Classification of Simple K*-Groups of Finite Morley Rank and Even Type: Geometric Aspects

Tuna Altınel   Alexandre V. Borovik*   G. Cherlin†

Introduction

According to a long-standing conjecture in model theory, simple groups of finite Morley rank should be algebraic. The present paper outlines some of the last in a series of results aimed at proving the following:

Even Type Conjecture. *Let G be a simple group of finite Morley rank of even type, with no infinite definable simple section of degenerate type. Then G is a Chevalley group over an algebraically closed field of characteristic* 2.

See [13] for a brief informal introduction to the subject, [1] for the most recent survey of the classification = programme, and [14] for further technical details on groups of finite Morley rank.

An infinite simple group G of finite Morley rank is said to be of *even type* if its Sylow 2-subgroups are infinite and of bounded exponent. It is of *degenerate type* if its Sylow 2-subgroups are finite. If the main conjecture is correct, then there should be no groups of degenerate type. So the flavour of the Even Type Conjecture is that the classification in the even type case reduces to an extended Feit-Thompson Theorem. Those who are skeptical about the main conjecture would expect degenerate type groups to exist. The Even Type Conjecture confirms that this is the heart of the matter.

In the present paper we outline some geometric arguments which play the crucial role at the final stages of analysis which has been undertaken in [3, 17, 4, 5, 7].

We work in the following context. Let G be a counterexample to the Even Type Conjecture of minimal Morley rank. This allows us to assume that every proper simple definable connected section of G is a Chevalley group over an algebraically closed field. We adopt the terminology of the classification of finite simple groups and say that G is a *K^*-group*. We take a 2-Sylow° subgroup S of G (that is, the connected component of a Sylow 2-subgroup), a Borel subgroup B containing S, and the set $\mathcal{M}$ of minimal 2-local° subgroups containing B

*Partially supported by the Royal Society Leverhulme Trust Senior Research Fellowship.
†Supported by NSF Grant DMS 0100794.

as a proper subgroup. It is shown at some point of our analysis [5, 7] that if $P \in \mathcal{M}$ then $O^{2'}(P/O_2(P)) \simeq \mathrm{SL}_2(K)$ for some algebraically closed field of characteristic 2 and $C_P(O_2(P)) \leqslant O_2(P)$.

We have the following natural case division:

Thin Groups: $|\mathcal{M}| \leqslant 1$. This case occurs in the nature only if $\mathcal{M} = 3D\emptyset$ and $= G \simeq \mathrm{SL}_2(K)$.

Quasithin Groups: $|\mathcal{M}| = 3D2$. In that case, we need to identify G with one of the Lie rank 2 Chevalley groups: $\mathrm{PSL}_3(K)$, $\mathrm{PSp}_4(K)$ or $\mathrm{G}_2(K)$ over an algebraically closed field of characteristic 2.

Generic Groups: $|\mathcal{M}| \geqslant 3$. In that case, G is a Chevalley group of Lie rank $|\mathcal{M}| \geqslant 3$.

Interestingly, each of these cases is resolved by an application of the amalgam method. In the case of *thin* groups, the crucial role is played by the Pushing-Up Theorem [5], proven, in our context, by essentially the same amalgam argument as its finite group prototype, due to Stellmacher [23].

Generic groups can be handled either by constructing a BN-pair = in G of (Tits) rank at least 3 [11] and the subsequent application of the classification of BN-pairs of finite Morley rank (Kramer, van Maldeghem and Tent [18]), or by the analysis of the centralisers of p-elements for odd primes p [12] which eventually leads to the construction in G of a system of "root SL_2-subgroups" and application of the Curtis-Phan-Tits Theorem; see the paper by Bennett and Shpectorov [10] in this = volume for the discussion of the underlying amalgams.

In this paper, we are dealing with *quasithin* groups and prove for = them the following

Identification Theorem *Let G be a simple K^*-group of finite Morley rank and even type. Suppose that G is generated by two 2-local$^\circ$ subgroups P_1, P_2 each containing the connected component of the normaliser of a fixed Sylow$^\circ$ 2-subgroup of G. Assume that $O^{2'}(P_i)/O_2(P_i) \simeq \mathrm{SL}_2(F_i)$ with F_i an algebraically closed field of characteristic 2, for $i = 3D1, 2$, and = that $C^\circ_{P_i}(O_2(P_i)) \leqslant O_2(P_i)$. Then G is a Lie rank 2 = Chevalley group over an algebraically closed field of characteristic 2.*

The proof of the Even Type Conjecture itself from these ingredients will be the subject of one further paper.

