*5. $R$ is* symmetric *if $R$ is invariant under the action of* $\mathrm{Sym}(n)$.

*6. $R$ is* antisymmetric *if $R$ is irreflexive, and for every transposition $\sigma$, we have $R^\sigma = -R$.*

In particular an antisymmetric relation $R$ in $n$ variables is invariant under the action of the alternating group $\mathrm{Alt}(n)$.

Two $L^n$-theories of particular interest are the theories of a "generic" irreflexive $n$-place relation, assumed to be either symmetric or antisymmetric:

**Definition 6**
*1. $S(n)$ is the theory of an irreflexive symmetric relation $R$ such that every ordered $(n-1)$-tuple extends to at least one $n$-tuple which satisfies $R$, and at least one which does not.*

*2. $A(n)$ is the theory of an irreflexive antisymmetric relation $R$ such that every ordered $(n-1)$-tuple extends to at least one $n$-tuple which satisfies $R$, and at least one which does not.*

As it happens [Po] $S(n)$ has no model of cardinality $n+2$ for any $n > 3$. To see this, consider the graph whose edges are the pairs whose complement, ordered as an $n$-tuple, satisfies $R$, and reformulate the extension property in terms of this graph.

On the other hand, $A(n)$ has models of cardinality $n+2$, which Poizat analyzes very neatly as follows [Po, p. 651]. Let $a, b, c$ be three elements, and $\mathbf{d}$ the remaining elements, arranged as an $(n-1)$-tuple. One of the relations $R$, $\neg R$ holds for two of these $n$-tuples, and the other relation holds for the third; we may therefore call one of three elements "exceptional" relative to the other two, and note that this relation is independent of the ordering of $\mathbf{d}$. But it is more useful to speak of the exceptional element as lying "between" the other two, because by further use of the axioms it can be verified that the three-place relation of betweenness defined in this way comes from a linear order. With this linear order $L$ fixed, or for that matter with the associated betweenness relation fixed (allowing a reversal of $L$), to determine the relation $R$ completely it suffices to specify one $n$-tuple for which $R$ holds. Thus there are at most two such relations, up to isomorphism: if $R$ is one such, then any other is isomorphic with $R$ or $-R$. Poizat's results therefore hinge on the determination of the values of $n$ for which $R$ and $-R$ are nonisomorphic.

For this it is convenient to focus on the behavior of the "central" $n$-tuple consisting of the middle $n$ elements of $L$; this $n$-tuple is determined by the betweenness relation associated with $R$ (and also with $-R$). If $n$ is congruent to 2 or 3 modulo 4 then reversal of $L$ carries $R$ to $-R$ since this function acts as an odd permutation on the central $n$-tuple and preserves the betweenness relation. Thus in this case there is a unique isomorphism type. In the remaining cases $R$ and $-R$ cannot be isomorphic, since such an isomorphism would preserve the betweenness relation, hence preserve or reverse $L$, and in any case act on the central $n$-tuple as an even permutation, thus preserving $R$.

So much for models of $A(n)$ of cardinality $n+2$. There is considerably more to be said about the models in general [Po].

We will see in Lemma 12 below how we eventually arrive at the theory $A(n)$ in the context of Proposition 2.

## 3 Primitivity

In this section we prove the following:

**Proposition 7**
*Let $T$ be a complete $L^n$-theory with $n > 2$ and $\mathcal{M}$ a model of $T$ of cardinality $n+k$ on which there is an $L^n$-definable equivalence relation having at least $k$ classes with more than one element. Then $T$ is categorical in power $n+k$.*

For $k=1$ there is always such a relation (with one class), so in this case the result is simply that a complete $L^n$-theory is categorical in power $n+1$. For $k=2$ the result becomes an important special case of Proposition 2:

**Corollary 8**
*Let $T$ be a complete $L^n$-theory with $n > 2$ and $\mathcal{M}$ an $L^n$-imprimitive model of $T$ of cardinality $n+2$. Then $T$ is categorical in power $n+2$.*

**Proof.** If $\mathcal{M}$ has a nontrivial $L^n$-definable equivalence relation $\sim$, and if $\sim'$ is the relation obtained from $\sim$ by collecting any classes with only one element into a single class, then $\sim'$ is again $L^n$-definable and is an equivalence relation with at least 2 classes having more than one element, unless there are only two classes and one of them is a singleton; in the latter case the element of the singleton class is definable, and it suffices to consider the induced structure on the remaining $n+1$ elements when the additional element is treated as a constant. $\square$

We now take up the proof of Proposition 7. For the remainder of this section we suppose that $k$, $n$, $T$, and $\mathcal{M}$ are fixed, satisfying the hypotheses of Proposition 7 with respect to a fixed $L^n$-definable equivalence relation $\sim$ on $\mathcal{M}$ having at least $k$ classes with more than one element. In particular $k \le n$ since there are at least $2k$ elements in the nontrivial equivalence classes.

Let $M$ be the underlying set of $\mathcal{M}$. Using the $L^n$-definition of $\sim$, fix an $L^n$ sentence $\phi_0$ expressing the property that $\sim$ is an equivalence relation. Call an $n$-tuple $\mathbf{a}$ of distinct elements in $M$ *standard* if $a_1, \ldots, a_k$ lie in distinct

$\sim$-classes of $M$, each of which contains more than one element, and $M \setminus \mathbf{a}$ is a system of representatives for these classes. There is an $L^n$-formula $\sigma(\mathbf{x})$ which states that the sequence $\mathbf{x}$ is standard, or more exactly which has that meaning when interpreted in models of cardinality $n+k$: $\sigma$ says that the elements $x_i$ are distinct, their first $k$ entries are inequivalent, and the equivalence class of $x_i$ is not exhausted by the elements in the sequence $x_1, \ldots, x_n$ for $1 \leq i \leq k$. The last clause requires a little attention: one reuses a variable at this point if there is more than one equivalence class, and if there is only one equivalence class the clause may be omitted, as in that case it is vacuous if the model is of cardinality greater than $n$.

With the exception of that degenerate case, the formula $\sigma(\mathbf{x})$ forces the cardinality of the model to be *at least* $n+k$, and in models of cardinality exactly $n+k$ this formula expresses the property of being standard. Furthermore, given two structures of cardinality $n+k$ satisfying $\phi_0$, and standard $n$-tuples $\mathbf{a}$, $\mathbf{b}$ in these structures, if the map $a_i \to b_i$ preserves the equivalence relation then this map has a unique extension to a bijection preserving the equivalence relation, and our problem is to find a formula $\phi(\mathbf{x})$ such that:

(1) $\phi(\mathbf{a})$ holds for some $n$-tuple in $\mathcal{M}$;
(2) If $\phi(\mathbf{b})$ holds for an $n$-tuple in a model $\mathcal{M}'$ of $\phi_0$ of cardinality $n+k$, then the map $a_i \mapsto b_i$ preserves the equivalence relation, and its canonical extension to a map from $\mathcal{M}$ to $\mathcal{M}'$ is an isomorphism.

