Infinite homogenous directed graphs

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ABSTRACT. The classification of the (countable) homogeneous directed graphs is complete. It has been known for almost two decades that there are $2^{\aleph_0}$ homogeneous directed graphs. We use methods devised by Lachlan and Woodrow, and applied by them to the case of homogeneous undirected graphs, and by Lachlan to the case of tournaments. Ours is the first case in which an uncountable collection of structures is handled by these methods. The methods used will first be sketched in the context of tournaments, then in our case. They combine a rather artificial inductive procedure with a timely use of Ramsey's theorem introduced by Lachlan in his work on tournaments. We also describe some examples, including a rather strange one uncovered by the classification procedure, and some related problems, for the most part open. We devote considerable attention to a decision problem connected with the classification which remains open. This problem has been studied by Brenda Latka, and is equivalent to a recognition problem for well quasi-ordered families of tournaments defined by finitely many constraints.

1. Introduction

1.1. HOMOGENEITY

1.1.1. *Definition.* A structure Γ is said to be *homogeneous* if any isomorphism between finitely generated substructures of Γ is induced by an automorphism. (Since we will be interested in combinatorial structures like directed graphs, and there are no functions present, we may say "finite" rather than "finitely generated" in our context.)

The condition of homogeneity is obviously very strong – it is more or less the strongest transitivity hypothesis one could place on the automorphism group of a countable structure. Accordingly the question of providing a complete classification of all the homogeneous structures of various kinds arises.

1.1.2. *The finite case.* As far as finite homogeneous structures are concerned, there is a completely general classification theory due to Lachlan which provides a classification by finitely many natural invariants in all cases in which the structures under consideration are equipped with finitely many relations and no functions. An exposition of this theory with substantial simplifications is found in [KL]. Lachlan's theory relies on the classification of the finite simple groups in what appears to be an essential way, though in the most common cases, where all relations are binary, this can be avoided [SL]. It should be said that if one is interested in generating completely explicit solutions to classification problems for finite homogeneous structures this theory does not provide a very practical method in the present state of knowledge, and more limited but efficient approaches have been used in special cases [Sh, Ga, GK, L?, L?].

In Lachlan's classification certain infinite examples are also captured as limits of finite structures, and Lachlan has shown that his classification covers exactly the homogeneous structures which are stable in Shelah's sense. In particular there are only countably many homogeneous stable structures for finite relational languages. (Here we are counting the isomorphism types of *countable* structures.)

1.2. CLASSIFICATION

1.2.1. *The classification problem.* When we consider the problem of classifying homogeneous structures in general, we run immediately into the following problem: there are $2^{\aleph_0}$ homogeneous directed graphs [He, Pe]. What then can we mean by a classification of an uncountable family of structures? As far as the particular case of directed graphs is concerned, I think the description of homogeneous directed graphs given below is a convincing classification. In the general case I find Lachlan's proposed criterion convincing. This makes use of Fraissé's theory of amalgamation classes. This theory will be reviewed in more detail below, but for our present purpose it suffices to say that each (countable) homogeneous structure is determined up to isomorphism by the isomorphism types of its finite substructures. Accordingly if $\mathcal{K}$ is a class of homogeneous structures then it is natural to study the entailment relation $\mathcal{A} \Longrightarrow_{\mathcal{K}} \mathcal{B}$ between two finite sets $\mathcal{A}, \mathcal{B}$ of finite structures defined by the condition:

Any homogeneous structure embedding all structures in $\mathcal{A}$ also embeds some structure in $\mathcal{B}$.
[Embedding signifies: embedding as an *induced* substructure.]

(The most striking case is that in which $\mathcal{B}$ contains only one structure.) Typically $\mathcal{K}$ will be taken to consist of all the countable homogeneous structures in some natural and fixed category, and we will drop the subscript $_{\mathcal{K}}$ from the notation.

We will call a collection $\mathcal{K}$ of homogeneous structures *classifiable* if the relation $\Longrightarrow_{\mathcal{K}}$ is decidable. There is no direct relation between the countability of $\mathcal{K}$ and its classifiability in this sense. In particular the class of homogeneous directed graphs is classifiable in this sense, but uncountable.

