#### ORIGINAL ARTICLE



# Resonances and Eigenvalues for the Constant Mean Curvature Equation

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#### Abstract

In this paper we study resonances and eigenvalues for the nonlinear constant mean curvature equation linearized around the bubbles found by Brezis–Coron. This nonlinear equation is also called a *H*-system equation. For degree one bubbles we only find resonances. For higher degree we prove eigenvalues occur. Our goal is to eventually obtain dispersive estimates for the wave equation associated to the linear and non-linear problem, a study of which was initiated by Chanillo–Yung.

**Keywords** *H*-systems  $\cdot$  Resonances  $\cdot$  Eigenvalues  $\cdot$  Dispersive estimates  $\cdot$  Mean curvature equation

Mathematics Subject Classification (2010) 35P25 · 35L77 · 35P30

## **1** Introduction

Let

$$u:\Omega\subset\mathbb{R}^2\to\mathbb{R}^3,$$

where  $\Omega$  is a domain. We will denote vector cross products of two vectors u, v by  $u \wedge v$ . Consider now the equation

$$\Delta u = 2u_x \wedge u_y, u = (u_1, u_2, u_3), \tag{1.1}$$

where  $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ .

If u satisfies (1.1) and in addition the Plateau conditions

$$|u_x|^2 = |u_y|^2, \quad u_x \cdot u_y = 0, \tag{1.2}$$

To Carlos Kenig in friendship and admiration.

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then the range of u defines a constant mean curvature surface in  $\mathbb{R}^3$ . In this work, we shall ignore the Plateau conditions (1.2). Equation (1.1) can be written in variational form

$$\mathcal{E}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{2}{3} \int_{\Omega} u \cdot u_x \wedge u_y.$$
(1.3)

Brezis and Coron [1] and Struwe [10] studied (1.3) and its Morse theory. Brezis and Coron [2] classified all the finite energy solutions of (1.1). These finite energy solutions will be referred to as bubbles in the sequel. [2] also obtained the first result for (1.1) as to how compactness in (1.3) fails. Equation (1.1) is a conformally invariant PDE and also invariant by translation and rotation in the target. Chanillo and Malchiodi [4] studied the Morse theory further for (1.1) and constructed multi-bubble solutions to (1.1). We point out that the Topology of  $\Omega$  plays a strong role in constructing solutions. For example, a result of Wente [11], another proof may be found in Proposition 3.1 of [4], shows that if  $\Omega$  is simply connected, then

$$\Delta u = 2u_x \wedge u_y, u|_{\partial\Omega} \equiv 0 \tag{1.4}$$

has only the solution  $u \equiv 0$ , while if  $\Omega$  is an annulus, Wente constructed non-trivial solutions to (1.4) in [11].

We now recall the classification result of [2]. Consider those solutions in all  $\mathbb{R}^2$  satisfying,

$$\Delta u = 2u_x \wedge u_y, \quad \int_{\mathbb{R}^2} |\nabla u|^2 < \infty.$$

Then, we have quantization, that is necessarily

$$\int_{\mathbb{R}^2} |\nabla u|^2 = 8\pi m, \quad m \in \mathbb{N}, \ m \ge 1,$$

and

$$u = \pi^{-1} \left( \frac{P(z)}{Q(z)} \right), \quad \pi : S^2 \to \mathbb{R}^2, \quad z = (x, y)$$

 $\pi$  is the stereographic projection to the plane from the Riemann sphere, P(z), Q(z) are polynomials which are holomorphic and  $m = \max\{\deg P, \deg Q\}$ . Thus the degree 1 bubbles which are basic have the form

$$(x, y) \rightarrow \left(\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|^2-1}{|z|^2+1}\right),$$

where z = x + iy, which we will abuse sometimes and think z = (x, y). Degree *m* bubbles,  $m \ge 2$ , can be written as (note these are special degree *m* bubbles)

$$\left(\frac{2\Re z^m}{1+|z|^{2m}},\,\frac{2\Im z^m}{1+|z|^m},\,\frac{|z|^{2m}-1}{|z|^{2m}+1}\right).$$
(1.5)

In [5] the study of the wave equation corresponding to (1.1)

$$\Delta u - u_{tt} = 2u_x \wedge u_y, \quad u|_{t=0} = u_0, \ u_t|_{t=0} = u_1 \tag{1.6}$$

was initiated. The right side of (1.6) is an example of a null form. Since we are in dimension 2, with poor Strichartz estimates and lack of Huygens principle, many problems concerning (1.6) remain unresolved. Some attempts have been made [3] to understand random data versions of (1.6). One of the main results of [5], in the spirit of a similar result for another conformally invariant equation, the Yamabe equation [8], is that if

$$\|\nabla u_0\|_2^2 > 8\pi$$

which is saying that if the initial data has more energy than a degree 1 bubble, and with an additional condition on  $u_1$ , involving energy trapping, then the solution to (1.6) blows up in finite time. Furthermore in [6], via unique continuation arguments it is proved that the blow up is not self-similar. No refined bubbling analysis for (1.6) is known to date.