The proof of the Identification Theorem relies very heavily on the amalgam method in the form used by Stellmacher in [22] and by Delgado and Stellmacher [15], particularly the former version. We have found that the type of arguments that are used in conjunction with the amalgam method can generally be adapted to the context of groups of finite Morley rank with comparatively little alteration, though some attention to detail is required, notably in conjunction with some basic facts of representation theory for which the analogs are obtained through some *ad hoc* arguments, and definability issues. Accordingly we will not devote much space here to the adaptation of those arguments, merely summarising the general flow, recording precisely the point to which they bring us, and pointing out a few issues that do require specific attention. A detailed account of the adapted argument will be found in the technical report [6]. A model for this sort of argument is also found in the appendix to [5], where an analog of the much shorter amalgam argument of [23] is presented.

The amalgam method delivers a great deal of information. We will show that once this information is in hand, the identification theorem can then be proved very efficiently on the basis of general principles, using two ingredients: a classification theorem for BN pairs of finite Morley rank and Tits rank 2 due to Kramer, Tent, and van Maldeghem [18] and a uniqueness result of Tits for parabolic amalgams for which Bennett and Shpectorov [9] have recently given a simple proof based on general principles.

1 Preliminaries

For general background on groups of finite Morley rank we refer to [13, 14]. A broader discussion of the problem to which the present paper is addressed is found in [1].

We will now present the main technical notions involved in the statement of the Identification Theorem.

Definition 1 Let G be a group of finite Morley rank.

1. A definable section of G is a quotient H/K with $= K \lhd H$ and K, H definable in G. The section is proper if $K > 1$ or $H < G$.

2. G is a *K-group* if every infinite simple definable section of G is a Chevalley group over an algebraically closed field.

3. G is a *K^*-group* if every infinite simple definable proper section of G is a Chevalley group over an algebraically closed field.

Definition 2 Let G be a group of finite Morley rank and S a Sylow$^\circ$ 2-subgroup (the connected component of a Sylow 2-subgroup).

1. G is degenerate if $S = 3D1$.

2. G is of even type if G is nondegenerate and S is of bounded exponent and definable.

3. G is of odd type if G is nondegenerate and S is divisible abelian.

A simple K^*-group of finite Morley rank is of one of these three types: degenerate, odd type, or even type [16, 2].

Definition 3 Let G be a group of finite Morley rank and of even type.

1. A 2-local° subgroup of G is a group of the form $N_G^\circ(Q)$ where Q is a connected definable 2-subgroup of G.
2. A Borel subgroup of G is a maximal connected solvable subgroup of G.
3. A standard Borel subgroup of G is a Borel subgroup which contains a Sylow° 2-subgroup.
4. $O^{2'}(G)$ is the minimal definable normal subgroup of G such that $G/O^{2'}(G)$ contains no involutions.
4. $O_2(G)$ is the largest definable normal 2-subgroup of G.

A few remarks are in order. First, with regard to standard Borel subgroups, if G is a K^*-group of even type then the standard Borel subgroups are those of the form $N_G(S)$ with S a Sylow° 2-subgroup of G. Secondly, it is not immediately clear that O_2 exists, but when G is of even type this is the case. In practice we will take G to be a connected K^*-group of even type and in this case $O_2(G)$ is itself connected [5].

With these definitions, the Identification Theorem has a precise meaning. The underlying idea is to work with an appropriate notion of parabolic subgroup, and for our purposes "parabolic" is best taken to mean: 2-local°, and containing a standard Borel. Earlier = papers have dealt with the existence of parabolics in this sense [5] = and with their structure [7].

2 The amalgam method

2.1 The issues

The basis for the proof of the Identification Theorem is the *amalgam method* as applied in [22] and in greater generality in [15]. We will indicate how this method is used in = our context, and what it produces. On the whole, this chapter of finite group theory goes over very smoothly to our context once the principles on which it relies are suitably translated. Accordingly we will not give the details of these arguments here; they may be found in [6]. On the whole we followed the line of [22] rather than the more general [15] as it is more = efficient in our particular case.