1.2.2. *Classification technique.* There is no general theory of homogeneous relational structures beyond Lachlan's theory for the stable case, but Lachlan and Woodrow have developed techniques which apply in some quite special cases. These techniques depend heavily on Fraissé's theory of amalgamation classes. They are best understood in the context of homogeneous tournaments, which I reworked in [Ch], incorporating some technical improvements I found while working on the case of directed graphs. The classification of homogeneous tournaments is now quite short, but in spite of technical improvements the classification of the homogeneous directed graphs is quite long and will be presented in the last four chapters of an AMS Memoir which should be in final form shortly.

1.3. CONTENTS

What I propose to do here is to review the Fraissé theory, the ideas developed by Lachlan and Woodrow, and the application to the classification of homogeneous tournaments, and to sketch briefly the adaptation of these methods to the case of directed graphs. Finally, there are a number of problems about homogeneous directed graphs which are not settled by the classification, but which can be reformulated in purely combinatorial terms as a result of the classification, and hence are now ripe for attack. One such problem is the determination of all the reducts of each homogeneous directed graph up to interdefinability, which presumably must wait (though perhaps not very long) on the further development of the techniques described by Simon Thomas in his article.

1.3.1. *A decision problem.* Another quite curious problem is the determination of the "finitely constrained" classes of directed graphs which contain only countably many homogeneous members. Typical examples of such classes would be the class of *tournaments* and the class of *partial orders*, both of which were given explicit classifications prior to the treatment of the general case – in fact, these two classifications are invoked at appropriate points in the treatment of the general case. This problem will lead us directly into the study of well quasi-ordered families of tournaments and the general theory of well quasi-orderings. At the end I will summarize the results of my student B. Latka on this problem.

1.3.2. *Pseudofinite structures.* Hrushovski's article describes the extension of Lachlan's theory in a quite distinct direction, remaining in the context of finite structures and weakening the homogeneity condition. The resulting theory is much closer in spirit to Lachlan's original theory (and considerably less anecdotal in spirit).

2. Amalgamation classes

2.1. FRAISSÉ'S THEORY

2.1.1. *Fraissé's correspondence.* It is fairly clear by a back-and-forth argument that a countable homogeneous structure is determined up to isomorphism by the isomorphism types of its finite substructures. Thus there is a bijective correspondence between homogeneous structures Γ and certain classes $\mathcal{A}$ of finite structures. Fraissé observed that the corresponding classes $\mathcal{A}$ can be characterized intrinsically by the obvious properties of closure under isomorphism and downward, together with joint embedding (any two members of $\mathcal{A}$ embed jointly in a third), and one further property – the amalgamation property – which captures the essence of the homogeneity condition.

2.1.2. *Definition.* The class $\mathcal{A}$ has the *amalgamation property* if for any structures $A_0, A_1, A_2 \in \mathcal{A}$ and any embeddings $f_i : A_0 \longrightarrow A_i$ $(i = 1, 2)$ there is a structure $A \in \mathcal{A}$ and there are embeddings $g_i : A_i \longrightarrow A$ with $g_2 f_2 = g_1 f_1$.

For example, a dense linear ordering without endpoints is the homogeneous tournament associated with the class of finite transitive tournaments.

2.1.3. *Application: Many models.* Fraïssé's correspondence can be used to good effect both to generate new examples of homogeneous structures, and to prove classification theorems. For example, following Henson, we will now use it to produce $2^{\aleph_0}$ countable homogeneous directed graphs. (Our directed graphs have at most one oriented edge between any pair of vertices, and no loops.)

2.1.4. *Construction.* Let $\mathcal{T}$ be a class of tournaments, and let $\mathcal{A}(\mathcal{T})$ be the class of directed graphs H containing no tournaments other than those which embed in some element of $\mathcal{T}$. Then $\mathcal{A}(\mathcal{T})$ is an amalgamation class, and the corresponding countable homogeneous structure will be denoted $\Gamma(\mathcal{T})$. Thus a finite tournament embeds in $\Gamma(\mathcal{T})$ if and only if it embeds in some element of $\mathcal{T}$.

Now let T_n be the tournament obtained from a linear ordering (transitive tournament) of length n by reversing the edges between successive vertices, as well as the edge from the first to last element. The collection $(T_n : n \geq 6)$ forms an *antichain*, or in other words: there is no embedding of T_m in T_n for $m \neq n$. To see this, examine the edges in T_n which belong to two distinct oriented 3-cycles in T_n; call these edges 2-edges. If we consider the 2-edges momentarily without their orientation, we find that T_n contains a unique circuit of 2-edges, of length $n - 2$. Hence these tournaments form an antichain. Accordingly, if $\mathcal{T} = (T_n : n \in X)$ with X a set of integers greater than 5, then:

$$X = \{n : T_n \text{ embeds in } \Gamma(\mathcal{T})\}.$$

Thus we have produced $2^{\aleph_0}$ homogeneous directed graphs $\Gamma(\mathcal{T})$.