Our goal here is to take an initial step for a linearized version of (1.6) and study dispersion and scattering like what was done in 3D for the Yamabe equation in [9]. To do so one has to study Born expansions. To perform such expansions one needs to know if resonances and eigenvalues exist for the linearized elliptic part. Appearance of eigenvalues in the spectrum complicates matters.

To state our theorems, we first linearize (1.1) around a bubble u, and get

$$\Delta w = 2w_x \wedge u_y + 2u_x \wedge w_y. \tag{1.7}$$

We now prefer to study (1.7) on the punctured sphere  $S^2 \setminus \{N\}$ , where N is the North Pole. So we study (1.7) on  $\mathbb{R}^2$ , equipped with the metric

$$\frac{4}{(1+|z|^2)^2}(dx^2+dy^2), \quad z=(x,y).$$

Equation (1.7) becomes due to a conformal change of the metric

$$\frac{1}{\varphi}\Delta w = \frac{2}{\varphi}(w_x \wedge u_y + u_x \wedge w_y), \quad \varphi = \frac{4}{(1+|z|^2)^2}.$$
(1.8)

Thus the eigenvalue equation for (1.8) is

$$\Delta w = 2(w_x \wedge u_y + u_x \wedge w_y) + \frac{4\lambda}{(1+|z|^2)^2}w,$$
(1.9)

where z = (x, y) and

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}.$$

The equation (1.9) when  $\lambda = 0$  is of course

$$\Delta w = 2(w_x \wedge u_y + u_x \wedge w_y). \tag{1.10}$$

We also notice by integration by parts the self-adjoint property

$$\mathcal{E}(v,w) = \mathcal{E}(w,v) = \int_{\mathbb{R}^2} \nabla v \cdot \nabla w + 2 \int_{\mathbb{R}^2} v \cdot (w_x \wedge u_y + u_x \wedge w_y)$$
(1.11)

for  $v, w \in C_0^{\infty}(\mathbb{R}^2, \mathbb{R}^3)$ . In view of (1.11) we search for solutions to (1.9), (1.10) for which  $w \in L^2(\mathbb{R}^2)$ .

**Definition 1.1** (a) We say  $\lambda$  is a resonance for (1.9), (1.10), if  $w \notin L^2(\mathbb{R}^2)$  and w satisfies (1.9), (1.10).

(b) We say  $\lambda$  is an eigenvalue if  $w \in L^2(\mathbb{R}^2)$  and w satisfies (1.9), (1.10).

Lastly, we shall only focus on w that is co-rotational. That is keeping in mind the expression for the bubbles (1.5), we assume for  $(r, \theta)$  polar coordinates,

$$w(r,\theta) = (f(r)\cos m\theta, f(r)\sin m\theta, g(r)).$$
(1.12)

**Theorem 1.2** (a)  $\lambda = 0$  is a resonance for (1.10) for u a degree 1 bubble.

(b)  $\lambda = 0$  is an eigenvalue for (1.9), (1.10), for u a degree m bubble of the form (1.5),  $m \ge 2$ , and with an eigenfunction of the form (1.12).

**Theorem 1.3** For u a degree 1 bubble, (1.9), (1.10) has no eigenfunctions of the form (1.12) for any  $\lambda$ . That is there are no co-rotational eigenfunctions for any choice of  $\lambda$ .

We end by recording a problem.

**Problem** In view of Theorem 1.2, when *u* is a degree *m* bubble,  $m \ge 2$  we know  $\lambda = 0$  is an eigenvalue. Does (1.9) have other eigenvalues besides  $\lambda = 0$ , where the eigenfunctions are co-rotational of the form (1.12), when *u* is a degree *m* bubble given by (1.5),  $m \ge 2$ ?

### 2 Proofs of the Results

We begin by proving Theorem 1.2.

**Proof** First we note that

$$\Delta u = 2u_x \wedge u_y$$

is invariant under dilations, that is under the map  $z \to \delta z$ ,  $\delta > 0$ , z = x + iy. We only consider dilations as we are focused on co-rotational solutions to our linearized equation

$$\Delta w = 2u_x \wedge w_y + 2w_x \wedge u_y + \frac{4\lambda}{(1+|z|^2)^2}w,$$
(2.1)

where u(x, y) is a bubble of degree *m*. Since we are examining eigenvalues and resonances at  $\lambda = 0$ , we set  $\lambda = 0$  in (2.1) and also

$$u(z) = \left(\frac{2\Re z^m}{1+|z|^{2m}}, \frac{2\Im z^m}{1+|z|^{2m}}, 1-\frac{2}{1+|z|^{2m}}\right).$$
(2.2)