The amalgam method has already been used in the context of groups of finite Morley rank in [5]; indeed, the original proofs of "pushing up" results in finite group theory [8, 19] do not seem to go over to our context, but the version given by Stellmacher in = [23], based squarely on the amalgam method, goes over quite smoothly, as seen in the appendix to [5], where the argument is given in detail in the context of groups of finite Morley rank. We also refer to [5] for a verification that some of the key properties of SL_2 which are used in amalgam arguments hold in our context. We will indicate below what sorts of adaptations generally need to be made in our context.

In the context of the Identification Theorem, the amalgam method begins by introducing the graph Γ associated with the right cosets of P_1 and P_2, where two distinct cosets are linked by an edge if they meet. Thus this is a bipartite graph on which G acts naturally, and it should be thought of as a labeled graph in which every vertex and edge is labeled by its stabiliser under the action of G. The universal cover $\hat{\Gamma}$ of this graph in the topological sense is a tree which is associated to the free product $\hat{G}$ of P_1 and P_2 over their intersection, which is easily seen to be the standard Borel subgroup B. The objective is to show, after a lengthy analysis which can take place either in Γ or in $\hat{\Gamma}$, that $\hat{\Gamma}$ has a quotient Γ^* on which $\hat{G}$ acts (not faithfully) with the following properties:

(*a*) Γ^* is a generalized n-gon, and the image G^* of $\hat{G}$ in $\mathrm{Aut}(\Gamma^*)$ has a (B, N)-pair of rank 2.

(*b*) The triple (B, P_1, P_2) in G is isomorphic with the corresponding triple in G^* (one says that G is *parabolic isomorphic* with G^*).

The intent of course is to apply a classification theorem to G^* in order to determine the possible isomorphism types, at which point the isomorphism type of the triple (B, P_1, P_2) in G is known, and one can return to G and complete the identification of G without further use of the amalgam method.

An obvious and potentially serious drawback of this approach from the point of view of groups of finite Morley rank is that the group $\hat{G}$ will not be definable and will not be a group of finite Morley rank, and hence *a priori* the same problem arises in G^*. This is handled by showing that $\hat{G}$ is "locally" of finite Morley rank and that G^* is actually of finite Morley rank. We will deal with this more explicitly below.

The other issues that arise are merely technical, and are of two sorts. On the one hand certain chapters of finite group theory that are applied in this context have to be developed appropriately in our category, the most problematic one being the representation theory of the group SL_2, which is handled in a largely *ad hoc* way as the representations involved are taken over the field of 2 elements and are infinite dimensional; Morley rank has to replace dimension as the measure of size here, and the representation theory is inevitably in a rudimentary state, but sufficient for the limited needs of the amalgam method. The other point that bears watching is the role of connectedness in the analysis. This is absent in the finite case, but comes up naturally in the transposition to the context of groups of finite Morley rank, as can be seen quite clearly already in [5], where some care had to be taken around this point.

We will say no more about these technical points, but we will discuss the definability issue, and also state explicitly the result delivered by the amalgam method, which serves as the point of departure for the identification of the group G.

2.2 Definability

Definition 4 Let G be a group acting on a graph Γ.

1. For any vertex δ and any $k \geqslant 0$, $= \Delta_k(\delta)$ is the set of vertices lying at distance at most k from δ in Γ, and $G_k(\delta)$ is the set of elements of G which can be expressed as a product of at most k elements of G, where each element stabilises some vertex in $\Delta_k(\delta)$.

2. The pair (G, Γ) is locally of finite Morley rank if for each k and δ the pair $(G_k(\delta), \Delta_k(\delta))$ has finite Morley rank, where the latter is a 2-sorted structure consisting of a partial group, a graph, and a partial action of the partial group on the graph. (A partial group is simply the restriction of a group, viewed as a relational system, to a subset.)

Because the amalgam method always works locally in the graph Γ, whatever can be done with groups of finite Morley rank can also be done with groups which are locally of finite Morley rank, in this context. As far as the universal cover is concerned, we have the following.