Looking ahead, we should point out that according to our classification of the homogeneous directed graphs, if one omits the graphs of the form $\Gamma(\mathcal{T})$ described above – which may be manufactured with total freedom – then it turns out that only countably many other countable homogeneous directed graphs exist.

2.1.5. *Application: Classification.* An early use of Fraïssé's correspondence to limit the possibilities, rather than to produce new examples, is found in Woodrow's [Wo]. Here triangle-free homogeneous graphs are classified, and the triangle-freeness is used heavily to control the outcome of various amalgamation problems. I found it useful to proceed in an analogous fashion in the case of homogeneous directed graphs omitting I_n for some n, where I_n is an independent set of n vertices, that is an edgeless directed graph on n vertices. The idea here is to study the amalgamation class "generated" by a finite set of graphs. More exactly, one studies the entailment relation $\Longrightarrow$ introduced above. (This can easily be redefined without mentioning homogeneity in terms of the consequences of a series of attempted amalgamations). In general the set of "consequences" of a finite set $\mathcal{A}$ of finite structures is not itself an amalgamation class, but on the other hand many interesting amalgamation classes can be usefully characterized in this manner as the set of consequences of a suitable finite set $\mathcal{A}$.

2.2. THE METHODS OF LACHLAN AND WOODROW

In [LW] a new idea enters the picture which makes more refined use of amalgamation classes. Consider the generic undirected graph Γ_n omitting the complete graph K_n, by which we mean the homogeneous graph associated to the amalgamation class $\mathcal{A}(K_{n-1})$ of all graphs which do not embed K_n. This is the "typical" homogeneous graph, and the main step in the classification of all homogeneous graphs is the characterization of the class $\mathcal{A}(K_{n-1})$ as the smallest amalgamation class containg K_{n-1} and two other small graphs which we will call A and B here. I am now going to deform the idea of [LW] rather badly, in a way that seems to me to preserve the essential idea.

If $\mathcal{A}$ is an amalgamation class containg K_{n-1}, A, and B, let $\mathcal{A}^*$ be the class of graphs $H \in \mathcal{A}$ such that for any extension H^+ of H by a single additional vertex, if H^+ does not contain K_n then H^+ is again in $\mathcal{A}$. Consider the following statement:

$$\text{If } \mathcal{A} \text{ is an amalgamation class containing } K_{n-1},\ A, \text{ and } B, \text{ then so is } \mathcal{A}^*. \tag{$*$}$$

Now it is fairly easy to see that $(*)$ is equivalent to the result that we are in any case aiming at, namely that any such amalgamation class $\mathcal{A}$ contains all graphs that do not contain K_n. In the first place, the assertion $(*)$ is a special case of this result. Conversely, it is not hard to see that given $(*)$, our main result follows by "undergraduate induction" (I refer to the kind of argument which normally involves confusion on the part of the author as to which value of k is under consideration). Indeed, we may prove by induction on k that any graph of order k omitting K_n is in any amalgamation class $\mathcal{A}$ containing K_{n-1}, A, and B. For the base step we take $k = 1$, or even $k = 0$. For the inductive step, if $|H| = k+1$ we remove a vertex, getting H_0 of order k. By induction H_0 is in $\mathcal{A}^*$, hence $H = H_0^+$ is in $\mathcal{A}$, and the proof is complete.

There is very little difficulty in checking that K_{n-1}, A, B satisfy the requirement for membership in $\mathcal{A}^*$, but unfortunately there is no obvious way to prove that the class $\mathcal{A}^*$ is again an amalgamation class. One responds to this difficulty by changing the definition of $\mathcal{A}^*$ so that with the new defintion it is obviously an amalgamation class; one then must work harder to verify that certain specific graphs also satisfy the new definition. The final inductive argument remains exactly as above.

I will not give any more of the details relating to the case of undirected graphs. Instead I prefer to pass directly to the case of tournaments, which was treated by Lachlan in [L], because in this case a second idea is introduced on top of this "cheap induction" argument using associated amalgamation classes.

