THERE IS NO UNIVERSAL COUNTABLE PENTAGON-FREE GRAPH

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Abstract. For $n \geq 4$, no countable C_n-free graph contains every countable C_n-free graph as a subgraph, for $n \geq 4$. This was proved for $n = 4$ by J. Pach.

1. introduction

It was R. Rado [Rad] who first remarked that among the countable graphs there exists a universal one: a countable graph which isomorphically embeds (as *induced* subgraphs) all countable graphs. On the other hand N. G. de Bruijn noticed that if we consider only those countable graphs in which all vertex degrees are finite, then there is no universal one, even in the weaker sense where we allow embeddings which are not necessarily embeddings as induced subgraphs (cf. [Rad]).

There has been some interest in the following more general problem: given a finite or countably infinite graph G, determine whether there is a universal, G-free, countable graph. It is easy to see that there is such a graph in the case $G = K_n$ is a complete graph, for any finite n. J. Pach showed that there is no universal G-free graph if $G = C_4$, the circuit of length 4 [HP], and in [KP84] the case in which G is a *complete bipartite* graph was settled. It is proved in [KMP] that there is a universal countable graph if all odd circuits up to a certain length are to be excluded.

One of the simplest cases of the problem which has not been resolved previously is the question of the existence of a universal countable pentagon-free graph. We will show here that if G is any circuit of length at least 4, then there is no universal countable G-free graph.

It would not be unreasonable to ask for a characterization of all the finite graphs G for which universal G-free graphs exist, or at least an algorithm which settles every instance of the problem. A case which

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seems even more basic than the case we treat here would be the characterization of the finite *trees* G for which a universal countable G-free graph exists.

We are grateful to an alert referee who saw that an earlier and slightly simpler version of the argument fails for cycle lengths greater than 5.

Definition 1.1.

1. A *graph* is a pair $G = (V; E)$ where the set V of vertices is arbitrary, and the set E of edges consists of unordered pairs from V; thus loops and multiple edges are excluded.

2. An *embedding* $f : G_1 \longrightarrow G_2$ between two graphs $G_1 = (V_1; E_1)$ and $G_2 = (V_2; E_2)$ is any injection $V_1 \to V_2$ such that $\{x, y\} \in E_1$ implies $\{f(x), f(y)\} \in E_2$; if the converse also holds for all $x, y \in V_1$, we call f a *strong* embedding.

3. For G, H graphs, we say that H is *G-free* if G does not embed into H (as a subgraph).

4. The *length* of a path or a circuit is the number of edges occurring in it.

Our proof will depend on the following lemma, a variation on a well-known result of G. Dirac on Hamiltonian circuits [Dir]. The proof of the lemma will be given at the end.

Lemma 1.2. *If G is a graph with all vertex degrees at most 12, and X is a set of $4N + 1$ vertices, then there is a circuit H of length $4N + 1$ on the set X such that in $G \cup H$ there is no circuit of length at most n which contains an edge of H, and G and H have no edge in common.*

Theorem 1. *If $n \geq 4$ there is no universal, countable, C_n-free graph.*

The case $n = 4$ was already settled by J. Pach [HP]. Our argument uses a similar method.

Proof. Fix $n \geq 4$. If $G = (V; E)$ is a C_n-free graph, and $S, X \subseteq V$, we will call the pair (S, X) *C_n-rigid* if the following holds.

> *For any C_n-free graph $H = (W; F)$, if $f, g : G \longrightarrow H$ are two embeddings which agree on S, then they agree on X.*

Below we will construct a C_n-free graph G of bounded vertex degree and a C_n-rigid pair (S, X) with S finite and X infinite. We first argue that this is sufficient.

We can select an infinite sequence of vertices $x_0, x_1, \ldots \in X$ such that $d(x_i, x_j) \geq n$ for $i \neq j$, since the vertex degrees in G are bounded. For every string $\epsilon = \epsilon_0 \epsilon_1 \ldots$ of zeros and ones, we define a graph $G(\epsilon)$ extending G as follows: the vertices x_{2i}, x_{2i+1} will be joined by an edge if $\epsilon_i = 0$, and by a path of length $n - 1$ if $\epsilon_i = 1$ (in particular each such path will contain $n-2$ new vertices, of degree 2). Clearly $G(\epsilon)$ is C_n-free. We will show that no countable C_n-free graph H embeds all of the graphs $G(\epsilon)$.

