Linear Algebra: Review for second Midterm (April 14, 2003)

While emphasis on the second midterm is on algorithms, there is also some theoretical component to the test. Be prepared to answer true/false questions, explaining why the true ones are true and making up suitable examples to show the false ones are false. You should also be prepared to explain the theory of eigenvalues and eigenvectors.

Topics and sample problems

1. Spaces

Basis, dimension

Problems: p. 248. 5,6

2. Matrices

Row and column spaces, nullspace.

Problems: p. 248. 7,8

Determinants, elementary row and column operations

Problems: p. 196. 14, 32

3. Eigenvalues and eigenvectors

Characteristic polynomial, eigenvalue, eigenvector, eigenspace

Problems: p. 309. 6, 7, 8, 9, 10

4. Diagonalization and powers

Diagonalization algorithm:

a. Compute the characteristic polynomial and find the roots, and their multiplicities. If any of them are complex, stop.

b. For each eigenvalue, find a basis for the corresponding eigenspace. If the dimension of some eigenspace is less than the algebraic multiplicity, stop.

c. Form the matrix P whose columns are the eigenvectors you found, and form the diagonal matrix D whose diagonal entries are the corresponding eigenvalues.

d. You have $P^{-1}AP = D$ and $A = PDP^{-1}$.

Power algorithm:

a. Diagonalize A: $A = PDP^{-1}$. If the algorithm fails, stop.

b. $A^k = PD^kP^{-1}$.

If you need to compute A^k, you will have to invert P^{-1}.

If, on the other hand, you need to compute $A^k\mathbf{u}$ for a single vector $\mathbf{u}$, then you only need to compute $P^{-1}\mathbf{u}$ which is a little bit easier; use Gaussian elimination to solve $P\mathbf{x} = \mathbf{u}$.

Problems: p. 309. 19, 20, 22

Exponential algorithm:

a. Diagonalize A: $A = PDP^{-1}$. If the algorithm fails, stop.

b. $e^A = Pe^DP^{-1}$.

(The same remarks about inverting P apply as in the previous case.)

5. Dot product, length, projection

Problems: p. 394. 2, 3, 6, 7

6. True/False, and terminology

Problems:

p. 196. 1 f,g,h,j

p. 248. 1 c,g,i,j,l,m,p; 2 a,b,c,d,e

p. 308. 1 b,c,d,h,i,j

p. 394. 1 a,b,c,d,e

7. General Theory

Theory of eigenvalues: Explain why each of the following is true.

1. The eigenvalues of a square matrix are the roots of its characteristic polynomial.

2. If $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are nonzero eigenvectors for the matrix A, and if the corresponding eigenvectors are all distinct, then $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

3. If $P^{-1}AP = B$ with A, B square and of the same size, and $\mathbf{u}$ is an eigenvector for B, then $P\mathbf{u}$ is an eigenvector for A.

4. If P is a matrix whose columns are eigenvectors for the matrix A, and if D is the diagonal matrix whose diagonal entries are the corresponding eigenvalues, then $AP = PD$.

5. If $P^{-1}AP = B$ then A and B have the same characteristic polynomial.

6. If A is square, then A and A^T have the same characteristic polynomial.

7. If A and B have the same characteristic polynomial then they have the same eigenvalues, with the same algebraic multiplicities, and also have the same determinant and trace.

8. The determinant of a matrix is the product of its eigenvalues.

Theory

I. Theorem 1.9 page 75: span and linear independence

If $k+1$ vectors are contained in the span of k vectors, then they are linearly dependent – as for example, $n+1$ vectors in $\mathbb{R}^n$.

II. Gaussian elimination and bases

After Gaussian elimination: the nonzero rows are a basis for the row space; the pivot columns in the original matrix form a basis for the column space; and the solutions to the homogeneous equation obtained by back substitution by setting each free variable in turn equal to 1 (and the other free variables to 0) form a basis for the nullspace.

III. Eigenvalues

The eigenvalues of A are the roots of the characteristic polynomial $\det(A - \lambda I)$. By the *Fundamental Theorem of Algebra*, if one allows both *complex roots* and *repeated roots*, then there are n of them (if the matrix is $n \times n$).

