NODAL GEOMETRY ON RIEMANNIAN MANIFOLDS

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1. Let $M^n$ be a smooth, compact, and connected Riemannian manifold with no boundary. Let $\Delta$ denote the Laplacian on $M$. Let $-\Delta u = \lambda u$, $u$ an eigenfunction with eigenvalue $\lambda$, $\lambda > 1$. Our main theorems are:

Theorem 1 (BMO estimate for $\log |u|$). For $u$, $\lambda$ as above,

$$||\log |u||_{\text{BMO}} \leq c^n \log \lambda,$$

where $c$ is independent of $\lambda$, and depends only on $n$ and $M$.

Theorem 2 (Geometry of nodal domains). Let $u$, $\lambda$ be as above, let $B \subset M$ be any ball, and let $\Omega \subset B$ be any of the connected components of $\{x \in B : u(x) \neq 0\}$. If $\Omega$ intersects the middle half of $B$, then

$$|\Omega| \geq c_1 \lambda^{-2n+2}/(\log \lambda)^{2n} |B|,$$

where $c$ is independent of $\lambda$ and $u$.

Similar theorems have been proved by H. Donnelly and C. Fefferman [1], [2] with $\lambda^n \log \lambda$ replaced by $\lambda^{n+2}/2$ in Theorem 1 and $\lambda^{-2n+2}(\log \lambda)^{-2n}$ replaced by $\lambda^{-2n+2} \log \lambda$ in Theorem 2. Of course, it is obvious that Theorems 1 and 2 above are not best possible.

Theorem 1 is the key to Theorem 2. We deduce Theorem 2 from Theorem 1 by essentially following the arguments in [2] with appropriate modifications in view of the better BMO estimate of Theorem 1.

We shall use the symbols $c$, $c_1$, $c_2$, $c_3$, $c_4$, and $c$ to denote generic constants which are independent of $\lambda$.

2. Before commencing the proof of Theorem 1, we recall two facts from [2]. We state these as Theorem 0.

Theorem 0. Let $M$, $u$, $\lambda$ be as above. Let $B(x, \delta)$ denote the ball centered at $x$ of radius $\delta$. Then

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\[ A \quad \int_{B(x,d(\log 2)^{1/\delta})} |u|^2 \leq c \int_{B(x,\delta)} |u|^2, \]
\[ B \quad \left( \int_{B(x,\delta)} |\nabla u|^2 \right)^{1/2} \leq c \frac{\sqrt{\delta}}{\delta} \left( \int_{B(x,\delta)} |u|^2 \right)^{1/2}. \]

We now begin the proof of Theorem 1.

**Lemma 1.** Let \( u, \lambda \) be as before. Then \( u \) satisfies the reverse-Hölder inequality,
\[ \left( \frac{1}{|B|} \int_{B} |u|^{2n/(n-2)} \right)^{(n-2)/2n} \leq c \sqrt{\lambda} \left( \frac{1}{|B|} \int_{B} |u|^2 \right)^{1/2}. \]

**Proof:** By the Poincaré-Sobolev inequality, for any ball \( B \),
\[ \left( \frac{1}{|B|} \int_{B} |u - \frac{1}{|B|} \int_{B} u|^2 \right)^{(n-2)/2n} \leq c |B|^{1/n} \left( \frac{1}{|B|} \int_{B} |\nabla u|^2 \right)^{1/2}. \]

We now apply Theorem 0(3) to the right side above, to get
\[ \left( \frac{1}{|B|} \int_{B} |u - \frac{1}{|B|} \int_{B} u|^2 \right)^{(n-2)/2n} \leq c \sqrt{\lambda} \left( \frac{1}{|B|} \int_{B} |u|^2 \right)^{1/2}. \]

By use of Minkowski's inequality, Lemma 1 follows. q.e.d.

Our theorem will follow from the lemma stated below.