3. Homogeneous tournaments

3.1. THE CLASSIFICATION

There are five homogeneous tournaments, namely I_1 with one vertex, C_3 the oriented 3-cycle, $\mathbb{Q}$ the rational order, $\mathbb{Q}^*$ the dense local order, and T^∞ the random tournament. We will describe the last two in more detail.

3.1.1. *Notation.* If T is a tournament, we say that a vertex a *dominates* a vertex b if the edge between them is oriented toward b, and we write $'a$ and a' for the sets of vertices dominating or dominated by a, respectively. A tournament is called a *local order* if for each vertex a, the induced tournaments on a' and $'a$ are linear orders. The class of finite local orders is an amalgamation class and the corresponding homogeneous tournament is called the dense local order. It is rather easy to check that the first four tournaments on our list are exactly the homogeneous local orders.

The last tournament on our list, the random tournament, is the homogeneous tournament corresponding to the amalgamation class of all finite tournaments. The classification of the homogeneous tournaments amounts to the following:

3.1.2. *Theorem.* A homogeneous tournament which is not a local order embeds all finite tournaments.

We may rephrase this at once in terms of amalgamation classes. Let $[I_1, C_3]$ denote the tournament consisting of one vertex dominating an oriented 3-cycle. The local orders are the tournaments omitting $[I_1, C_3]$ and its dual $[C_3, I_1]$. Taking this duality into account, the classification theorem becomes:

3.1.3. *Theorem.* If $\mathcal{A}$ is an amalgamation class of tournaments and $[I_1, C_3] \in \mathcal{A}$, then all finite tournaments are in $\mathcal{A}$.

3.2. THE PROOF

The proof can be carried out for the most part using relatively general principles. The outstanding item of specific information required is the following:

3.2.1. *Lemma.* If $\mathcal{A}$ is an amalgamation class of tournaments and $[I_1, C_3]$ is in $\mathcal{A}$, then $[C_3, I_1]$ is in $\mathcal{A}$.

This can of course be proved by exhibiting some amalgamation diagrams. One can also argue as follows. With some effort it can be checked that a tournament which omits $[C_3, I_1]$ has the form $[L, S]$ where L is a linear order, S is a local order embedding C_3, and as the notation suggests, every vertex in L dominates every vertex in S. If such a tournament is homogeneous it clearly reduces either to L or to S, and hence omits $[I_1, C_3]$ as well.

Now we may prepare our cheap induction argument as follows.

3.2.2. *Definition.* Let $\mathcal{A}$ be an amalgamation class of tournaments. Then $\mathcal{A}^* =$:

$$\{H \in \mathcal{A} : \text{ Any linear extension of } H \text{ lies in } \mathcal{A}\},$$

where a linear extension of H is a tournament $H \cup L$ with the induced subtournament on L linear (i.e., transitive).

What we need is then:

3.2.3. *Proposition.* Let $\mathcal{A}$ be an amalgamation class of tournaments containing $[I_1, C_3]$. Then:
1. $\mathcal{A}^*$ is an amalgamation class.
2. $\mathcal{A}^*$ contains $[I_1, C_3]$.

If this proposition is granted, our rather silly inductive argument succeeds. Furthermore, the first claim is purely formal. In brief, if A_0, A_1, A_2, f_1, f_2 are the data for an amalgamation problem in $\mathcal{A}^*$ and we list the (finitely many) possible solutions B_i to this problem, then the assumption that no B_i is in $\mathcal{A}^*$ quickly produces a contradiction, as each B_i would then have a linear extension $B_i \cup L_i$ lying outside A, and after gluing the L_i together into one long linear order L, we would find that $A_0 \cup L, A_1 \cup L, A_2 \cup L$ with the natural embeddings define an amalgamation problem in $\mathcal{A}$ which has no solution in $\mathcal{A}$!

This very formal analysis has reduced the classification problem to the following:

3.2.4. *Lemma.* If $\mathcal{A}$ is an amalgamation class of tournaments containing $[I_1, C_3]$ and H is a finite linear extension of $[I_1, C_3]$, then $H \in \mathcal{A}$.

The major innovation in [La] comes at this point. Using Ramsey's theorem, Lachlan reduces the preceding lemma to one of the following form (I use a variant of his original version). We will use the notation $L[C_3]$ for the composition of a linear tournament L with the tournament C_3, that is a series of copies of C_3 with each copy dominating later ones.