Dilation of the bubble given by (2.2) yields

$$u^{\delta}(x, y) = \left(\frac{2\delta^m \Re z^m}{1 + \delta^{2m} |z|^{2m}}, \frac{2\delta^m \Im z^m}{1 + \delta^{2m} |z|^{2m}}, 1 - \frac{2}{1 + \delta^{2m} |z|^{2m}}\right).$$
(2.3)

By invariance we have

$$\Delta u^{\delta} = 2u_x^{\delta} \wedge u_y^{\delta}. \tag{2.4}$$

Differentiating (2.3), (2.4) in  $\delta$ , and setting  $\delta = 1$ , we get,

$$w = \frac{\partial u^{\delta}}{\partial \delta}\Big|_{\delta=1} = \left(\frac{2m\Re z^m (1-|z|^{2m})}{(1+|z|^{2m})^2}, \ \frac{2m\Im z^m (1-|z|^{2m})}{(1+|z|^{2m})^2}, \ \frac{2m|z|^{2m}}{(1+|z|^{2m})^2}\right)$$
(2.5)

and

 $\Delta w = 2w_x \wedge u_y + 2u_x \wedge w_y.$ 

We see from (2.5), that there exists  $c_1, c_2 > 0$  such that

$$|w(x, y)| \le c_1 (1+|z|)^{-m}$$

and

$$c_2|z|^{-m} \le |w(x, y)| \text{ for } |z| \ge 10.$$

Thus  $w \in L^2(\mathbb{R}^2)$  for  $m \ge 2$ , and  $w \notin L^2(\mathbb{R}^2)$  when m = 1. Hence, we conclude that  $\lambda = 0$  is an eigenvalue for the linearized problem around a bubble of degree  $\ge 2$ , and  $\lambda = 0$  is a resonance for the linearized problem when the CMC equation is linearized around a bubble of degree one. When  $\lambda = 0$  and u a degree one bubble, we will conclusively rule out eigenvalues in the next theorem. This ends the proof.

We now prove Theorem 1.3.

**Proof** Our first goal is to compute the ODE for the co-rotational solution for the linearized problem, where we linearize at a degree *m* bubble. We will later specialize to m = 1. Since we are considering the punctured sphere  $S^2 \setminus \{N\}$ , where *N* is the North pole, as explained in the introduction, our eigenvalue equation is

$$\Delta w = 2w_x \wedge u_y + 2u_x \wedge w_y + \frac{4\lambda}{(1+|z|^2)^2}w,$$
(2.6)

where *u* is a degree *m* bubble. *w* being co-rotational has the form in polar coordinates  $(r, \theta)$ ,  $w(r, \theta) = (f(r) \cos m\theta, f(r) \sin m\theta, g(r))$ . By changing variables it is easily seen

$$w_x \wedge u_y + u_x \wedge w_y = \frac{1}{r} w_r \wedge u_\theta + \frac{1}{r} u_r \wedge w_\theta.$$
(2.7)

Next we perform a Kelvin transformation by setting

$$r = \frac{1}{t}, \quad \theta \to -\theta$$

and routine computation yields that (2.7) becomes

$$t^{3}(w_{t} \wedge u_{\theta} + u_{t} \wedge w_{\theta}).$$

$$(2.8)$$

The Laplacian transforms to

$$t^4\left(\frac{\partial^2}{\partial t^2} + \frac{1}{t}\frac{\partial}{\partial t} + \frac{1}{t^2}\frac{\partial^2}{\partial \theta^2}\right) = t^4\Delta.$$

After performing the Kelvin transformation

$$\tilde{w}(t,\theta) = (\tilde{f}(t)\cos m\theta, -\tilde{f}(t)\sin m\theta, \tilde{g}(t)),$$

where

$$\tilde{f}(t) = f\left(\frac{1}{t}\right), \quad \tilde{g}(t) = g\left(\frac{1}{t}\right).$$

Likewise the bubble becomes

$$\tilde{u} = \left(\tilde{F}(t)\cos m\theta, -\tilde{F}(t)\sin m\theta, \tilde{G}(t)\right).$$

We conserve notation and drop the tildes.

$$w_t = (f'(t)\cos m\theta, -f'(t)\sin m\theta, g'(t)),$$
  

$$w_\theta = (-mf\sin m\theta, -mf\cos m\theta, 0),$$
  

$$u_t = (F'\cos m\theta, -F'\sin m\theta, G'),$$
  

$$u_\theta = (-mF\sin m\theta, -mF\cos m\theta, 0).$$

We obtain by elementary computation,

$$w_t \wedge u_\theta = (mFg' \cos m\theta, -mFg' \sin m\theta, -mFf'),$$
  
$$u_t \wedge w_\theta = (mfG' \cos m\theta, -mfG' \sin m\theta, -mfF').$$

Hence

$$2(w_t \wedge u_\theta + u_t \wedge w_\theta) = 2\left((fG' + Fg')\cos m\theta, -m(fG' + Fg')\sin m\theta, -m(Ff' + fF')\right).$$