The first part of clause (2), namely the requirement that the map given by $a_i \mapsto b_i$ preserves $\sim$, presents no problem at all. The remainder of clause (2) would be easy if we could define all the $n$-tuples in $\mathcal{M}$ explicitly in terms of the given $n$-tuple $\mathbf{a}$, without exceeding the allotted number of variables, but this is not quite possible. However we will see that all $n$-tuples of distinct elements can be generated from a single standard $n$-tuple by combining the following operations:

a. Given a standard $n$-tuple $\mathbf{a}$ and indices $i, j$ with $1 \leq i \leq k$ and $a_j \sim a_i$, let $\mathbf{a}^j$ be the result of replacing $a_j$ by the unique element of its equivalence class not represented in $\mathbf{a}$.
b. Given a standard $n$-tuple $\mathbf{a}$, replace some of the elements of $\mathbf{a}$ by elements not in $\mathbf{a}$, in such a way that none of the elements removed is equivalent to any of the elements adjoined.

In case $(a)$ we will say that the second $n$-tuple is obtained by a *switch*; in case $(b)$ we will speak of a *shift*.

If $\mathbf{a}$ is a standard $n$-tuple, and $\mathbf{b}$ is any $n$-tuple with distinct entries, then up to permutations of variables, $\mathbf{a}$ can be converted into $\mathbf{b}$ by a series of switches followed by a shift (and it will be useful to notice that the $n$-tuple remains standard as switches are applied). Indeed, if $A, B$ are respectively the complement of $\mathbf{a}$ and of $\mathbf{b}$ in $\mathcal{M}$, then after appropriate switches we may suppose that any element of $A$ which is equivalent to an element of $B$ belongs to $A \cap B$. After that it suffices to shift $B \setminus A$ into $A \setminus B$.

The reader may notice that we define switches very narrowly and treat shifts more broadly. This is because we need to consider the issue of $L^n$-definability of these operations, and shifting is an $L^n$-definable operation, whereas switching is not in general, and requires a considerably more subtle analysis (which, however, is not new – it was used already in the case $k = 1$).

In any case it will suffice to prove the following two definability lemmas relating to shifts and switches (we ignore permutations of variables, which raise no issues):

**Lemma 9** *Let* $\mathbf{a}$ *be a standard* $n$*-tuple in a model of* $\phi_0$ *of cardinality* $n + k$*. Then there is an* $L^n$*-formula* $\phi_{\text{sh}}(\mathbf{x})$ *holding for* $\mathbf{a}$ *such that for any other model of* $\phi_0$ *of cardinality* $n + k$ *with an* $n$*-tuple* $\mathbf{b}$ *satisfying* $\phi_{\text{sh}}(\mathbf{b})$*, the canonical mapping extending* $a_i \mapsto b_i$ *preserves* $\sim$ *and induces an isomorphism between the structures induced on any pairs of* $n$*-tuples* $\mathbf{a}'$*,* $\mathbf{b}'$ *which arise from* $\mathbf{a}$*,* $\mathbf{b}$ *by corresponding shifts.*

**Proof.** This is a definability result. Each of the possible shifts $\mathbf{y}$ of a standard $n$-tuple $\mathbf{x}$ is explicitly definable in $L^n$ from the common part of $\mathbf{x}$ and $\mathbf{y}$, since a shift of $l$ elements frees up the corresponding $l$ variables, and these variables may be reused to stand for the new elements of $\mathbf{y}$. It then suffices to assert that these $l$ variables represent the missing elements of certain specific equivalence classes, which have representatives in the common part of $\mathbf{x}$ and $\mathbf{y}$: one simply writes $x_i \sim x_j$ and $x_i \neq x_{i'}$ where the $x_i$ are recycled variables, and the $x_j, x_{i'}$ represent elements common to $\mathbf{x}$ and $\mathbf{y}$. $\square$

For switches the matter is more subtle, but as we have said, the issue arises already for the case $k = 1$, and can be dealt with in the same way in general:

**Lemma 10** *For any* $L^n$*-formula* $\phi(\mathbf{x})$*, and any indices* $i, j$ *with* $1 \leq i \leq k$ *and* $1 \leq j \leq n$*, there is a formula* $\phi^{i,j}(\mathbf{x})$ *which expresses the following:*

> $\mathbf{x}$ *is standard,* $x_i \sim x_j$*, and* $\phi(\mathbf{x}^j)$ *holds.*

**Proof.** Using a universal quantifier $\forall x_j$ one can easily express the condition "$\phi(\mathbf{x}) \& \phi(\mathbf{x}^j)$"; similarly, using an existential quantifier $\exists x_j$ one may express the condition "$\phi(\mathbf{x}) \vee \phi(\mathbf{x}^j)$". Denote the corresponding $L^n$-formulas by $\phi_1^j(\mathbf{x})$ and $\phi_2^j(\mathbf{x})$ respectively. The desired formula $\phi^j$ is then: $\phi_1^j(\mathbf{x}) \vee [\neg\phi(\mathbf{x}) \& \phi_2^j(\mathbf{x})]$. $\square$

It should be emphasized that in the above, although $\mathbf{x}$ stands for a sequence of variables, the expression $\mathbf{x}^j$ does not have any syntactical meaning, and is used in an informal sense; howver the expression $\phi(\mathbf{x}^j)$ does have a meaning, and $\phi(\mathbf{x}^j)$ is viewed as a property of $\mathbf{x}$, then this property can be expressed in the logic $L^n$.

With Lemmas 9 and 10 in hand, one can produce the final sentence which characterizes $\mathcal{M}$, in the following elaborate form:

> "There is a standard sequence $\mathbf{x}$ such that:
> For each sequence $\sigma$ of up to $k$ switches, the standard sequences $\mathbf{y}_\sigma$ obtained by performing those switches satisfy certain formulas $\phi^\sigma_{\text{sh}}(\mathbf{y}_\sigma)$, which determine the induced structure on shifts of $\mathbf{y}_\sigma$."

Here of course one cannot actually quantify over sequences of switches, so one writes out an enormous conjunction, and then the appropriate formulas $\phi^\sigma_{\text{sh}}(\mathbf{y}_\sigma)$ are chosen in accordance with Lemma 9, so that $\phi^\sigma_{\text{sh}}(\mathbf{y}_\sigma)$ is true in $\mathcal{M}$ for the standard sequence $\mathbf{y}_\sigma$, and determines the structure induced on each $n$-tuple obtainable from $\mathbf{y}_\sigma$ by a shift. Then repeated applications of Lemma 10, one for each switch, allow us to rewrite the formulas $\phi^\sigma_{\text{sh}}(\mathbf{y}_\sigma)$ as formulas $\psi_\sigma(\mathbf{x})$ in the original variables $\mathbf{x}$; this is important since we do not have enough variables to speak directly about the sequences $\mathbf{y}_\sigma$.

