Linear Algebra: Final Review

Main Algorithms

1. Test for linear independence
2. When vectors are linearly dependent, express one of them as a linear combination of the others.
3. Basis for row space, column space, null space, eigenspace, orthogonal complement.
4. Determination of eigenvalues.
5. L, U-decomposition and its application to the solution of linear systems
6. Matrix inversion
7. Computation of determinant via a combination of row and column operations, cofactor expansions, and explicit formulas.
8. Computation of eigenvalues.
9. Diagonalization, powers
10. Projection on a line
11. Gram-Schmidt orthogonalization procedure
12. Projection on a subspace
13. Least squares, data-fitting.
14. Determination of the type of a conic section.

While one should bear in mind the applications to differential equations, I will not be testing this material, which is another application of the idea of diagonalization.

Theory

Concepts

You should able to give reasonably precise and explicit definitions or explanations of each of the following, with one exception: we never gave a proper definition of the *determinant*, so the best you can do in this case is explain in general terms what it is, and what it is good for.

matrix multiplication, inverse, power, matrix times vector, rank, nullity, row echelon form
elementary row operation, Gaussian elimination
column space, row space
linear combination, independence, span, subspace, basis, dimension
determinant, trace
eigenvalue, eigenvector, characteristic polynomial, algebraic multiplicity, geometric multiplicity
symmetric matrix, diagonalizable matrix, orthogonal matrix, rotation matrix, invertible matrix, similar matrices
dot product, orthogonal vectors, orthogonal set of vectors, orthogonal basis, orthogonal complement, orthogonal matrix,
projection onto a line, a subspace

We might also add: "linear transformation"—but this is not, officially, a part of the course.

Facts

I have put up a fairly exhaustive summary of the theory elsewhere. What follows below is a list of a substantial number of theoretical points for which it is possible to give reasonably straightforward and brief explanations. Assuming you have all the essential algorithms under control, and can explain the terms listed above, the next stage is to consider how one knows each of the following. Some of the apparently simple facts lie fairly deep, while others that may seem sophisticated follow directly from the definitions.

1. The rank of a matrix is equal to the rank of the transpose.
2. The rank of a matrix gives the dimension of both the row space and the column space.
3. The nullity of a matrix gives the dimension of its null space;
4. If A is a matrix and V is its column space, then the dimension of $V^{\perp}$ is the nullity of A^T.
5. For any subspace V of $\mathbb{R}^n$ we have $\dim(V) + \dim(V^{\perp}) = n$.
6. Eigenvectors of a symmetric matrix which correspond to *distinct* eigenvalues are orthogonal.
7. An orthogonal set of nonzero vectors must be linearly independent.
8. A diagonalizable matrix has all of its eigenvalues real.
9. The sum of the algebraic multiplicities of the eigenvalues of an $n \times n$ matrix is n.
10. The product of the eigenvalues of a matrix gives its determinant.
11. If all the eigenvalues of a square matrix are real, and if the geometric multiplicity of each eigenvalue is its algebraic multiplicity, then the matrix is diagonalizable.
12. If A is an $n \times n$ matrix and $\mathbb{R}^n$ has a basis consisting of eigenvectors for A, then A is similar to a diagonal matrix.
13. If A is an invertible matrix then its determinant is nonzero.
14. The set of eigenvectors for a square matrix A corresponding to a *particular* eigenvalue, with the zero vector *included*, forms a subspace.
15. The span of any set of vectors in $\mathbb{R}^n$ is a subspace.
16. The set of solutions for a system of homogeneous linear equations in n unknowns forms a subspace of $\mathbb{R}^n$.
17. A set of homogeneous linear equations involving more variables than equations must have a nonzero solution.
18. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.
19. Similar matrices have the same characteristic polynomial, the same eigenvalues, and the same algebraic multiplicities.
20. If A, B, C are $n \times n$ matrices with both B and C similar to A, then B and C are similar to each other.

Examples

The standard example of a matrix which is not diagonal is $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ where one picks a value for λ; then the algebraic multiplicity will be two and the geometric multiplicity 1. Similar, but larger, examples would consist of the eigenvalue down the diagonal, and a sequence of 1's just above the diagonal.

As the zero vector is not considered an eigenvector, it provides a counterexample to some statements which would otherwise be true. The reason for this exclusion is the following: each eigenvector is associated to a *unique* eigenvalue—this would not hold for the zero vector.

For understanding Gaussian elimination at a theoretical level, one should consider matrices with linearly dependent rows, both square and nonsquare, such as $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. (In the second example, the shape of the matrix forces the rows to be linearly dependent, but the columns are linearly independent.)

Sometimes rotation matrices or symmetric matrices provide good examples. In extreme cases the consideration of diagonal matrices can be helpful (including the identity matrix and the zero matrix)—though these can also be misleading.

Bear in mind the failure of commutativity, which can usually be illustrated by random 2×2 matrices; this is what makes the expression PDP^{-1} interesting. If we had commutativity, this would just be D.

The standard basis $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ in $\mathbb{R}^n$ is perhaps the best example of a basis (and the first couple are good for thinking about the "span"). But it has special properties: this is an *orthogonal basis* and the vectors are *unit* vectors.

Projection onto the span of $\bar{e}_1, \bar{e}_2$ in $\mathbb{R}^4$ is a good example of orthogonal projection. (Exercise: write down the projection matrix, and check that it is symmetric and idempotent.)

Food for thought

Prove $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$

What are the eigenvalues of a projection matrix, and what are their algebraic and geometric multiplicities? Is every projection matrix diagonalizeable?