Lemma 1 *Let $\mathcal{P} = 3D(P, Q, B)$ be a structure consisting of two groups P, Q. Let $G = 3DP *_B Q$ be the free product with amalgamation and let $= \Gamma$ be the associated tree of cosets, on which G acts naturally. Then the structure $\mathcal{G} = 3D(G, \Gamma)$ consisting of G acting on Γ is locally interpretable in $\mathcal{P}$ in the following sense: for any vertex $\delta \in V(\Gamma)$ and any $k \geqslant 0$ the graph $\Delta_k(\delta)$, the partial group $G_k(\delta)$, and the partial action of $G_k(\delta)$ on $\Delta_k(\delta)$ are all interpretable in $\mathcal{P}$.*

Proof. Let $X = 3DP \cup Q$ and let $R_k(x_1, \ldots, x_k)$ be the relation on X defined by: "$x_1 \cdots x_k = 3D1$ in G". Everything comes down to the definability of this relation in $\mathcal{P}$, which is proved by induction based on the following property of free products with amalgamation: if $x_1, \ldots, x_k$ are alternately from $P \setminus B$ and $Q \setminus B$ then the product is nontrivial. In the remaining cases, either the product can be shortened, and induction applies, or else $k = 3D1$. Bearing in mind that the natural maps of P and Q into G are embeddings, the claim follows. □

Corollary 2 *Under the stated hypotheses, if $\mathcal{P}$ has finite Morley rank then (G, Γ) is locally of finite Morley rank.*

We must also look at the passage from the universal cover to a generalized n-gon. This is handled at the outset in [15] by two results, (3.6) (p. 77) and (3.7) (p. 79), most of which involve no finiteness hypothesis:

Fact 1 [15] *Let Γ be a tree and K a Cartan subgroup of G with apartment $T = 3DT(K)$. Suppose that T fulfills the uniqueness and exchange conditions and that $s \geqslant 3$. Then there exists an equivalence relation $\approx$ on Γ which is compatible with the action of G so that:*

1. *$\tilde{\Gamma} = 3D(\Gamma / \approx)$ is a generalised $(s-1)$-gon;*

2. *$G_{\tilde{\Gamma}} \cap G_\delta = 3D1$ for each $\delta \in V(\Gamma)$; here $G_{\tilde{\Gamma}}$ is the kernel of the action of G on $\tilde{\Gamma}$.*

3. *$G/G_{\tilde\Gamma}$ has a (B,N)-pair of rank 2.*

The precise meaning of the hypotheses is not really relevant here; for the most part they represent the conditions which must be verified in the course of a detailed analysis. Also, in quoting this statement verbatim, we have omitted the context, which is more general than that of the Identification Theorem, apart from a finiteness hypothesis that plays no role here. However it may be remarked that in the case which actually concerns us, the Borel subgroup B splits as $S \rtimes K$ with S a Sylow° 2-subgroup and K a complement which may be called a "maximal torus", and the apartment T may be defined as the fixed-point set of K, which will be a 2-way infinite path on which the normaliser of K acts, with two orbits. We will also enter somewhat more into the details below, in discussing the Moufang property.

What needs to be added to this fact is the following:

Lemma 3 *In the context of Fact 1, if (G,Γ) is locally = of finite Morley rank then the "quotient" $(G/G_{\tilde\Gamma},\tilde\Gamma)$ has finite Morley rank.*

Proof. This requires an examination of the construction of $\tilde\Gamma$ as given in [15]. There are two points to be observed.

In the first place, the quotient $\tilde\Gamma$ is covered by $\Delta_{s-1}(\delta)$ for any vertex $\delta \in V(\Gamma)$. Secondly, with δ fixed, it needs to be seen that the equivalence relation $\approx$, which we factor out, is definable on $\Delta_{s-1}(\delta)$. In the proof of Fact 1 it is shown that equivalent pairs in $= \tilde\Gamma$ lie at distance at least $2(s-1)$, and the argument shows that on $\Delta_{s-1}(\delta)$ the equivalence relation is given by:

$$\alpha \approx \beta \text{ iff } d(\alpha,\beta) = 3D2(s-1) \text{ and } \gamma(\alpha,\beta) \text{ is conjugate under } G \text{ to a subpath of } T$$

where $\gamma(\alpha,\beta)$ is the path from α to β. Here we may replace T by two fixed subpaths of T of length $2(s-1)$, and the problem of definability reduces to the relation: "(α,β) is conjugate to (α_0,β_0)" where the four vertices $\alpha,\beta,\alpha_0,\beta_0$ lie in a set of the form $\Delta_k(\delta)$.