This completes the proof of Proposition 7.

## 4 $\tau(\mathcal{M})$

We know that if $\mathcal{M}$ is a model of a complete $L^n$-theory which is not categorical in power $n+2$ then $\mathcal{M}$ is primitive in the (weak) sense that it carries no non-trivial $L^n$-definable equivalence relation. This notion is considerably weaker than the group theoretic notion; for example, a sufficiently random graph will be primitive in this sense, and rigid.

**Notation 11** *For $\mathcal{M}$ a structure and $n$ fixed, let $\tau(\mathcal{M})$ be the greatest $t \leq n$ for which all $t$-tuples of distinct elements of $\mathcal{M}$ have the same $L^n$-type.*

Note that in speaking of $L^n$-types, we do not allow the introduction of any constants; the elements whose type is under consideration are taken to be represented by variables taken from the available set of $n$ variables. $L^n$-types will frequently be referred to simply as "types" – and no other types will occur.

The letters $t$ and $\tau$ are supposed to suggest (mildly) the word "transitivity" – our definition involves a weak definable analog of $t$-fold transitivity, in the same way that our notion of primitivity corresponds to the group theoretic one. In particular primitivity implies "transitivity": i.e., $\tau \geq 1$.

Now $\tau(\mathcal{M}) = n$ if and only if the complete $L^n$-theory of $\mathcal{M}$ is the theory of equality. The critical case for us is that in which $\tau(\mathcal{M}) = n - 1$, as this is the case which corresponds to Poizat's theory $A(n)$. Namely:

**Lemma 12**
If $n > 2$, $\mathcal{M}$ is an $L^n$-structure of order $n + 2$, and $\tau(\mathcal{M}) = n - 1$, then the complete $L^n$-theory of $\mathcal{M}$ is one of the following, up to interpretability:

1. The theory $A(n)$ of an antisymmetric $n$-ary relation such that every $(n-1)$-tuple of distinct elements extends to at least one $n$-tuple which satisfies the relation, and at least one which does not;
2. The analogous theory $S(3)$ of a symmetric irreflexive relation on triples such that every pair of distinct elements extends to at least one triple which satisfies the relation, and at least one which does not;
3. The $L^3$-theory $B(3)$ of the "betweenness" relation on a dense linear order;
4. The $L^4$-theory $F(4)$ of a binary relation $R$ on disjoint unordered pairs of points, encoded as a 4-place relation, such that every triple of points extends to a quadruple satisfying this relation, and the triple of relations $(R, R^\sigma, R^{\sigma^2})$ is a partition of the set of 4-tuples, where $\sigma \in \mathrm{Sym}(4)$ is a 3-cycle.

We remark that the first three theories were all studied in [Po]. We choose the notation "B" to suggest "betweenness", and "F" to suggest "factorization", the latter for reasons put forward below.

**Proof.** Consider the Boolean algebra of $n$-place irreflexive relations which are $L^n$-definable on $\mathcal{M}$. The symmetric group on $n$ symbols acts naturally on this algebra and permutes the atoms in some fashion. As $\tau(\mathcal{M}) < n$, there is more than one atom, and as $\tau(\mathcal{M}) = n - 1$ and $|\mathcal{M}| = n + 2$, there are at most three atoms.

If there are two atoms, then let $R$ denote one of them; the other is $-R$. The symmetric group either stabilizes both atoms or switches them; thus $R$ is either antisymmetric, or symmetric. Thus we are dealing either with the theory $A(n)$ or the analogous theory $S(n)$ of a symmetric relation $R$ with similar extension properties. In the case of $S(n)$, as remarked in [Po], and explained above, there is no model of cardinality $n + 2$ for $n > 3$.

It remains to consider the case in which there are exactly three atoms. Then for each $(n-1)$-tuple $\mathbf{a}$ in $\mathcal{M}$ and each atom $R$, there is a unique extension of $\mathbf{a}$ to an $n$-tuple satisfying $R$. If one of these atoms is symmetric, then after fixing $n-3$ points we have a set of distinguished triples taken from a set of 5 elements with the property that any pair of points lies in a unique distinguished triple. As any two distinguished triples meet in a unique point, we have a projective plane of order 2 on a set of 5 points, which is absurd. Accordingly none of the relations is symmetric, and as there are three atoms it follows that $\mathrm{Sym}(n)$

permutes them transitively. So the stabilizer of an atom under this action is a subgroup of $\mathrm{Sym}(n)$ of index 3, and therefore $n \leq 4$.

If $n = 3$ this is one of the situations considered in Proposition 8 of [Po] (and the note added in proof), leading to the third example mentioned above. Finally if $n = 4$ we have three atoms, each stabilized by a Sylow 2-subgroup of $\mathrm{Sym}(4)$, and permuted transitively by $\mathrm{Sym}(4)$. There are three equivalence relations having two classes of size two on a set of four elements, and the Sylow 2-subgroups of $\mathrm{Sym}(4)$ are their stabilizers. (For our present purposes such an equivalence relation should be viewed as a partition, specifically as an unordered pair of disjoint unordered pairs.) It follows that the atoms, which are sets of ordered 4-tuples with distinct entries, may be be viewed as relations on unordered pairs of disjoint pairs in $\mathcal{M}$. Picking one such atom $R$ we arrive at our fourth theory; the axioms given for $F(4)$ clearly hold, and the theory so axiomatized is clearly complete for $L(4)$. We will describe it more explicitly in the corollary following. $\square$

**Corollary 13**
*Let $T$ be a complete $L^n$-theory which has a model $\mathcal{M}$ of order $n+2$ for which $\tau(\mathcal{M}) = n-1$. Then either $T$ is categorical in power $n+2$ or $T = A(n)$.*

**Proof.** By the foregoing lemma we have to consider the theories $S(3)$, $B(3)$, $F(4)$. The first two were dealt with in [Po] in the course of an analysis of all the relevant $L(3)$-theories, so we need only examine $F(4)$. Let $\mathcal{M}$ be a model of $F(4)$ of order 6, which we think of as the complete graph on 6 vertices. The relation $R$ in question is thought of as a set of pairs of disjoint edges. The extension property forces each edge of $\mathcal{M}$ to be paired by $R$ with at least two other edges on the remaining 4 points; since this holds not only for $R$ but also for $R^\sigma$ and $R^{\sigma^2}$, each edge must be paired by $R$ with exactly two edges, covering the remaining 4 points; in other words for each edge $e$ the set $F_e = \{e\} \cup \{f : R(e,f)\}$ is a *1-factor* of $\mathcal{M}$, a covering of its vertices by disjoint edges.