**Lemma 2.** Suppose \( \nu > 0 \),
\[ (2.1) \quad \int_{B(x,\delta)(1+\sqrt{\lambda})} w \leq c_\delta \int_{B(x,\delta)} w, \]
and
\[ (2.2) \quad \left( \frac{1}{|B|} \int_{B} w^{n/(n-2)} \right)^{(n-2)/n} \leq c_\lambda \frac{1}{|B|} \int_{B} w. \]

Then \( \| \log w \|_{L^{\infty}} \leq c(n)^{1/n} \log \lambda \).

Theorem 1 follows by choosing \( w = |u|^2 \).

**Lemma 3.** Fix any \( \delta > 0 \), with \( \delta < 1/2 \). Let \( \{ B_i \}_{i=1}^\infty \) be any finite collection of balls \( B_1, B_2, \ldots, B_N \) such that
\[ a) \quad \bigcup_{i=1}^N \bigcap_{\delta} (1+\delta)B_i, \]
\[ b) \quad \sum_{i=1}^N \lambda B_i(x) \leq 4^a \beta^{-n}. \]
Proof. Select a ball $B_i$ with the largest radius from the collection $(B_i)_{i \in \mathbb{N}}$. Having selected $B_1, \ldots, B_{i-1}$, select $B_i$ so that $B_i \not\subseteq \bigcup_{j=1}^{i-1} (1+\delta)B_j$ and $B_i$ has the largest possible radius out of the balls in the collection $(B_i)_{i \in \mathbb{N}}$. Our choice of the subcollection $B_1, \ldots, B_i$ clearly satisfies (a). We now prove (b). Let $x \in M \setminus B_i, x = M(x_0)$. By a translation we may suppose $x_0 = 0$. For $x \in \mathbb{R}^n$, define $T_x(y) = x/\|x\|, r > 0$. By our selection procedure $T_x(B_i) \not\subseteq \bigcup_{j=1}^{i-1} (1+\delta)T_x(B_j)$, where $r_i$ denotes the radius of $B_i$. Now $T_x(B_i)$ is also a ball containing the origin, and since $r_i \geq r_i$ for $i < k$, we have $T_x(B_i) \subset T_x(B_k)$, and we conclude

\[ (*) \quad T_x(B_k) \subset \bigcup_{i=1}^{k-1} (1+\delta)T_x(B_i). \]

Let $z_i$ denote the center of $T_x(B_i)$. We note that each of the balls $T_x(B_i)$ has radius 1 and $0 \in \bigcap_{i=1}^{k} T_x(B_i)$. We will show that $|z_i - z_j| \geq \delta$. For if $|z_i - z_j| < \delta$, and assuming $r_j \geq r_i$, we get $(1+\delta)T_x(B_i) \supset T_x(B_j)$, a contradiction of $(*)$. Thus the balls $B(z_i, \delta/2)$ are all disjoint. Furthermore $T_x(B_i) \subset \{x : |x| < 2\}$ for all $i = 1, 2, \ldots, M$. Hence, $M(\delta/2)^n \leq 2^n$, i.e., $M \leq 4^\delta/\delta^n$, and (b) follows.

Lemma 4. Let $w$ satisfy the hypothesis of Lemma 2, let $B$ be a fixed ball, and let $E \subset B$ such that $|E| \geq (1 - c_k\lambda_k^{-1/2})|B|$. Then

\[ \int_{|x| \leq c_k \lambda_k^{-1/2}} f \leq \frac{c_k \lambda_k^{-1/2}}{c_k \lambda_k^{-1/2}} \int_{|x| \leq c_n \lambda_n^{-1/2}} f \leq \frac{c_k \lambda_k^{-1/2}}{c_k \lambda_k^{-1/2}} \int_{|x| \leq c_n \lambda_n^{-1/2}} f. \]

where $c_k = c_k(n, \cdot, c_n)$ and $c_k = c_k(n, \cdot, c_n)$.