3.2.5. *Lemma.* If $\mathcal{A}$ is an amalgamation class of tournaments containing $[I_1, C_3]$ and H is an extension of a tournament of the form $L[C_3]$ with L finite and linear by a *single* vertex, then $H \in \mathcal{A}$.

The main point here is that $L[C_3]$ contains a potentially large number of disjoint copies of $[I_1, C_3]$. The connection between the two versions of the lemma is provided by Ramsey's theorem and an explicit amalgamation argument. In addition to [La], where the argument is presented with all details, a rather detailed sketch of the argument is given in [Ch].

3.3. THE FINAL STAGE OF THE PROOF.

It should be said that a third idea comes into play at this point in the argument. The foregoing lemma is quite concrete in its content, apart from our total lack of understanding as to how a certain vertex of H sits over the others. It turns out that this Lemma can easily be proved by induction if we set it up properly. In fact the summary up to this point seems to cover the main ideas adequately, but this final point is rather important in practice and is really the main issue that really requires attention when one is working out these proofs. Accordingly we will go into a little more detail concerning the proof of the last lemma.

In the first place, having gotten so much mileage out of the "amalgamation class" viewpoint, we find that it is now useful to shift our viewpoint back from amalgamation classes to homogeneous tournaments. So we translate the last lemma back to the context of homogeneity, getting:

3.3.1. *Lemma.* Let Γ be a homogeneous tournament embedding $[I_1, C_3]$. Let H be an extension of $L[C_3]$ by a single vertex v, with L finite and linear. Then H embeds in Γ.

In this form the proof proceeds by induction on the length of L. Let C be the first copy of C_3 in $L[C_3]$, and let p be the type of the vertex v over C, that is p consists of a specification, for each vertex of C, as to whether v is to dominate or be dominated by that vertex. Let ${}^pC$ be the set of vertices in $\Gamma - C$ which have this same type p over C, and let C' be the set of vertices in Γ dominated by all vertices of C. (The possibility ${}^pC = C'$ turns out to be illusory after one makes the modifications alluded to below.) We consider the structure $\mathbb{T} = ({}'C, C')$ consisting of two tournaments together with some oriented edges connecting them. A structure of this type will be called a 2-tournament. To embed H into Γ, it is sufficient to embed the 2-tournament $\mathrm{H} = (\{v\}, L_0[C_3])$ into $\mathbb{T}$, where L_0 is L with its first vertex removed, and with v relating to $L_0[C_3]$ in H as it does in H.

3.3.2. *A shift of category.* Unfortunately as we began with tournaments and we are now working in a new category, the category of 2-tournaments, even though L_0 is shorter than L the induction fails. However if we start over, and rephrase our lemma in terms of homogeneous 2-tournaments (thereby strengthening it a little), then the indicated induction succeeds. To carry this out involves writing down (and proving) some specific properties of the 2-tournament $\mathbb{T}$. In [Ch] I introduce the notion of an *ample* 2-tournament at this point and carry out the appropriate inductive proof in a couple of easy pages.

4. Some homogeneous directed graphs

It is always more pleasant if a classification project turns up a few new examples along the way. I gave a catalog of the known homogeneous directed graphs in 1983, including one slightly odd one that turned up in the process of making that list. Much later (Summer 1988) I found one more, which turned out to be the last. This last example may perhaps be called the dense local partial order, by analogy with the dense local order.

4.1. EXAMPLES

4.1.1. *The dense local order.* The dense local order may be realized by partitioning the dense linear order $\mathbb{Q}$ into two dense sets Q_0, Q_1 and reversing the edges between Q_0, Q_1. This may also be described as follows: identify the two possible orientations of an edge with the integers modulo 2, and shift the orientations between Q_i, Q_j by $j - i$. There is another homogeneous directed graph that can be manufactured in a very similar way by partitioning $\mathbb{Q}$ into three dense sets Q_i

labelled by the integers modulo 3, and identifying the two possible orientations of an edge with the numbers ± 1, while 0 represents the absence of an edge: if we shift the edges between Q_i and Q_j by $j-i$ we get a directed graph that may be pictured as a circle along which edges point up to 1/3 of the way around in the positive (say, counterclockwise) direction.