Accordingly the relation $R$ corresponds to a partition of all the edges of $\mathcal{M}$ into 5 disjoint 1-factors. By inspection or [CL, Proposition 8.1, p. 56], there is a unique such partition, up to isomorphism. It follows easily that the theory $F(4)$ has a unique model of order 6. $\square$

The combinatorics associated with the theory $F(4)$ and its connection with various unusual combinatorial phenomena is discussed in [CL, Chapter 8], in particular it is closely connected with the existence of outer automorphisms of the symmetric group on 6 letters

## 5 Preliminary analysis

From now on we fix the following notation and hypotheses:

I. $\mathcal{M}$ is an $L^n$-structure of order $n+2$, and $n \geq 3$.
II. $\mathcal{M}$ is $L^n$-primitive.
III. $\tau(\mathcal{M}) < n-1$. We will write $\tau$ for $\tau(\mathcal{M})$.
IV. $R$ is an irreflexive $L^n$-definable $(\tau+1)$-ary relation which is nontrivial in the sense that both $R$ and $\neg R$ are satisfied in $\mathcal{M}$.

Our objective is to show that the theory of $\mathcal{M}$ is categorical in power $n+2$, primarily by making a close study of $\tau$. First however we require some general considerations.

**Notation 14** *Let* $\mathbf{c}$ *be a* $\tau$*-tuple in* $\mathcal{M}$ *with distinct entries. Set* $\mu_{\mathbf{c}}(R) = |\{c' \neq c_1, \ldots, c_\tau : R(\mathbf{c}, c')\}|$.

**Lemma 15**
*For* $\mathbf{c} \in \mathcal{M}$ *of length* $\tau$ *with distinct entries, and* $R(x_1, \ldots, x_{\tau+1})$ *an* $L^n$*-definable relation on* $\mathcal{M}$*,* $\mu_{\mathbf{c}}(R)$ *is independent of the choice of* $\mathbf{c}$.

**Proof.** Note that in the course of the proof we may replace $R$ by $-R$ if we so choose.

If $\mu_{\mathbf{c}}(R) < n-\tau$ then the value of $\mu_{\mathbf{c}}(R)$ is part of the $L^n$-type of $\mathbf{c}$ and hence is independent of the choice of $\mathbf{c}$. In any case $\mu_{\mathbf{c}}(R) + \mu_{\mathbf{c}}(-R) = n+2-\tau$, so if $\mu_{\mathbf{c}}(R) \geq n-\tau$ then after replacing $R$ by $-R$ we have $\mu_{\mathbf{c}}(R) \leq 2$. As $\tau < n-1$, this results in $\mu_{\mathbf{c}}(R) < n-\tau$, as desired, unless:

$$\mu_{\mathbf{c}}(R) = 2 \text{ and } \tau = n-2$$

In this case we have $\mu_{\mathbf{c}}(R) = \mu_{\mathbf{c}}(-R) = 2$ and as the inequalities $\mu_{\mathbf{c}}(R) \geq 2$ and $\mu_{\mathbf{c}}(-R) \geq 2$ are both part of the $L^n$-type of $\mathbf{c}$, also in this case $\mu$ is independent of $\mathbf{c}$. $\square$

**Notation 16** *In view of the foregoing, we write* $\mu(R)$ *in place of* $\mu_{\mathbf{c}}(R)$ *in the sequel.*

**Remark 17**
*Since* $|R| = |R^\sigma|$ *for any permutation* $\sigma$ *of the variables, it follows that* $\mu(R) = \mu(R^\sigma)$ *by a counting argument.*

In one more general case it will be possible to show the categoricity of the theory of $\mathcal{M}$ in power $n+2$ by writing out explicit axioms, as we did in the imprimitive case, so we deal with this before turning to a closer study of the parameter $\tau$.

**Definition 18**
*Let $A$ be a subset of $\mathcal{M}$ of cardinality at most $n-1$. We say that $A$* separates points *in $\mathcal{M}$ if the $L^n$-types realized by elements of $\mathcal{M}$ over $A$ are all distinct.*

**Lemma 19**
*Suppose that in $\mathcal{M}$ the following two conditions are satisfied:*

*(1) Every subset of $\mathcal{M}$ of cardinality $n-1$ separates points.*
*(2) There is a subset $X$ of $\mathcal{M}$ of cardinality $n-1$ such that every subset of $X$ of cardinality $n-2$ separates points.*

*Then the theory of $\mathcal{M}$ is categorical in power $n+2$.*

**Proof.** As in our previous argument along these lines, we write out the diagram of $\mathcal{M}$ explicitly.

We fix an $n$-tuple $\mathbf{c} = (c_1, \ldots, c_n)$ in $\mathcal{M}$ so that for $X = \{c_1, \ldots, c_{n-1}\}$ condition (2) is fulfilled. Let $a, b$ be the remaining two elements of $\mathcal{M}$. Note that $a$ and $b$ realize distinct $L^n$-types over $c_1, \ldots, c_{n-1}$. Hence in any model $\mathcal{M}'$ of the theory of $\mathcal{M}$ of order $n+2$, we can pick a realization $\mathbf{c}'$ of the type of $\mathbf{c}$ and label the remaining elements $a'$, $b'$ correspondingly, according to their types over $c'_1, \ldots, c'_{n-1}$. This produces a canonical map $\mathcal{M} \to \mathcal{M}'$, which we now fix. We claim that this map $\mathcal{M} \to \mathcal{M}'$ is an isomorphism.

In any case this map preserves atomic formulas when restricted to $\mathbf{c}$. Let $\mathbf{d}$ be any $n$-tuple of distinct elements of $\mathcal{M}$ which contains a subsequence $\mathbf{d}_0$ consisting of $n-2$ elements out of $c_1, \ldots, c_{n-1}$, and let $\mathbf{d}', \mathbf{d}'_0$ be the corresponding sequences in $\mathcal{M}'$. Then the $L^n$-types of the individual elements of $\mathcal{M}$ over $\mathbf{d}_0$ are clearly preserved by our canonical map, and as $\mathbf{d}_0$ separates points in $\mathcal{M}$, the theory of $\mathcal{M}$ then determines the $L^n$-type of the sequence $\mathbf{d}$, and correspondingly $\mathbf{d}'$. This applies in particular to any sequence obtained from $\mathbf{c}$ by replacing exactly one element by $a$ or $b$. In particular the canonical map preserves atomic formulas that do not involve both $a$ and $b$.