The action of a vertex stabiliser G_δ is transitive on the set of neighbors $\Delta(\delta)$, so if α and α_0 are in fact conjugate and at distance $2d$, and δ is the midpoint of the path joining α and α_0, then there is an element of $G_d(\delta)$ carrying α to α_0. Thus the following serves as a definition of conjugacy, for such pairs: "α,α_0 lie at distance $2d$ for some (*bounded*) d, and with δ the midpoint of the path joining them, there is $g \in G_d(\delta)$ such that $\alpha^g = 3D\alpha_0$ and β^g is conjugate to β_0 under G_{α_0}"; in the final clause we have a bound on $d(\beta^g,\beta_0)$, so this condition is also definable. □

This disposes of all definability issues: when the amalgam method succeeds, the group G^* (or $G/G_{\tilde\Gamma}$, in the current notation) has finite Morley rank, and for that matter is interpretable in the original group G, in the notation of the Identification Theorem.

2.3 Application of the amalgam method (the Moufang property)

We have already indicated the main thrust of the amalgam method in our context, namely:

Fact 2 [6] *Under the hypotheses of the Identification Theorem, there is a group G^* of finite Morley rank which is parabolic isomorphic to $= G$, and which has a rank* 2 (B, N)*-pair.*

This leaves something to be desired however. We would like to apply the classification of Moufang (B, N) pairs of Tits rank at least 2 and of finite Morley rank, given in [18].

Definition 5 Let Γ be a generalised n-gon, $G = 3D\mathrm{Aut}\Gamma$.

1. For $\gamma = 3D(\delta_0, \ldots, \delta_{n-1})$ a path of length $n-1$ in Γ, let $U(\gamma)$ be the intersection of $G_{\Delta_1(\delta)}$ for $\delta = 3D\delta_1, \ldots, \delta_{n-1}$.
2. Γ is Moufang if for every path $\gamma = 3D(\delta_0, \ldots, \delta_{n-1})$ in Γ, the group $=$ $U(\gamma)$ operates transitively on $\Delta(\delta_0) \setminus \{\delta_1\}$.

As it happens the Moufang property follows from general principles for the generalised $(s-1)$-gons delivered by Fact 1 in the context of the Identification theorem.

At this point one should actually invoke the definition of s:

Definition 6 In the context of the amalgam method (e.g., the Identification Theorem):

1. A path $\gamma = 3D(\delta_0, \ldots, \delta_n)$ in Γ is regular if G_γ (the pointwise stabiliser) operates transitively on $\Delta(\delta_0) \setminus \{\delta_1\}$ and on $\Delta(\delta_n) \setminus \{\delta_{n-1}\}$.
2. s is the minimum length of a non-regular path.

It follows easily from the definition of s, and induction, that any two paths of equal length l, with $l \leqslant s$, are conjugate under the action of G.

Now the following is contained in the analog of [15, (14.1)]:

Fact 3 *Let $\gamma = 3D(\delta_0, \ldots, \delta_{s-1})$ be a path of length $= s - 1 \geqslant 2$. Then $O_2(G_\gamma)$ acts transitively on $\Delta(\delta_0) \setminus \{\delta_0\}$ and $\Delta(\delta_{s-1}) \setminus \{\delta_{s-1}\}$.*

Here the notation $O_2(G_\gamma)$ simply represents a Sylow 2-subgroup of G_γ since G_γ is contained in a Borel subgroup, of the form $S \rtimes K$ with S a Sylow 2-subgroup and K a torus; furthermore some conjugate of K is a complement to $O_2(G_\gamma)$ (any path of length at most $s-1$ is conjugate to a path contained in $T = 3DT_K$).