4.1.2. *The dense local partial order.* A similar approach starting with the generic partial order $\mathcal{P}$ produces another homogeneous directed graphs. We partition $\mathcal{P}$ into three dense subsets P_i indexed by the integers modulo 3, we identify the three binary relations holding in $\mathcal{P}$ with the integers modulo 3 as above, and we shift the relations between P_i and P_j by $j-i$. The resulting homogeneous directed graph is rather hard to visualize and I would not claim to grasp it.

In any case, this is the most subtle example of a homogeneous directed graph, and it is not too far removed from a linear order.

5. The classification argument

As I have said, the classification of all homogeneous directed graphs involves a particularly long and detailed analysis. I will sketch the argument briefly here.

After disposing of the finite and imprimitive cases, it seems necessary to divide the primitive infinite case in two, according as our homogeneous directed graph Γ does or does not embed I_∞, an infinite set of independent vertices.

5.1. THE FIRST CASE

If Γ does not contain I_∞, then in fact it omits I_n for some n. If $n=2$ we are of course dealing with tournaments. For any finite value of n the argument simply generalizes the argument for tournaments, and indeed this more general argument was the direct source for the presentation of the case of tournaments in [Ch]. One simplifying factor in the case of directed graphs omitting I_n is that any application of Ramsey's theorem to a sufficiently large set of vertices inevitably produces a long linear order, and for this reason the argument stays fairly close to the model argument given for tournaments.

5.2. THE SECOND CASE

The study of homogeneous directed graphs embedding I_∞ seems harder. In spirit the situation is rather similar to the classification of homogeneous undirected graphs by Lachlan and Woodrow, but they found a trick exploiting the symmetry of the relations in an essential way. I use an argument which is much closer to the analysis of homogeneous tournaments. This argument can also be used for undirected graphs, but is harder than the one used by Lachlan and Woodrow in that case. I will go into this case in substantially more detail.

5.2.1. *Formulation in terms of amalgamation classes.* Let Γ be a homogeneous directed graph embedding I_∞, and let $\mathcal{T}$ be the set of tournaments embedding in Γ. Let A be the directed graph consisting of the disjoint union of an isolated vertex and an oriented path of length 2 (order 3), and define $\mathcal{A}(\mathcal{T})$ as the set:

$$\{A\} \cup \{I_n : \text{ all } n\} \cup \mathcal{T}.$$

Let $\Gamma(\mathcal{T})$ be the homogeneous directed graph associated with the amalgamation class $\mathcal{A}(\mathcal{T})$. Our problem is to show that if Γ embeds A as well, then $\Gamma \simeq \Gamma(\mathcal{T})$; if Γ instead omits A then we easily fall back into one of a small number of known exceptional cases. In terms of amalgamation classes our goal has become:

5.2.2. *Theorem.* Let $\mathcal{A}$ be an amalgamation class of finite directed graphs with $\mathcal{A}(\mathcal{T}) \subseteq \mathcal{A}$. Then $\mathcal{A}$ contains every finite directed graph H such that every subtournament of H embeds in a tournament in $\mathcal{T}$.

This will be proved for finite sets $\mathcal{T}$, from which the general case follows instantly. The advantage of working with finite sets is that we may proceed by induction on the size n of the largest element of $\mathcal{T}$, as well as on the number of nonisomorphic tournaments of this maximal size in $\mathcal{T}$.

5.2.3. *The class $\mathcal{A}^*$.* Just as in the case of tournaments, we will introduce an auxiliary amalgamation class $\mathcal{A}^*$, but there is a small difference at this point. In our earlier treatment we made use of linear tournaments, but now we may make a similar definition using either linear tournaments or independent sets of vertices I_n. We will have to consider both possibilities, since we at some point will have to invoke Ramsey's theorem, and we simply cannot predict in advance which of the two configurations will be produced. In spite of this ambiguity, much of the proof goes as in the case of tournaments until we arrive at the relatively concrete stage corresponding to our final lemma on 2-tournaments, itself proved by an inductive argument. At this stage we require a similar lemma on homogeneous 2-digraphs, which we will now state explicitly.

5.2.4. *Ample 2-directed graphs.* A 2-digraph $\mathrm{H} = (\Gamma_1, \Gamma_2)$ will be called *ample* if it embeds the specific directed graph A introduced above in its second component Γ_2, and if every possible configuration of the form $(\{v\}, I_n)$ embeds in H. With the class of finite tournaments $\mathcal{T}$ and the 2-directed graph H fixed, a configuration $(\{v\}, H)$ will be called *restricted* if for every subtournament T of H we have: T embeds in some element of $\mathcal{T}$, and $(\{v\}, T)$ embeds in H.