Now consider a sequence $\mathbf{d}$ obtained from $\mathbf{c}$ by replacing two elements, $c_i$ and $c_j$, by $a$ and $b$ respectively. Let $A$ be the set of entries $d_i$ of $\mathbf{d}$ other than $b$. Then $A$ separates points in $\mathcal{M}$ and hence the elements $c_i, c_j, b$ realize three distinct types over $A$. By what we have done so far, $c'_i$ and $c'_j$ realize the corresponding types over the image of $A$. Hence $b'$ can only realize the third of these types in $\mathcal{M}'$, and it follows that the natural map preserves atomic formulas when restricted to $\mathbf{d}$ as well. This completes the analysis. $\Box$

**Corollary 20**
*Suppose that the theory of $\mathcal{M}$ is not categorical in power $n+2$. Then one of*

*the following holds:*

*(Va) There is a subset $A$ of $\mathcal{M}$ of cardinality $n-1$ and a complete 1-type over $A$ realized by two elements of $\mathcal{M}$; or*

*(Vb) For every subset $A$ of $\mathcal{M}$ of cardinality $n-1$ there is a subset $B \subseteq A$ of cardinality $n-2$, such that the remaining 4 points of $\mathcal{M}$ either realize the same type over $B$, or realize exactly two types over $B$, with each type realized twice.*

**Proof.** This is not quite a simple repetition of Lemma 19 because of the slightly more precise description in our case $(Vb)$, corresponding to case (2) previously. The possibility that we are now excluding in this case is that of a type realized by a unique element $a$ over $B$, with $a \notin B$. In this case we consider the set $A' = B \cup \{a\}$, and notice that two elements which realize the same type over $B$ also realize the same type over $A'$, so we find ourselves back in case $(Va)$. $\square$

Now what remains to be proved is the following:

**Proposition 21**
*Under hypotheses $(I-IV)$, case $(Va)$ does not occur, and in case $(Vb)$ we have one of the following:*

1. *$n=3$, $\tau=1$, and up to definable equivalence $\mathcal{M}$ is the pentagon graph.*
2. *$n=4$, $\tau=2$, and up to definable equivalence $\mathcal{M}$ is the projective line over the field $F_5$, equipped with all relations invariant under the group* $\mathrm{PSL}(2,5)$*; or*
3. *$n=6$, $\tau=3$, and up to definable equivalence $\mathcal{M}$ is a 3-dimensional affine space over a field of 2 elements, with the 4-place relation of "coplanarity".*

It is easy to see that the theories exhibited are categorical in power $n+2$, and so this will conclude our analysis.

We divide the proof into two parts. Case $(Va)$ is treated in the next section, and Case $(Vb)$ is the subject of the final section. The proofs in the two cases are very similar, with the structure of the argument clearest in Case $(Va)$.

## 6 Case $(Va)$

**Lemma 22**
*Under hypotheses $(I-IV)$, case $(Va)$ does not occur.*

**Proof.** Assume toward a contradiction that $A$ is a subset of order $n-1$ and that $M \setminus A = \{b_1, b_2, c\}$ where $b_1$ and $b_2$ have the same type over $A$.

Let $R(x_1, \ldots, x_k)$ be a nontrivial relation with $k = \tau + 1$, and with $\mu_R \leq (n + 2 - \tau)/2$.

Suppose first:

(1) $R$ is symmetric in its last two variables

We may also suppose that $\mu_R$ is minimal, subject to (1).

We deal first with the case in which $\tau = 1$ and $R$ is binary. Thus we have the structure of a graph $\Gamma$ on $\mathcal{M}$. If $(b_1, b_2)$ is not an edge of $\Gamma$, then $b_1$ and $b_2$ have the same neighbors in $\Gamma$, since this holds in the restriction of $\Gamma$ to $A \cup \{b_1, b_2\}$ by assumption, and the degree is constant. Thus in this case we have a nontrivial equivalence relation defined on $\mathcal{M}$, which is a contradiction. The same applies if $(b_1, b_2)$ is an edge of $\Gamma$, by passing to the complementary graph.

Now suppose $\tau > 1$. For $\mathbf{a} = (a_1, \ldots, a_{k-2})$ distinct, let $\Gamma_{\mathbf{a}}$ be the graph on the remaining elements with edge relation given by $R(\mathbf{a}, x, y)$. We will find $\mathbf{a} \subseteq A$ so that $(b_1, b_2)$ is not an edge of $\Gamma_{\mathbf{a}}$. First choose $\mathbf{a}' = (a_1, \ldots, a_{k-3})$ in $A$, distinct and arbitrary. Then $|\{a_1, \ldots, a_{k-3}\} \cup \{x : R(\mathbf{a}', x, b_1, b_2)\}| \leq (\tau - 2) + (n + 2 - \tau)/2 < |A|$ so we may choose $a_{k-2} \in A \setminus \{a_1, \ldots, a_{k-3}\}$ so that with $\mathbf{a} = (\mathbf{a}' a_{k-2})$, $b_1$ and $b_2$ are nonadjacent in $\Gamma_{\mathbf{a}}$.

Arguing as in the case $\tau = 1$, $b_1$ and $b_2$ have the same neighbors in the graph $\Gamma_{\mathbf{a}}$, so the relation $R^E(x_1, \ldots, x_{k-2}, x, y)$ defined by "$x, y$ have the same neighbors in $\Gamma_{\mathbf{x}}$ and $x \neq y$" is nontrivial. Furthermore with $x_1, \ldots, x_{k-2}$ and $x$ fixed, and with $z$ a fixed neighbor of $x$ in $\Gamma_{\mathbf{x}}$, we have $R^E(\mathbf{x}, x, y) \Rightarrow y$ is adjacent to $z$ and $y \neq x$, hence $\mu_{R^E} < \mu_R$, since $\mu_R$ is the degree of $z$ in the graph $\Gamma_{\mathbf{x}}$. This contradicts the choice of $R$.

Now suppose that (1) fails in the sense that

> There is no nontrivial $k$-ary relation defined on $\mathcal{M}$ which is symmetric in two of its variables.

If $R$ is a nontrivial $k$-ary relation, then by considering the relations

$$R(x_1, \ldots, x_{k-2}, x, y) \& R(x_1, \ldots, x_{k-2}, y, x)$$
$$\neg R(x_1, \ldots, x_{k-2}, x, y) \& \neg R(x_1, \ldots, x_{k-2}, y, x)$$

we conclude that $R$ is antisymmetric in its last two variables, so that for $\mathbf{a} = \{a_1, \ldots, a_{k-2}\}$ distinct, the relation $R(\mathbf{a}, x, y)$ defines a tournament $\Gamma_{\mathbf{a}}$ on the remaining elements, with constant indegree and outdegree, both equal to $(n + 2 - \tau)/2$.

Fix $\mathbf{a} = (a_1, \ldots, a_{k-2})$ in $A$ distinct and suppose that $(b_1, b_2)$ is an arc of the tournament $\Gamma_{\mathbf{a}}$. For $u$ a vertex of $\Gamma_{\mathbf{a}}$ let $u' = \{v : (u, v) \text{ is an arc of } \Gamma_{\mathbf{a}}\}$. Then $b_1' \cap A = b_2' \cap A$ and as $b_1, b_2$ have equal outdegrees we find that $b_1' = \{b_2\} \cup (b_2' \cap A)$. This leads us to consider the relation $R^1(\mathbf{x}, x, y)$ defined by:

$$R(\mathbf{x}, x, y) \& \forall z \neq y \, [R(\mathbf{x}, x, z) \Rightarrow R(\mathbf{x}, y, z)]$$

Evidently $R^1$ is a nontrivial definable $k$-ary relation with $\mu_{R^1} = 1$: if $R^1(\mathbf{x}, x, y_1)$ and $R^1(\mathbf{x}, x, y_2)$ both hold, then $y_1, y_2 \in x'$ and hence by the definition of $R^1$, one would have both $y_2 \in y_1'$ and $y_1 \in y_2'$.