Now to verify the Moufang property for the generalised $(s-1)$-gon furnished by Fact 1 in the context of the Identification Theorem, let $\gamma = 3D(\delta_0, \ldots, \delta_{s-2})$

be a path of length $s-2$ in $\hat{\Gamma}$, and extend it to a path $\tilde{\gamma} = 3D(\delta_0, \ldots, \delta_{s-1})$ of length $s-1$. Let $Q = 3DO_2(G_\gamma)$. It suffices to show that Q fixes the neighbors of each δ_i for $1 \leqslant i \leqslant s-2$. Or, more simply:

Lemma 4 *If $\delta \in V(\Gamma)$ and α, β are distinct neighbors of δ, then $O_2(G_{\alpha\beta})$ fixes $\Delta(\delta)$.*

Now $G_{\alpha\beta} = 3DG_{\alpha\delta} \cap G_{\beta\delta}$ is the intersection of two Borel subgroups of G_δ, and $O_2(G_{\alpha\beta})/O_2(G_\delta)$ is the intersection of two distinct Sylow subgroups of SL_2, hence trivial, that is: $O_2(G_{\alpha\beta}) \leqslant O_2(G_\delta)$, and as G_δ acts transitively on $\Delta(\delta)$, it follows that $O_2(G_\delta)$ fixes all neighbors of δ. Thus the lemma is immediate.

3 Identification

We apply a very general result from [18, Theorem 3.14]:

Fact 4 *Let G^* be an infinite simple group of finite Morley rank with a = spherical Moufang BN-pair of Tits rank 2. Then $G^* \simeq \mathrm{PSL}_3(F)$, $\mathrm{PSp}_4(F)$, or $G_2(F)$ for some field F.*

This field must of course be algebraically closed as it will also have finite Morley rank.

Thus we now have as a corollary of the amalgam analysis sketched in the previous section:

Lemma 5 *Under the hypotheses of the identification theorem, the triple (B, P_1, P_2) is isomorphic to a triple consisting of a Borel subgroup and the two minimal parabolic subgroups containing it, in one of the groups $G^* \simeq \mathrm{PSL}_3(F)$, $\mathrm{PSp}_4(F)$, or = $\mathrm{G}_2(F)$ for some algebraically closed field F (of characteristic 2, as we work with even = type).*

Evidently we now want to identify G itself with the appropriate one of these three groups. We use a theorem of Tits found in [20, Chapter II, Theorem 8]; for an alternative proof, based on Tits' Lemma [24], see Bennett and Shpectorov [9].

Fact 5 *Let G^* be a Chevalley group of Lie rank 2 and let P_1, P_2 = be minimal parabolics containing a common Borel subgroup B. Let N be the normaliser of a Cartan subgroup of B. Then G^* is the universal closure of the amalgam of P_1, P_2, = and N.*

(The idea of the proof given by Bennett and Shpectorov is to adjoin to the natural point/line geometry associated with G^* a third kind of object, the set of apartments, where an apartment is incident with its elements. This has the effect of making the geometry simply connected, and a very general result of Tits [24] on groups acting flag-transitively on simply connected geometries then applies.)

To complete the proof of the Identification Theorem we may therefore proceed as follows. Let G^* be the target group $\mathrm{PSL}_3(F)$, $\mathrm{PSp}_4(F)$, or $\mathrm{G}_2(F)$. Working in the original group G, fix a maximal torus K in B and let $N = 3DN_G(K)$.

Lemma 6 *If $C_G(K) = 3DK$ then $G \simeq G^*$.*

Proof. Let K be a maximal torus in B. $P_i = 3DO_2(P_i) \rtimes (L_i \times K_i)$ with $L_i \simeq \mathrm{SL}_2(F)$ and $K = 3D(K \cap L_i)K_i$. Let $w_i \in L_i$ be an involution inverting $K \cap L_i$ and let $W = 3D\langle w_1, w_2\rangle$, $= a = 3Dw_1w_2$. Evidently the structure of P_1 and P_2 determine the map $W \to \mathrm{Aut}(K)$, so as G and G^* are parabolic isomorphic, W acts on K like D_{s-1}. In particular $a^{s-1} \in C_G(K) = 3DK$, and a is inverted by both w_1 and w_2. It follows that $a^{s-1} = 3D1$.

Thus $KW \simeq N_{G^*}(K)$. By Fact 5 the subgroup of G generated by P_1, P_2, KW is = isomorphic with G^* and as P_1, P_2 already generate G, we have $G \simeq G^*$. □

In the proof of the next lemma we make use of information on centralisers of semisimple elements in semisimple algebraic groups found in [21].