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Hence, we arrive at a pair of second order ODE,

$$t^{4}\left(f'' + \frac{1}{t}f' - \frac{m^{2}}{t^{2}}f\right) = 2mt^{3}(fG' + Fg') + \frac{4\lambda t^{4}}{(1+t^{2})^{2}}f,$$
(2.9)

$$t^{4}\left(g'' + \frac{1}{t}g'\right) = -2mt^{3}(Ff' + fF') + \frac{4\lambda t^{4}}{(1+t^{2})^{2}}g.$$
 (2.10)

Before we proceed further we note that if u is a degree m bubble,

 $|\nabla u| \leq C.$ 

Thus from (2.8) and (2.6), any eigenfunction will satisfy

$$t^{4} |\Delta w| \le t^{4} |\nabla w| + \frac{4t^{4}}{(1+t^{2})^{2}} |\lambda| |w|$$
  
$$\le C(|\nabla w| + |w|)t^{4}.$$

We have

$$|\Delta w| \le C(|\nabla w| + |w|).$$

If we show that w vanishes to infinite order at t = 0, the origin, then by applying the unique continuation Lemma 2.6.1, p. 70 of [7], we may conclude that  $w \equiv 0$  in  $\mathbb{R}^2$ . This will complete the proof of the theorem. To show that w does indeed vanish to infinite order, we need to assume by contradiction that  $f, g \in L^2(\mathbb{R}^2)$  and f, g are of course smooth. Thus both f, g have a formal power series expansion.

$$f(t) \sim \sum_{n\geq 2} a_n t^n$$
,  $g(t) \sim \sum_{n\geq 2} b_n t^n$ .

Obviously  $n \ge 2$ , for we are assuming  $f, g \in L^2(\mathbb{R}^2)$ ; and t is the variable after Kelvin transformation. The expression for the m = 1 bubble in  $(t, \theta)$  coordinates is

$$\left(\frac{2t}{1+t^2}\cos\theta, -\frac{2t}{1+t^2}\sin\theta, \frac{1-t^2}{1+t^2}\right).$$

Thus

$$F(t) = \frac{2t}{1+t^2}, \quad G(t) = -1 + \frac{2}{1+t^2}.$$

Thus,

$$F'(t) = \frac{2(1-t^2)}{(1+t^2)^2}, \quad G' = -\frac{4t}{(1+t^2)^2}$$

Now

$$Ff' + fF' = \left(\sum_{n \ge 2} na_n t^{n-1}\right) \frac{2t}{1+t^2} + 2\left(\sum_{n \ge 2} a_n t^n\right) \frac{(1-t^2)}{(1+t^2)^2}.$$

Thus, (2.10) becomes after cancelling off  $t^4$  from both sides; for m = 1,

$$g'' + \frac{1}{t}g' = -\frac{4}{1+t^2}\sum_{n\geq 2}na_nt^{n-1} + 4\left(\sum_{n\geq 2}a_nt^{n-1}\right)\frac{(1-t^2)}{(1+t^2)^2} + \frac{4\lambda}{(1+t^2)^2}g.$$

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Since  $g(t) = \sum_{n \ge 2} b_n t^n$ , the left side above, is

$$\sum_{n\geq 2} n^2 b_n t^{n-2} = \sum_{n\geq 0} (n+2)^2 b_{n+2} t^n.$$

We have using (2.10)

$$(1+t^2)^2 \sum_{n\geq 0} (n+2)^2 b_{n+2} t^n$$
  
=  $-4(1+t^2) \sum_{n\geq 2} na_n t^{n-1} + 4\left(\sum_{n\geq 2} a_n t^{n-1}\right) (1-t^2) + 4\lambda \sum_{n\geq 2} b_n t^n.$  (2.11)

It is easily seen from (2.11), that

$$(n+2)^2 b_{n+2} = \sum_{k \le n+1} \alpha_k b_k + \sum_{k \le n+1} \gamma_k a_k, \quad n \ge 0,$$

for suitable constants  $\alpha_k$ ,  $\gamma_k$ . We may apply induction to conclude that  $b_k = 0$ , for all  $k \ge 0$ . Thus g(t) vanishes to infinite order at t = 0. A similar argument but now using (2.9), allows us to express

$$((n+2)^2-1)a_{n+2} = \sum_{k\leq n+1} \sigma_k b_k + \sum_{k\leq n+1} \delta_k a_k, \quad n\geq 0.$$

We conclude  $a_k = 0$ , for all  $k \ge 0$ . Thus f(t) vanishes to infinite order at t = 0. This then proves our theorem.

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