Proof. The proof of Lemma 4 rests on an induction on $k$, the inductive step being accomplished by Lemma 3. We verify Lemma 4 for $k = 1$. To do so note that if $|E| \geq (1 - c_\lambda^{-1/2})|B|$ (for some appropriate choice of $c = c(n, \cdot, c_n)$), then $\int_{|x| \leq \frac{1}{2} \lambda_1} f \leq \frac{1}{2} \lambda_1$. To see this, observe $|B \setminus E| \leq c_{\lambda_1^{-1/2}}|B|$. Thus by (2.2),

\[ \int_{|x| \leq \frac{1}{2} \lambda_1} w \leq \left( \int_{|x| \leq \frac{1}{2} \lambda_1} w^{-n(n-1)/2} \right)^{2/n-1} |B \setminus E|^{n/2} \leq \lambda_1^{2/n-1} \int_{|x| \leq \frac{1}{2} \lambda_1} w. \]

We make the choice $\lambda_1^{2/n-1} < 1/2$ and inserting this choice into the inequality above we get $\int_{|x| \leq \frac{1}{2} \lambda_1} w \leq \frac{1}{4} \lambda_1$.

Thus $\int_{|x| \leq \frac{1}{2} \lambda_1} w \leq \frac{1}{4} \lambda_1$.

If $c_n \leq 1$ and $|E| \geq (1 - c_n \lambda_n^{-1/2})|B|$, then $|E| \geq (1 - c_n \lambda_n^{-1/2})|B|$. Therefore $\int_{E \setminus B} w \geq \frac{1}{4} \lambda_1 \lambda_n w \geq \lambda_n^{2/n-1/2} \lambda_1 \lambda_n w$, and we are done with the case $k = 1$. 

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So we assume the statements are valid for \( k = \frac{1}{2} \). Clearly we can assume \(|E| \leq (1 - c_k \lambda^{-n/2})|B|\), or else there is nothing to prove. For each point of density \( x \) of \( E \) we can thus select a ball \( B_x \subseteq B \) such that \( x \in B_x \) and

\[
|B_x \cap E|/|B_x| = 1 - c_k \lambda^{-n/2}.
\]

We apply the covering lemma, Lemma 3, to the \( B_x \)'s with the choice \( \lambda = \lambda_{\frac{1}{2}}, \) and also assume without loss of generality that the \( B_x \) are finitely many. Define \( E_t = (\bigcup_{x \in X} (1 + \lambda^{-1/2})B_x) \cap B \). Then \( E_t \subseteq B \), and to complete the induction we will show that

\[
(2.3) |E| \leq (1 - c_2 \lambda^{-n/2})|E_t|, \quad (2.4) \int_{E} w \geq c_2 \lambda^{-n/2} \int_{E_t} w.
\]

We prove (2.3) first. Now,

\[
|E_t| = |E| + \left| \left( \bigcup_{x \in X} (1 + \lambda^{-1/2})B_x \right) \cap B \setminus E \right| \geq |E| + \left| \left( \bigcup_{x \in X} B_x \right) \cap B \setminus E \right| = |E| + \left| \bigcup_{B \setminus E} B \right|.
\]

By the covering lemma, Lemma 3, the expression above is bounded below by

\[
|E| + c_2 \lambda^{-n/2} \sum |B|, \quad (2.5)
\]

By our selection, \( |B_x \setminus E| = c_k \lambda^{-n/2}|B_x| \), thus (2.5) is bounded below by

\[
|E| + \sum c_k \lambda^{-n} \lambda^{-n/2} \lambda^{-n} = |E| + c_k \lambda^{-n} (1 + \lambda^{-1/2}) \lambda^{-n} \sum (1 + \lambda^{-1/2})B_x.
\]

Set \( c_k \lambda^{-n} (1 + \lambda^{-1/2}) \lambda^{-n} = c_2, \) and note \( c_2 \leq c \). From the expression above we deduce

\[
|E_t| \geq c_2 \lambda^{-n/2} |E| + |E|,
\]

and (2.3) follows.