Fact 6 [21, Corollary 4.6] *Let G^* be a semisimple algebraic group and x a semisimple element of G^* of prime order p. Let $\pi : \tilde{G} \to G^*$ be the canonical map from the simply connected cover. If p does not divide $|\ker \pi|$ then $C_{G^*}(x)$ is connected.*

Fact 7 [21, 3.19] *Let G^* be a semisimple algebraic group and and y any semisimple element. Then $C_{G^*}(y)$ is reductive.*

Combining these two:

Corollary 7 *With the hypotheses and notation of Fact 6, $C_{G^*}(x)$ is connected and reductive. In particular if G^* is one of the groups SL_3, Sp_4, or G_2 over an algebraically closed field of characteristic 2 and x is a semisimple element of prime order $p > 3$, then $C_{G^*}(x)$ is a torus or the = product of a torus with SL_2.*

Proof. $C_{G^*}(x)$ is reductive of Lie rank 2, and contains a central element = of order greater than 3. The claim follows. □

Lemma 8 *$C_G(K) = 3DK$.*

Proof. We will make free use of the parabolic isomorphism of G and G^*.

With K_i, L_i as in the preceding proof, $C_{P_i}(K_i) = 3DL_i \times K_i$ with $L_i \simeq \mathrm{SL}_2(F)$, with F the base field of G^*. More exactly, $L_i \simeq \mathrm{SL}_2(F_i)$ with F_1 and F_2 definably isomorphic, but this amounts to the same thing.

Take $a \in K$ of order greater than 3. As observed above $C_{G^*}(a)$ is reductive and is either a torus, or the product of a torus with $\mathrm{SL}_2(F)$.

In particular the rank of $C_S(a)$ is at most $f = 3D\mathrm{rk}(F)$ for any such element a. Accordingly the same applies to $C_Q(a)$ for any Sylow° 2-subgroup Q of G, and any a normalising Q of order greater than 3. Let U be a Sylow° 2-subgroup of $C_G(K_i)$ ($i = 3D1$ or 2). It follows that $\mathrm{rk}(U) \leqslant f$. As $\mathrm{rk}(S\cap = L_i) = 3Df$, we conclude that $S \cap L_i$ is a Sylow° 2-subgroup of $C_G^\circ(K_i)$.

Let $U_i = 3DS \cap L_i$. Then we have

$$(*) \qquad U_i \leqslant L_i \leqslant C_G^\circ(K_i)$$

and $C_G^\circ(K_i)$ is a connected K-group, with U_i as a Sylow 2-subgroup.

By $(*)$ we have $O_2(C_G^\circ(K_i)) = 3D1$, and by an elementary result on K-groups [4, 2.33] it follows that $C_G^\circ(K_i) = 3DE(C_G^\circ(K_i)) * O(C_G^\circ(K_i))$. Here $E = 3DE(C_G^\circ(K_i))$ is a central product of quasisimple algebraic = groups, U_i is a Sylow 2-subgroup of E, and $U_i \leqslant L_i \leqslant E$. It is then easy to see that $L_i = 3DE$. As a result, L_i is normalised by $C_G(K_i)$ for $i = 3D1, 2$ and hence:

Both L_1 and L_2 are normalised by $C_G(K)$.

The groups $L_i \simeq \mathrm{SL}_2(F)$, $i = 3D1, 2$, do not allow definable groups of outer automorphisms [14, Theorem 8.4]. Hence $C_G(K)$ must act on L_i via inner automorphisms commuting with $K \cap L_i$ and hence $C_G(K) = 3D(K \cap L_i) \times C_G(KL_i)$. Let $H_i = 3DC_G(KL_i)$. Since $(K \cap L_1)(K \cap L_2) \leqslant K$, it follows that $C_G(K) = 3DK(H_1 \cap H_2)$.

Now $H = 3DH_1 \cap H_2$ centralises $\langle U_1, U_2\rangle = 3DS$ and H centralises each L_i, hence also each P_i, hence G. As G is simple, $H = 3D1$ and $C_G(K) = 3DK$. □

This completes the identification of G.

References

[1] T. Altınel, *Simple groups of finite Morley rank of even type*, in "Model Theory of Groups and Tits Buildings" (K. Tent, ed.), London Math. Soc. Lect. Notes Ser. vol. 291, 2002.