5.2.5. *Lemma $\mathcal{T}$.* Let $\mathcal{T}$ be a finite set of finite tournaments and let H be an ample homogeneous 2-digraph. Then any restricted configuration $(\{v\}, H)$ embeds in H.

Now the point is that both this lemma and the main result must be proved simultaneously by induction over $\mathcal{T}$. The order of steps is as follows. First Lemma $\mathcal{T}$ is proved for configurations in which H is a disjoint union of tournaments and oriented paths of length 2. This provides the key step to complete the main result for $\mathcal{T}$. Then using the main result for $\mathcal{T}$, the general case of Lemma $\mathcal{T}$ follows, and we are ready to proceed to a new $\mathcal{T}$.

6. Decision problems

6.1. CLASSIFICATION PROBLEMS

With the classification of the directed graphs in hand we can consider various special cases. In general it is natural to consider *finitely constrained* classes of directed graphs, in which we study all the directed graphs which embed all directed graphs in one finite set $\mathcal{A}$ and omit all directed graphs in a second finite set $\mathcal{B}$. Typically $\mathcal{A}$ is empty, though one might for example put I_2 in $\mathcal{A}$. In particular the classification of homogeneous partial orders [Sch] falls into this framework.

6.1.1. *Three problems.* Let $\mathcal{K}$ be a finitely constrained class of directed graphs. Our classification result for directed graphs specializes directly to $\mathcal{K}$. Accordingly one may well expect the following natural problems to become trivial:

$$\text{Are there any homogeneous directed graphs in } \mathcal{K}? \tag{A}$$

$$\text{Are there infinitely many homogeneous directed graphs in } \mathcal{K}? \tag{B}$$

$$\text{Are there uncountably many homogeneous directed graphs in } \mathcal{K}? \tag{C}$$

We also have Lachlan's original question: is the entailment relation $\Longrightarrow$ decidable for finite structures in $\mathcal{K}$? Our classification immediately yields a positive answer to this question, as any real classification must, and problem (A) is a special case of this.

6.1.2. *Problem B.* Problem (B) is a little less trivial. If some negative constraint H $(H \in \mathcal{B})$ is linear, then all but finitely many of the homogeneous directed graphs of the form $\Gamma(\mathcal{T})$ are excluded by this constraint. This immediately reduces problem (B) to a manageable special case. If on the other hand the negative constraints $H \in \mathcal{B}$ all involve C_3, then (B) is settled: there are indeed infinitely many homogeneous structures in $\mathcal{K}$, generated by linear tournaments of different sizes. So in either case (B) is settled quickly.

6.1.3. *Problem C.* Problem (C) is open, but can be rephrased in purely combinatorial terms. In the first place, a direct application of the classification theorem reduces the problem to the very special case in which there are no positive constraints, and the set $\mathcal{B}$ of negative constraints consists exclusively of tournaments. In this case we consider the class $\mathcal{T}_{\mathcal{B}}$ of all finite tournaments which contain no member of $\mathcal{B}$. If $\mathcal{T}_{\mathcal{B}}$ contains no infinite antichain (with respect to embeddability), then $\mathcal{T}_{\mathcal{B}}$ is said to be well quasi-ordered, and $\mathcal{B}$ is said to be *tight* – that is, $\mathcal{B}$ is a tight constraint. Our special case of problem (C) is equivalent to the problem of determining whether $\mathcal{T}_{\mathcal{B}}$ is well quasi-ordered.

It is certainly not out of the question that such a decision problem could turn out to undecidable, either as stated or perhaps in a somewhat more general formulation for slightly richer combinatorial structures. However there is no obvious encoding to show this.

6.2. RESULTS ON PROBLEM (C)

The known results in this area, due to my student B. Latka, all concern the classes $\mathcal{T}_{\{B\}}$ determined by just one negative constraint. She has shown that the problem is decidable in this case, and has also given a very simple decision procedure – the latter being much harder to obtain, as it involves the explicit solution of a number of critical cases. The soft approach to decidability does not seem to work in the context of more than one constraint.

Latka shows that all sufficiently large nonlinear tournaments constitute loose constraints, which certainly yields the decidability result. She then gives an explicit (and very short) list of the exceptions; this latter information would seem to be a prerequisite for a useful analysis of the case of two constraints, in our present state of knowledge.