We now prove (2.4). By (2.1),

\[
\int_{E} w \leq \sum \int_{B \cap E} w \leq c_2 \sum \int_{B} w.
\]

But \( (1 - c_k \lambda^{-n/2})|B| = |E \cap B| \), thus \( \int_{B \cap E} w \leq 2 \int_{B \cap E} w \). Therefore, by Lemma 3,

\[
c_0 \sum \int_{B} w \leq 2c_0 \sum \int_{B \cap E} w \leq 2c_0 \int_{E} w \sum \int_{B \cap E} w \leq 2 \cdot 4^c c_0 \lambda^{-n/2} \int_{E} w.
\]

We select \( c_{1/2} = 2 \cdot 4^c c_0 \) and (2.4) follows. Q.E.D.
We now prove Lemma 2. It will be enough to assume $|B|^{-1} \int_B w = 1$, and to show for $t > 0$

$$\| \{ x \in B : w^{-1}(x) > t \} \| \leq \frac{|B|}{\rho^d \log^c \rho},$$

which is equivalent to showing

$$\| \{ x \in B : w(x) < t \} \| \leq \rho^{d \log^c \rho} |B|.$$ 

Let us denote by $E$ the set $\{ x \in B : w(x) < t \}$. Select $\lambda_0$ such that $|E| \sim (1 - c_2 \lambda^{-n})^{|B|}$. Thus $\lambda_0 \sim c_2^{|x^*(\log |B|)/|E|}$. and so by Lemma 4 and the normalization $|B|^{-1} \int_B w = 1$, we get,

$$|B| = \int_B w \leq (c_2^{|x^*(\log |B|)/|E|}) \int_B w \leq (c_2^{|x^*(\log |B|)/|E|}) |E|.$$ 

Hence,

$$|B|/|E| \leq \rho^{d \log^c \rho} \leq |(B)/|E|| \leq \rho^{d \log^c \rho},$$

and it follows easily that $|E| \leq \rho^{d \log^c \rho} |B|$. q.e.d.

We now prove Theorem 2. We will be brief and only indicate those points in our argument which differ substantially from the argument presented in [2]. Before commencing we note an equivalent formulation of Theorem 1:

Theorem 1'. Let $u, \lambda$ be as before and let $E \subset B$. Then

$$\sup_B |u| \leq \left( c |B|/|E| \right)^{d \log^c \rho} \sup_E |u|,$$

The Lemma stated below is proved in [2] (Lemma 2 there).

Lemma 5. Suppose $\Omega$ is a component of $\{ x \in B(x_0, \delta) : u(x) > 0 \}$ and assume $\Omega \subset B(x_0, \delta)$ and $0 < \delta < \lambda^{-n/2}$. Suppose further $|\Omega|/|B(x_0, \delta)| < \eta^n < \frac{1}{\delta^2}$. Then there is a positive number $r_0$ satisfying

(a) $0 < r_0 < \frac{\eta}{r_0}$,

(b) $|\Omega \cap B(x_0, r_0)| \geq \eta^n r^d_0$,

(c) $\sup_{\Omega \cap B(x_0, r)} |u| \leq \left( \frac{r_0}{r} \right)^{d \log^c \rho} \sup_{B(x_0, \delta)} |u|,$

where $c_4$ depends on the "bounded geometry" estimates.
We also need the estimate below which is Theorem 1 in [1].

**Lemma 6.** Let \( u \), \( x \) be as above. Then

\[
\sup_{B(x, r')} |u| \leq \left( \frac{r'}{r} \right)^{c_1/2} \sup_{B(x, r)} |u|, \quad 0 < r' < r.
\]

Lemma 6 may be deduced from Theorem 0(A) above by an iteration and a use of the mean value inequalities for \( u \).