Now by our remarks above applied to a tournament associated with $R^1$ in place of $R$, we find $(n + 2 - \tau)/2 = 1$, and this is a contradiction. $\square$

When this analysis is repeated in case $(Vb)$, it will lead to the identification of some exceptional cases.

## 7 Case $(Vb)$

Our remaining claim is that under hypotheses $(I - IV)$ and $(Vb)$, the theory of $\mathcal{M}$ is one of the following:

1. $n = 3$, $\tau = 1$, and up to definable equivalence $\mathcal{M}$ is the pentagon graph.
2. $n = 4$, $\tau = 2$, and up to definable equivalence $\mathcal{M}$ is the projective line over the field $F_5$, equipped with all relations invariant under the group $\mathrm{PSL}(2, 5)$; or
3. $n = 6$, $\tau = 3$, and up to definable equivalence $\mathcal{M}$ is a 3-dimensional affine space over a field of 2 elements, with the 4-place relation of "coplanarity".

Actually we will operate with a very special case of condition $(Vb)$:

$(Vb')$ There is $A \subseteq M$ with $|A| = n - 2$ and $M \setminus A = \{b_1, b_2, c_1, c_2\}$ so that the type of $b_i$ over $A$ is independent of $i$, and the type of $c_i$ over $A$ is independent of $i$ ($i = 1$ or $2$).

In this section hypotheses $(I - IV)$ and $(Vb')$ will be assumed throughout. Lemma 23 will provide a nontrivial irreflexive $(\tau + 1)$-place relation $R$, symmetric in its last two variables, with $\mu_R \leq 2$. Lemmas 24 and 25 dispose of the case in which $\mu_R = 1$. Lemmas 27 and 28 dispose of the case in which $\mu_R = 2$, either by reducing to the previous case, or outright.

**Lemma 23**
*There is a nontrivial $(\tau+1)$-ary relation $R$, symmetric in its last two variables,*

*with* $\mu_R \leq 2$.

**Proof.** Fix $A, b_1, b_2, c_1, c_2$ as in condition $(Vb')$ and let $R$ be a nontrivial $(\tau+1)$-ary relation defined on $\mathcal{M}$ with $\mu_R \leq (n+2-\tau)/2$. Set $k = \tau + 1$. Assume that $\mu_R \geq 3$, so that $n \geq \tau + 4$.

Suppose first:

(1) $R$ is symmetric in its last two variables.

We may also suppose that $\mu_R$ is minimal, subject to (1).

We deal first with the case in which $R$ is binary and $\tau = 1$. Thus we have the structure of a graph $\Gamma$ on $\mathcal{M}$. If $b_1$ and $b_2$ have the same neighbors in $\Gamma$ then there is a nontrivial equivalence relation defined on $\mathcal{M}$, a contradiction. If $(b_1, b_2)$ is not an edge of $\Gamma$, then the remaining possibility is that they each have one neighbor in $\{c_1, c_2\}$, and the remaining $\mu_R - 1$ neighbors in $A$ are common to both. Accordingly we consider the relation $R'(x, y)$ defined by:

$$\neg R(x,y) \& \exists z \, \forall z' \neq z \, [R(x,z') \Rightarrow R(y,z')]$$

In other words: $x$ and $y$ are nonadjacent, and have at least $\mu_R - 1$ common neighbors.

We estimate $\mu_{R'}$ by comparing upper and lower bounds on the number of edges which are adjacent to some neighbor of $x$ (including the ones containing the vertex $x$), where $x$ is fixed. This yields the estimate:

$$\mu_R + \mu_{R'}(\mu_R - 1) \leq \mu_R^2$$

and thus $\mu_{R'} \leq \mu_R$. As $R'$ is symmetric, the choice of $R$ yields $\mu_{R'} = \mu_R$ and hence every vertex $y$ other than $x$ which is adjacent to a neighbor of $x$ must satisfy $R'(x, y)$. Thus the relation "$y = x$ or $R'(x, y)$" is a nontrivial equivalence relation on $\mathcal{M}$, a contradiction.

If on the other hand $(b_1, b_2)$ is an edge of $\Gamma$, then one concludes that $b_1$ and $b_2$ have at least $\mu_R - 2$ common neighbors. Then we consider the relation $R''(x, y)$ defined by:

$$R(x,y) \& \exists^{\geq \mu_R - 2} z \, [R(x,z) \& R(y,z)]$$

which is definable using at most $n$ variables. Evidently $R''$ is nontrivial, symmetric, and satisfies $\mu_{R''} \leq \mu_R$, and hence $\mu_{R''} = \mu_R$, so $R'' = R$. Thus any two adjacent vertices have at least $\mu_R - 2$ common neighbors. It follows that for any three neighbors of a vertex in $\Gamma$, at least one of them is adjacent to the other two.

Now let $R^2(x, y)$ be the relation "the distance from $x$ to $y$ in $\Gamma$ is 2". We will show that $\mu_{R^2} \le 2$. Suppose that $a$ is a vertex of $\Gamma$ and that $v_1, v_2, v_3$ are distinct vertices lying at distance 2 from $a$ in $\Gamma$. Let $u_i$ be chosen so that $(a, u_i, v_i)$ is a path for $i = 1, 2, 3$. Then the $u_i$ are distinct, as otherwise one of them has $a$ and two of the $v_i$ as neighbors, forcing $a$ to be adjacent to some $v_i$. As the $u_i$ are distinct we may suppose that $u_1$ is adjacent to $u_2$ and then by considering the neighbors of $u_2$ conclude that $u_1$ is adjacent to $v_2$. Consideration of the neighbors of $u_1$ yields a contradiction. Thus $\mu_{R^2} \le 2$ and $\mu_{R^2}$ is clearly nontrivial, so we have a relation of the desired type.

Now assume $\tau > 1$. As in the proof of Lemma 22 we can produce a sequence $\mathbf{a}$ of distinct elements of $A$, of length $k - 2$, so that $(b_1, b_2)$ is not an edge in the corresponding graph $\Gamma_{\mathbf{a}}$. This depends on the inequality

$$(\tau - 2) + (n + 2 - \tau)/2 < n - 2,$$

i.e. $n > \tau + 2$, which holds as noted at the outset.