If H is a countable graph embedding all of the graphs $G(\epsilon)$, then as S is finite, there are two different strings ϵ, ϵ' such that the embeddings of $G(\epsilon), G(\epsilon')$ agree on S. If H is C_n-free as well, then the two embeddings agree on X. In particular we can denote the image of x_i under either of these embeddings by x_i again. If $\epsilon_i \neq \epsilon'_i$, then x_{2i}, x_{2i+1} are joined by both a path of length $n-1$ and an edge in H; so $C_n \leq H$, as claimed.

It remains to construct the desired initial graph G. Let T_n be a graph with three distinguished vertices a, b, c, with c adjacent to a and b, and with c joined to a by two paths of length $\lfloor n/2 \rfloor - 1$, and to b by two paths of length $\lfloor (n-1)/2 \rfloor$. When n is 4 or 5, modify this rule by omitting all "paths" of length 1. The additional paths are of course taken disjoint except at their extremities. The graph T_n is clearly C_n-free.

We claim:

$$(*) \qquad (\{a,b\},\{c\}) \text{ is a } C_n\text{-rigid pair in } T_n.$$

In other words, if H is a C_n-free graph containing vertices a, b, c', c'' such that a, b, c' and a, b, c'' are the images of a, b, c under two embeddings f, g of T_n into H, we claim that $c' = c''$. If $c' \neq c''$, let U_1 and U_2 be paths of length $[n/2]-1$ and $[(n-1)/2]$ joining c' to a and b respectively in H, coming from the embedding f of T_n into H, and chosen so that c'' does not lie on U_1 or U_2. Then $U_1 \cup U_2$ is a path of length $n-2$ from a to b, and together with c'' this creates a circuit of length n in H, a contradiction. This proves (*).

Now our construction proceeds as follows. Let $N = (14^n - 1)/13$. We will construct a strongly increasing sequence of C_n-free graphs G_s (that is, G_s is an induced subgraph of G_t for $s \leq t$), together with a distinguished set X_s of $4N+1$ vertices in G_s, so that:

$$(1) \qquad \text{The vertex degrees in } G_s \text{ are all at most 12.}$$

$$(2) \qquad \text{The vertex degrees in } G_s \text{ for vertices in } X_s \text{ are at most 6.}$$

$$(3) \qquad \text{The pair } (X_{s-1}, X_s) \text{ is } C_n\text{-rigid in } G_s \text{ for } s \geq 1.$$

$$(4) \qquad \text{There is a circuit } H_s \text{ on the set of vertices } X_s \text{ such that:}$$

$$G_s \cup H_s \text{ contains no circuit of length at most } n\text{, and}$$

$$G_s, H_s \text{ have no edges in common}$$

Assuming this construction has been carried out, we then set

$$G = \bigcup_s G_s, \qquad S = X_0, \qquad X = \bigcup_s X_s.$$

So it now suffices to construct suitable G_s and X_s for all s. We will rely here on Lemma 1.2.

Initially, X_0 is a set of $4N+1$ vertices, and G_0 has X_0 as its vertex set, with no edges.

For the inductive step, suppose that G_s and X_s have been constructed satisfying conditions (1-4). Enumerate the vertices of X_s in cyclic order as $(x_i^s : i \in \mathbb{Z}/(4N+1)\mathbb{Z})$. For each $i \in \mathbb{Z}/(4N+1)\mathbb{Z}$, adjoin (freely) a copy $T_n^{i,s}$ of T_n to G_s, identifying the vertices a, b with x_i^s, x_{i+1}^s respectively. Let G_{s+1} be the resulting graph, and let $c^{i,s}$ denote the vertex of G_{s+1} corresponding to c in $T_n^{i,s}$. We will take X_{s+1} to be

$$\{c^{i,s} \mid i \in \mathbb{Z}/(4N+1)\mathbb{Z}\}.$$

It is clear that G_{s+1}, X_{s+1} then have properties (1-3). Furthermore our lemma shows that condition (4) can be met on the basis of condition (1). The only other point that needs to be checked is that G_{s+1} is again C_n-free.