**Proof of Theorem 2.** By Theorem 1',

\[
\sup_{B(x, r' \cup \Omega)} |u| \leq \left( \frac{c_1 |B(x_0, r_0)|}{|B(x_0, r_0) \cap \Omega|} \right)^{c_1/2} \sup_{B(x_0, r_0 \cup \Omega)} |u| \leq (c_n \eta^{-c_1 \log 2})^{c_1/2} \sup_{B(x_0, r_0 \cup \Omega)} |u|.
\]

The estimate above follows by assuming \( \Omega/|B(x_0, \delta)| \leq \eta^\delta \) and then using Lemma 3(b). We shall arrive at a contradiction for a suitable choice of \( \eta \), and thus for this choice of \( \eta \) we will have \( \Omega/|B(x_0, \delta)| \geq \eta^\delta \) which will prove Theorem 2. Using Lemma 5(c) we get

\[
(c_n \eta^{-c_1 \log 2})^{c_1/2} \sup_{B(x_0, r_0 \cup \Omega)} |u| \leq (c_n \eta^{-c_1 \log 2})^{c_1/2} \sup_{B(x_0, \delta)} |u|.
\]

Thus,

\[
\sup_{B(x_0, r_0 \cup \Omega)} |u| \leq (c_n \eta^{-c_1 \log 2})^{c_1/2} \sup_{B(x_0, \delta)} |u|.
\]

Applying Lemma 6 to the left side above yields

\[
\sup_{B(x_0, \delta)} |u| \leq \left( \frac{\delta}{r_0} \right)^{c_1/2} (c_n \eta^{-c_1 \log 2})^{c_1/2} \sup_{B(x_0, \delta)} |u| \leq \left( \frac{\delta}{r_0} \right)^{c_1/2} (c_n \eta^{-c_1 \log 2})^{c_1/2} \sup_{B(x_0, \delta)} |u|.
\]

Therefore \( (c_n \delta)^{c_1/2} (c_n \eta^{-c_1 \log 2})^{c_1/2} \geq 1 \).

Let us assume that our choice of \( \eta \) is such that \( c_1/\eta - c_\lambda \geq 0 \). So using Lemma 5(a) we see easily

\[
1 \leq \left( \frac{r_0}{\eta} \right)^{c_1/2} (c_n \eta^{-c_1 \log 2})^{c_1/2}.
\]

We now choose \( \eta = n_0^2 \) and \( n_0 = c_\lambda^{c_1 \eta} (\log \lambda)^{-c_1} \). This choice forces \( c_1/\eta - c_\lambda \geq 0 \) and also yields

\[
1 \leq (c_n \eta^{-c_1 \log 2} - c_\lambda \eta^{c_1 \log 2}).
\]
This is a contradiction as \( c_0 \eta_0 < 1 \) for small \( \varepsilon \). Thus, \( |\Omega \cap B(x_0, \delta)| > \varepsilon^2 = c \lambda^{-3n} (\log \lambda)^{-2n} \). We now get rid of the restriction \( \delta < \lambda^{-1/2} \). Suppose \( B \subset M \) is any ball with radius \( r > \lambda^{-1/2} \). Assume \( x_\theta \in \Omega \) belongs also to the middle half of \( B \). We apply our previous conclusion to \( \Omega \cap B(x_\theta, \lambda^{-1/2}) \) to get

\[
|\Omega \cap B| \geq |\Omega \cap B(x_\theta, \lambda^{-1/2})| \geq c \lambda^{-2n} (\log \lambda)^{-2n} |B(x_\theta, \lambda^{-1/2})| = \frac{c \lambda^{-2n} (\log \lambda)^{-2n} |B|}{r^2},
\]

But \( r \leq \eta_0 \) as the manifold is compact and we hence arrive at

\[
|\Omega \cap B| \geq c \lambda^{-2n} \eta_0 (\log \lambda)^{-2n} |B|.
\]

**References**