If $b_1$ and $b_2$ have the same neighbors in $\Gamma_{\mathbf{a}}$ we arrive at a contradiction as in the proof of Lemma 22. The alternative is that $b_1$ and $b_2$ have exactly $\mu_R - 1$ neighbors in common. Then we consider the relation $R'(\mathbf{a}, x, y)$ analogous to $R'(x, y)$ as defined above, and argue as in that case that $\mu_{R'} = \mu_R$ and that any two vertices with a common neighbor in $\Gamma_{\mathbf{a}}$ have exactly $\mu_R - 1$ common neighbors.

It follows that $\Gamma_{\mathbf{a}}$ is a disjoint union of bipartite graphs, where each component is the result of deleting a matching from a complete bipartite graph $K_{\mu_R+1,\mu_R+1}$. Accordingly we define a relation $R^3(\mathbf{x}, x, y)$ by: "the distance from $x$ to $y$ in $\Gamma_{\mathbf{x}}$ is 3". Then $\mu_{R^3} = 1$ and we have a relation of the desired type.

This disposes of case (1). Now suppose:

> There is no nontrivial $k$-ary relation defined on $\mathcal{M}$ which is symmetric in two of its variables.

Arguing as in the proof of Lemma 22, if $R$ is any nontrivial $k$-ary relation then it is antisymmetric in its last two variables, and for $\mathbf{a} = (a_1, \ldots, a_{k-2})$ distinct, the relation $R(\mathbf{a}, x, y)$ defines a tournament $\Gamma_{\mathbf{a}}$ on the remaining elements, of indegree and outdegree constant and equal to $(n + 2 - \tau)/2$.

Fix $\mathbf{a}$ in $A$ of length $k-2$ and suppose that $(b_1, b_2)$ is an arc in the tournament $\Gamma_{\mathbf{a}}$. As in the proof of Lemma 22 it follows by inspection that $b_1' \subseteq \{b_2\} \cup b_2'$ and hence the relation $R^1(\mathbf{x}, x, y)$ which expresses this relationship is nontrivial and satisfies $\mu_{R^1} = 1$. Thus the relation $R^1(\mathbf{x}, x, y) \vee R^1(\mathbf{x}, y, x)$ has $\mu = 2$. $\square$

**Lemma 24**
*If $\tau \geq 4$ then $\tau = n - 2$.*

**Proof.** Fix $A, b_1, b_2, c_1, c_2$ as in condition $(Vb')$.

We have assumed $\tau \leq n - 2$. Suppose toward a contradiction that $4 \leq \tau < n - 2$. Fix a nontrivial $(\tau + 1)$-ary relation $R$ which is symmetric in its last two variables, with $\mu_R \leq 2$.

Consider the relation $R'(x_1, x_2, y_1, y_2)$ defined as follows:

$$\forall z_1, \ldots, z_{\tau-1} \notin \{x_1, x_2, y_1, y_2\} \, [R(\mathbf{z}, x_1, y_1) \Rightarrow R(\mathbf{z}, x_2, y_1) \vee R(\mathbf{z}, x_2, y_2)]$$

Note that at most $n$ variables are used in this definition.

Now $R'(b_1, b_2, c_1, c_2)$ holds, and since $\tau \geq 4$ the same holds for all quadruples $x_1, x_2, y_1, y_2$ of distinct elements. We claim:

$$(*) \qquad R(\mathbf{z}, x_1, y) \Rightarrow R(\mathbf{z}, x_2, y)$$

for all choices of $\mathbf{z}, x_1, x_2, y$ distinct, from which it follows easily that $R$ is trivial. If the claim $(*)$ fails, take $\mathbf{z}, x_1, x_2, y$ distinct for which we have

$$R(\mathbf{z}, x_1, y) \& \neg R(\mathbf{z}, x_2, y)$$

and apply the property $R'(x_1, x_2, y, y')$ for $y' \notin \{z_1, \ldots, z_{\tau-1}, x_1, x_2, y\}$ to conclude that $R(\mathbf{z}, x_2, y')$ holds for all such $y'$, and thus $\mu_R \geq 3$, a contradiction. $\square$

**Lemma 25**
*If there is a $(\tau + 1)$-ary relation $R$ defined on $\mathcal{M}$ with $\mu_R = 1$, then either $n = 5$ and $\mathcal{M}$ is $L^n$-definably equivalent with the projective plane over $F_2$, or $n = 6$ and $\mathcal{M}$ is $L^n$-definably equivalent with affine geometry of dimension 3 over $F_2$, given by the coplanarity relation.*

**Proof.** Fix $A, b_1, b_2, c_1, c_2$ as in condition $(Vb')$.

If $\tau = n - 2$ then let $\mathbf{a}$ be an enumeration of $A$; consideration of $R(\mathbf{a}, x)$ then yields a contradiction. Thus $\tau < n-2$ and hence by the previous lemma $\tau \leq 3$.

If $\tau = 1$ it follows easily that $\mathcal{M}$ is imprimitive, so assume $\tau \geq 2$.

If $\tau < n-3$ then let $\mathbf{a}'$ be a sequence of $\tau - 2$ distinct elements of $A$ and choose $a \in A$ so that none of the relations

$$R(\mathbf{a}', a, b_1, b_2), R(\mathbf{a}', a, b_1, c_1), R(\mathbf{a}', a, b_1, c_2)$$

holds. Then $R(\mathbf{a}', a, b_1, a_0)$ holds for some $a_0 \in A$ hence also $R(\mathbf{a}', a, b_2, a_0)$ holds, contradicting $\mu_R = 1$. Thus $\tau = n-3$.

Let $\mathbf{a}$ be a sequence of $\tau$ distinct elements of $A$. Then $R(\mathbf{a}, x)$ determines a unique element of $\mathcal{M}$ which can only be the remaining element of $A$. As the same applies to any permutation of $R$, it follows that $R$ is symmetric when restricted to $A$. Hence the symmetrization $R^s$ of $R$, obtained by intersecting $R^\sigma$ for all permutations $\sigma$ of the variables, is nontrivial. Since $\mu_R = 1$ it follows that $R$ itself is symmetric.

Now $\tau = 2$ or 3 and correspondingly $n = 5$ or 6. If $\tau = 2$ then it is easily seen that $R$ picks out the lines of a projective plane over $F_2$, and also that there is no further structure on $\mathcal{M}$.

If $\tau = 3$ and $n = 6$ then $R$ picks out certain subsets of $\mathcal{M}$ of order 4, which we may call *planes*. One checks that our conditions force this to be the expected affine geometry with no further structure. As $\mu_R = 1$ any three points lie in a unique plane. So there are 14 planes, each containing 6 pairs of points, and thus on the average a pair of points lies in 3 planes. In particular some pair of points lies in at least 3 planes, and as this property can be expressed in $L^n$ and $\tau \geq 2$, the same applies to any pair. It follows that each pair of points lies in exactly 3 planes, and hence any two planes which meet will intersect in two points. Now fixing a point 0, set $x + y = z$ if either $\{0, x, y, z\}$ is a plane, or one of $x, y, z$ is 0 and the other two are equal. One checks that this is the desired group structure on $\mathcal{M}$ and that the geometry on $\mathcal{M}$ agrees with the associated affine geometry.