IV. Eigenspaces

The λ-eigenspace for A is the nullspace of $A - \lambda I$; this is how one determines a basis for it.

V. Diagonalization

A matrix A is *diagonalizable* if the equation $P^{-1}AP = D$ has a solution with P invertible and D diagonal.

Here are two characterizations of diagonalizability for an $n \times n$ matrix A:

(1) The whole space $\mathbb{R}^n$ has a basis made up of eigenvectors for A.

(2) The eigenvalues of A are all *real*, and furthermore for each root its algebraic multiplicity equals its geometric multiplicity.

VI. Dot products

$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

$\mathbf{u} \cdot \mathbf{v}$ is $||\mathbf{u}||||\mathbf{v}|| \cos\theta$ with θ the relative angle.

Two special cases: $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$; $\mathbf{u} \cdot \mathbf{v} = 0$ if $\mathbf{u} \perp \mathbf{v}$.

Explanations – Summarizing the book and class discussion

I. Span and linear independence

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are in the span of $\mathbf{u}_1, \ldots, \mathbf{u}_m$ with $n > m$. Let A be the matrix whose columns are the $\mathbf{u}_i$ and B the matrix whose columns are the $\mathbf{v}_i$. Then the relation is $B = AX$ with X some $m \times n$ matrix. By part I we can solve $X\mathbf{c} = \vec{0}$ with $\mathbf{c}$ nonzero. Then $B\mathbf{c} = AX\mathbf{c} = \vec{0}$ so $c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \vec{0}$.

II. Gaussian elimination and bases

Elementary row operations do not change the row space or nullspace.

Therefore, to get the row space or null space for a matrix, we can first put it in row echelon form, and then it is fairly easy to see what to do.

For the column space, we must work with the original matrix. However, since the nullspace is unchanged, and the nullspace consists of all *linear relations* among the columns, it follows that the relations of linear dependence and independence in the original matrix are the same as in the row echelon form, and therefore the pivot columns are the ones we want.

III. Eigenvalues

For λ to be an eigenvalue of A, one wants a nonzero eigenvector, in other words a solution to $(A - \lambda I)\mathbf{x} = \vec{0}$.

By the theory from the first part of the course, this means that the matrix $A - \lambda I$ should *not* be invertible, and the condition for this is $\det(A - \lambda I) = 0$.

IV. Eigenspaces

The equations $A\mathbf{x} = \lambda\mathbf{x}$ and $(A - \lambda I)\mathbf{x} = \vec{0}$ are equivalent.

V. Diagonalization

The main point that one wants to check is this: if a matrix A has n real roots (with repetitions) and if the dimension of each eigenspace is the corresponding algebraic multiplicity, then the matrix is diagonalizable. None of this is easy.

1) One first shows that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are eigenvectors corresponding to distinct eigenvalues, then they are linearly independent.

2) From this it follows that if we take a basis for each eigenspace for A, we get a linearly independent set.

Now suppose that the $n \times n$ matrix A has all of its eigenvalues real, and all its eigenspaces of the correct dimension. Then:

3) The linearly independent set constructed at stage 2 contains n vectors, so it is a basis for $\mathbb{R}^n$.

4) If P is the matrix of eigenvectors and D is the corresponding diagonal matrix of eigenvalues, then we have two facts:

a. $AP = PD$

b. P is invertible.

For (a) one shows that on both the left and the right hand sides, the product shown gives P with each column multiplied by the corresponding eigenvalue.

For (b) one uses the fact that a matrix whose columns are linearly independent must be invertible.

VI. Dot products

This is largely computational. One fundamental point is the following:

$(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$. This is fundamental.

Notice that it implies that the value of the dot product is unaffected by a rotation (this fact can be used to simplify computations).

Here's the proof: if A is a rotation then $A^TA = I$ and hence $(A\mathbf{u}) \cdot (A\mathbf{v}) = \mathbf{u} \cdot (A^TA\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.