[2] T. Altınel, A. Borovik and G. Cherlin, *Groups of mixed type*, J. Algebra 192 (1997), 524–571.

[3] T. Altınel, A. Borovik and G. Cherlin, *On groups of finite Morley rank with weakly embedded subgroups*, J. Algebra 211 (1999), 409–456.

[4] T. Altınel, A. Borovik and G. Cherlin, *Groups of finite Morley rank and even type with strongly closed abelian subgroups*, J. Algebra 232 (2000), 420–461.

[5] T. Altınel, A. Borovik and G. Cherlin, *Pushing up and $C(G,T)$ in groups of finite Morley rank of even type*, J. Algebra 247 (2002), 541–576.

[6] T. Altınel, A. Borovik and G. Cherlin, *An analog of a theorem of Stellmacher for groups of finite Morley rank*, Dimacs Technical Report, in preparation.

[7] T. Altınel, A. Borovik, G. Cherlin and L.-J. Corredor, *Parabolic 2-local subgroups in groups of finite Morley rank of even type*, submitted.

[8] B. Baumann, *Über endliche Gruppen mit einer zu $L_2(2^n)$ isomorphen Factorgruppe*, Proc. Amer. Math. Soc. 74 (1979), 215–222.

[9] C. D. Bennett and S. Shpectorov, *A remark on a theorem of J. Tits*, Proc. Amer. Math. Soc. 129 (2001), 2571–2579.

[10] C. D. Bennett, R. Gramlich, C. Hoffman and S. Shpectorov, *Curtis-Phan-Tits theory*, these proceedings.

[11] A. Berkman and A. Borovik, *An identification theorem for groups of finite Morley rank and even type*, to appear in J. Algebra.

[12] A. Berkman and A. Borovik, *A generic identification theorem for groups of finite Morley*, submitted.

[13] A. Borovik, *Tame groups of odd and even type*, in "Algebraic groups and their representations, Cambridge 1997" (R. Carter, ed.), NATO Adv. Sci. Inst. Ser. C, vol. 517, Kluwer, Dordrecht, 1998, = pp. 341–366.

[14] A. Borovik and A. Nesin, "Groups of Finite Morley Rank", Oxford University Press, 1994.

[15] A. Delgado and B. Stellmacher, *Weak (B, N)-pairs of rank 2*, in "Groups and Graphs: New Results and Methods", by A. Delgado, D. Goldschmidt, and B. Stellmacher, Birkhaüser, DMV Seminar 6, Basel, 1985.

[16] E. Jaligot, *Groupes de type mixte*, J. Algebra 212 (1999), 753–768.

[17] E. Jaligot. *Groupes de type pair avec un sous-groupe faiblement inclus*, J. Algebra 240 (2001), 413–444.

[18] L. Kramer, H. van Maldeghem and K. Tent, *Simple groups of finite Morley rank and Tits buildings*, Israel J. Math. 109 (1999), 189–224.

[19] R. Niles, *Pushing-up in finite groups*, J. Algebra 57 (1979), 26–63.

[20] J.-P. Serre, "Trees", Springer-Verlag, New York, 1980.

[21] T. A. Springer and R. Steinberg, *Conjugacy classes*, in "Seminar on Algebraic Groups and related finite groups", (A. Borel et al., eds.), Lecture Notes Math. 131, Springer Verlag, New York, 1970, pp. 167–266.

[22] B. Stellmacher, *On graphs with edge-transitive automorphism groups*, Illinois J. Math. 28 (1984), 211–266.

[23] B. Stellmacher, *Pushing Up*, Arch. Math. 46 (1986), 8–17.

[24] J. Tits, *Ensembles ordonnés, immeubles et sommes amalgameés*, Bull. Soc. Math. Belg. A38 (1986), 367–387.

Tuna Altınel, Institut Girard Desargues, Université Claude Bernard Lyon-1, Bâtiment Braconnier, 21 Avenue Claude Bernard, 69622 Villeurbanne CEDEX, France; e-mail: = `altinel@desargues.univ-lyon1.fr`

Alexandre V. Borovik, Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, United Kingdom; e-mail: `borovik@umist.ac.uk`

Gregory Cherlin, Department of Mathematics, Rutgers University, Hill Center, Piscataway, New Jersey 08854, USA; e-mail: = `cherlin@math.rutgers.edu`