Suppose that C is a circuit of length n in G_{s+1}. As G_s is C_n-free, C does not lie wholly in G_s. Therefore if we replace all the edges of C which occur in some $T_n^{i,s}$ by the corresponding edge in the circuit H_s, the result is a contraction C' of C lying in $G_s \cup H_s$, and containing an edge of H_s. C' is a circuit (or perhaps a doubled edge). By our inductive hypothesis, $G_s \cup H_s$ contains no circuit of length at most n, and no repeated edge. Thus G_{s+1} is C_n-free. This completes the construction. $\square$

We turn to the proof of Lemma 1.2, which is modeled on the proof of a theorem of Dirac on hamiltonian circuits [Dir]. To see the connection, associate to the pair G, X of the lemma an auxiliary graph $\hat{X}$ on the vertex set X, in which two vertices are linked by an edge just in case their distance in G is at least n. Then the circuit provided by the lemma is, in particular, a hamiltonian circuit in this graph.

Proof of Lemma 1.2. We observe at the outset that in any graph whose vertex degrees are bounded by 14, there can be at most N vertices lying at distance at most $n-1$ from any fixed vertex.

For the purposes of the present argument, any path or circuit lying in the complete graph on X will be called a path or circuit *in* X. By an abuse of notation, we will identify a circuit with a cyclic ordering of its vertex set. A circuit will be called "short" if has length at most n.

A path or circuit H in X will be *admissible* if $G \cup H$ contains no short circuit with an edge in H, and G and H have no edges in common.

We must build an admissible circuit with vertex set X. We begin by building an admissible path $H_1 = (x_1, \ldots, x_{3N+1})$ of length $3N$. For this we simply select the vertices x_i successively: if $H = (x_1, \ldots, x_i)$ with $i < 3N + 1$ is admissible, we can choose $x_{i+1} \in X - V(H)$ lying at distance at least n from x_i in $G \cup H$.

Similarly, after H_1 is constructed we can find an index i in the range $[2N + 1, 3N + 1]$ such that the distance from x_1 to x_i in $G \cup H_1$ is at least n. Then the circuit $H_2 = (x_1, \ldots, x_i, x_1)$ is an admissible circuit of length at least $2N + 1$.

Now starting from an admissible circuit H in X of length at least $2N + 1$, and any point x of $X - V(H)$, we will argue that there is an admissible circuit with vertex set $V(H) \cup \{x\}$; this is then sufficient to complete the argument. We use Dirac's idea here. More than half of the vertices of H lie at distance at least n from x in $G \cup H$, so there are two adjacent vertices y, z in H with this property. Form a new circuit H' by inserting x between y and z. One may then check that H' is again admissible: for example, if a circuit in $G \cup H'$ contains *both* of the new edges (y, x) and (x, z), then replacing these two edges by (x, z) we get a shorter circuit lying in $G \cup H$. $\square$

References

[Dir] G. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3) **2** (1952), 69-81.

[HP] A. Hajnal, J. Pach, *Monochromatic paths in infinite graphs*, in: Finite and Infinite Sets, Coll. Math. Soc. J. Bolyai, (Eger, Hungary,1981), 359-369.

[KMP] P. Komjáth, A. Mekler, J. Pach, *Some universal graphs*, Israel Journal of Math. **64** (1988), 158-168.

[KP84] P. Komjáth, J. Pach, *Universal graphs without large bipartite subgraphs*, Mathematika **31** (1984), 282-290.

[KP] P. Komjáth and J. Pach, *Universal elements and the complexity of certain classes of infinite graphs*, to appear.

[Rad] R. Rado, *Universal graphs and universal functions*, Acta Arithmetica **9** (1964), 331-340.

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