Finally one must check that there is no further structure on $\mathcal{M}$. It suffices to consider a formula $S(x_1, x_2, x_3, x_4, x_5, x_6)$ defining a complete type (with $x_1, \ldots, x_6$ distinct) and to show that it is definable in the affine geometry. We may suppose that $S$ implies that the sets $\{x_1, x_2, x_3, x_4\}$ and $\{x_1, x_2, x_5, x_6\}$ are planes.

Now if more than one type is realized over $A$ then by hypothesis there are exactly two such types, and one can define a nontrivial ternary relation $T(x, y, z)$ expressing that $x$ and $y$ realize the same type over the plane complementary to the plane containing $x, y, z$; this is done using three auxiliary variables to determine the complementary plane, and then reusing the variable $z$ to represent its fourth element. As this is a contradiction, $\mathcal{M}$ realizes only one type over $A$, and as $\tau \geq 3$ the same applies to any plane.

Returning to $S$, we consider the relation $\exists x_6\, S(x_1, \ldots, x_5, x_6)$. By our last remark, for distinct $x_1, \ldots, x_5$, this relation is equivalent to the coplanarity of $x_1, x_2, x_3, x_4$. Hence the coplanarity conditions which follow from $S$ are equivalent to $S$. $\square$

**Corollary 26**
*Under the hypotheses of Lemma 25, condition $(Vb)$ holds if and only if $n = 6$ and $\mathcal{M}$ is $L^n$-definably equivalent to 3-dimensional affine geometry over $F_2$.*

**Proof.** In the projective plane over $F_2$ condition $(Vb)$ fails for four points with no three on a line. In the affine case any five points contain an affine plane. $\square$

We have to deal also with the case $\mu_R = 2$.

**Lemma 27**
*Suppose that $R$ is a nontrivial $(\tau + 1)$-ary relation definable on $\mathcal{M}$, which is symmetric in its last two variables, with $\mu_R = 2$, and suppose that there is no such relation with $\mu = 1$. Then $\tau = n - 2$.*

**Proof.** Suppose $\tau \leq n - 3$. Fix $A, b_1, b_2, c_1, c_2$ as in condition $(Vb')$. If $\tau = 1$ it is easy to get a contradiction: as $\mathcal{M}$ is primitive it must then consist of a single cycle of prime order, and as $\tau \leq n - 3$ we have $n \geq 4$, $n + 2 \geq 6$, and this easily contradicts the assumptions on the $b_i$ and $c_i$.

So we will suppose $\tau \geq 2$. We take $\mathbf{a}$ a sequence of $\tau - 1$ distinct elements of $A$, choosing the first $\tau - 2$ elements arbitrarily, but taking the last element $a_{\tau-1}$ so that $R(\mathbf{a}, b_1, b_2)$ does not hold; since there are exactly 2 elements $x$ for which $R(a_1, \ldots, a_{\tau-2}, x, b_1, b_2)$ holds, and at least 3 available in $A \setminus \{a_1, \ldots, a_{\tau-2}\}$, this can be done.

Let $\Gamma_{\mathbf{a}}$ be the associated graph. Then the connected components of $\Gamma_{\mathbf{a}}$ are cycles. If one of these cycles has even order then the "antipodality" relation $R^A$ is a $(\tau+1)$-ary relation defined on $\mathcal{M}$, symmetric in its last two variables, with $\mu_{R^A} = 1$, a contradiction. Thus the connected components of $\Gamma_{\mathbf{a}}$ have odd order.

From this, applying $(Vb')$, it follows easily that $b_1, b_2, c_1, c_2$ are contained in a component of order 5, with the fifth vertex $a$ of this component lying in $A$, and adjacent to $b_1$ and $b_2$. The relation $R^5(\mathbf{x}, x)$ given by: "the connected component of $x$ in $\Gamma_{\mathbf{x}}$ is a 5-cycle" is expressible in $L^n$ and as this relation is $\tau$-ary and nonempty it holds everywhere, so $\Gamma_{\mathbf{a}}$ is a union of 5-cycles. If $\tau < n - 2$ then at least one such cycle is contained in $A$ and this yields $\tau \leq n - 7$. Accordingly we may select the final entry $a_{\tau-1}$ of $\mathbf{a}$ in $A$ in such a way that neither $R(\mathbf{a}, b_1, b_2)$ nor $R(\mathbf{a}, c_1, c_2)$ holds. This then violates our analysis of the component of $\Gamma_{\mathbf{a}}$ containing $b_1$. We conclude therefore that

$\tau = n - 2$. $\square$

**Lemma 28**
*Under the hypotheses of the preceding lemma, either $n = 3$ and $\mathcal{M}$ is definably equivalent to the pentagon graph, or $n = 4$ and $\mathcal{M}$ is definably equivalent to the projective line over $F_5$ equipped with all $PSL(2,5)$-invariant relations.*

**Proof.** Fix $A, b_1, b_2, c_1, c_2$ as in condition $(Vb')$. By the preceding lemma, $|A| = \tau$. Let $\mathbf{a}$ be a sequence of $\tau - 1$ distinct elements of $A$ and let $a_\tau$ be the remaining element of $A$. As seen in the previous proof, the graph $\Gamma_{\mathbf{a}}$ is a 5-cycle.

As we assume there is no $L^n$-definable $(\tau+1)$-ary relation $S$ symmetric in its last two arguments with $\mu_S = 1$, for each permutation $\sigma$ of the arguments, $R^\sigma$ is either $R$ or $-R$. Since a transposition of the last two arguments preserves $R$, it follows that $R$ is symmetric. Thus we may think of $R$ as a set of distinguished subsets of $\mathcal{M}$ of size $\tau + 1$ such that each subset of size $\tau$ has exactly two extensions to one of these distinguished subsets. Passing to complements, let $\mathcal{B}$ be

$$\{M \setminus \{a_1, \ldots, a_{\tau+1}\} : R(\mathbf{a})\}$$

Then $\mathcal{B}$ is a family of triples such that every subset of $M$ of order 4 contains exactly two triples. Fix an element $a$ in $\mathcal{M}$ and define a graph $\Gamma(\mathcal{B}, a)$ on the remaining points by the rule: $(b_1, b_2)$ is an edge if and only if $\{a, b_1, b_2\} \in \mathcal{B}$. The conditions on $\mathcal{B}$ imply that $\Gamma(\mathcal{B}, a)$ contains no triangle and no independent set of three vertices. Hence $\Gamma(\mathcal{B}, a)$ has order at most 5, that is $n \leq 4$.

Thus either $\tau = 1$, $n = 3$ or $\tau = 2$, $n = 4$. It is easy to reconstruct the relation $R$ in these two cases, uniquely up to isomorphism, and to verify that there is no additional structure. $\square$

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