6.2.1. *Soft arguments.* To get Latka's first, comparatively soft result, one first exhibits two infinite antichains of finite tournaments which serve to show that a number of specific tournaments are loose. Let $[L_2, C_3]$ and $[C_3, L_2]$ be respectively tournaments in which two vertices dominate or are dominated by a 3-cycle C_3, and let $C_3[L_1, L_1, L_4]$ be a tournament in which one vertex of C_3 has been replaced by a linear tournament of length 4. One of Latka's antichains, a modification of a linear order in the spirit of Henson's example, is made up of tournaments omitting $C_3[L_1, L_1, L_4]$, while the other is made up of tournaments omitting $[L_2, C_3]$ and $[C_3, L_2]$. On the other hand every sufficiently large nonlinear tournament contains one of these three tournaments, by the pigeon-hole principle.

6.2.2. *Hard arguments.* A closer look at the information afforded by Latka's two antichains reveals that every nonlinear tournament on at least seven vertices constitutes a loose constraint, and that among non-linear tournaments for which the problem is not settled by these antichains, there are only two maximal ones, of order 5 and 6 respectively. Let N_5 be the tournament obtained from the linear tournament of order 5 by reversing the arrows representing the successor relation, and let $C^+ = [L_1, C_3[L_1, L_1, L_3]]$ be the tournament in which one vertex dominates the tournament derived from C_3 by replacing one vertex by a linear tournament of order 3. Latka completes the analysis of the case of a single constraint by showing that tournaments omitting N_5 or C^+ admit a structure theorem of a type that allows us to apply Kruskal's Tree Theorem to conclude that the class is well quasi-ordered. (It should however be noted that the analysis of the irreducible tournaments omitting N_5 is still being checked.) In the case of N_5 the full Kruskal theorem is needed, basically because the class is closed under composition, while in the case of C^+ Kruskal's theorem for trees of bounded height – which is essentially the same as Higman's lemma for finite words in a well quasi-ordered alphabet – is adequate.

Thus in a certain precise sense, this completes the classification of all singly constrained classes of finite tournaments which admit a structure theorem.

7. Whither?

7.1. REDUCTS

I have mentioned the work of Simon Thomas and Jim Bennett on reducts of homogeneous structures, and Hrushovski's work extending Lachlan's theory, both presumably well documented in this volume.

7.2. 0-1 LAWS

Another interesting line is the study of 0-1 laws for probabilities in spaces consisting of the labelled structures in a finitely constrained amalgamation class. The very striking results of Kolaitis, Prömel, and Rothschild on the case of finite undirected graphs omitting a fixed K_{n+1} have not been generalized to the directed case, and perhaps unexpectedly involve new difficulties. Consider the case $n = 2$ (treated earlier in [EK]): a random finite undirected graph with no triangles is bipartite, with probability approaching 1 as the number of vertices goes to infinity. The directed version of this result would apply to directed graphs omitting one of the two tournaments L_3, C_3 of order 3. For directed graphs omitting L_3, a similar result may hold, but not with the same proof. For directed graphs omitting C_3 the analogous statement is clearly false: there are more graphs of a given size obtained from a linear tournament by omitting some edges (as in the Albert-Frieze model [1] for random partial orders) than there are bipartite graphs. So all of this remains quite mysterious.

7.3. THE FINITE MODEL PROBLEM

A somewhat related problem is the finite model problem. We know that the random (or generic) undirected graph has the finite model property: any first order property of this graph is true of some (actually, most) finite graphs. The analogous question for the generic graph omitting K_{n+1} is open, even for $n = 2$. By the 0-1 law, most finite triangle-free graphs contain no cycle of length 5, and are thus very different from the generic infinite triangle-free graph; but it is possible that this graph has (a few) good finite approximations. The most successful exploration of this case that I know of is due to Michael Albert (unpublished), but is still in its early stages.

7.4. OTHER CLASSIFICATION PROBLEMS

In principle the methods used to classify homogeneous graphs apply to any finite binary language. In practice these methods are quite cumbersome and appear to be near the point of collapsing under their own weight. At the same time the actual classification is strikingly simple: one essentially has the directed graphs corresponding to the most obvious amalgamations classes ($\Gamma(\mathcal{T})$ and Γ_n) and a handful of exceptions, finite, imprimitive, or linked to partial orders. It remains to be seen whether our classification techniques break down at the point where radically new examples appear.

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