LOCAL VERSION OF COURANT'S NODAL DOMAIN THEOREM

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Abstract

Let (M^n, g) be a compact n-dimensional Riemannian manifold without boundary, where $g = (g_{ij})$ is C^1-smooth. Consider the sequence of eigenfunctions u_k of the Laplace operator on M. Let B be a ball on M. We prove that the number of nodal domains of u_k that intersect B is not greater than

$$C_1 \frac{\text{Volume}_g(B)}{\text{Volume}_g(M)} k + C_2 k^{\frac{n-1}{n}},$$

where C_1, C_2 depend on M. The problem of local bounds for the volume and for the number of nodal domains was raised by Donnelly and Fefferman, who also proposed an idea how one can prove such bounds. We combine their idea with two ingredients: the recent sharp Remez type inequality for eigenfunctions and the Landis type growth lemma in narrow domains.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold without boundary and with C^1-smooth g_{ij}. The spectrum of the Laplace operator on M is discrete. There is a sequence of eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$$

that tend to ∞ and a sequence of (real) eigenfunctions u_k such that

$$\Delta_g u_k + \lambda_k u_k = 0.$$

Our enumeration of eigenvalues is non-standard. We start with $\lambda_1 = 0$ and $u_1 = 1$ on M. The nodal domains of u_k are the connected components of $M \setminus Z_{u_k}$, where Z_{u_k} is the zero set of u_k (Z_{u_k} is called the nodal set of u_k). The Courant nodal domain theorem states that the k-th eigenfunction u_k has at most k nodal domains. If the multiplicity of an eigenvalue is more than 1, one may enumerate the eigenfunctions corresponding to this eigenvalue in any order. Our main result is the local version of Courant's theorem.

Theorem 1.1. *Consider a ball $B = \{x \in M : d_g(x, x_0) < r\}$ with center at $x_0 \in M$ and radius $r < r_0(M)$. For any eigenfunction u_k, the number of connected components of $B \setminus Z_{u_k}$ that intersect $\frac{1}{2}B$ (the local number of nodal domains in B) is not greater than*

$$C_1 \frac{\mathrm{Volume}_g(B)}{\mathrm{Volume}_g(M)} k + C_2 k^{\frac{n-1}{n}},$$

where C_1, C_2 depend only on (M, g) and are independent of k and B.

The main issue addressed in this paper is why a nodal domain cannot be very long and narrow. A local bound on the volume of a nodal domain is proved in Theorem 1.2.

The starting point is the idea due to Donnelly and Fefferman [7] that one can use growth estimates for eigenfunctions to prove the local bounds for the volume and for the number of nodal domains. Donnelly and Fefferman showed in [7] that if B is a ball of radius $1/\sqrt{\lambda_k}$, then the number of connected components of $B \setminus Z_{u_k}$ that intersect a twice smaller ball with the same center is at most $C_1(M)k^{C_n}$, where C_n is an explicit constant strictly bigger than 1, which depends on the dimension n only. Donnelly and Fefferman conjectured that C_n can be improved to 1 (like in the Courant theorem). The constant C_n was improved by Chanillo and Muckenhoupt [5] and then by Lu [19], and by Han and Lu [13]. The arguments involved subtle versions of covering lemmas and improvements to BMO bounds for $\log |u_k|$. However the improved C_n was still bigger than 1. In this work we don't use the language of BMO norms, and argue in terms of closely related Remez type inequality. We discuss growth estimates of eigenfunctions in Section 2.1.

In the proof of Theorem 1.1 we combine the idea of Donnelly and Fefferman with two ingredients. The first ingredient is the sharp Remez type inequality for eigenfunctions, which is related to a (resolved) conjecture by Landis on a three balls theorem for wild sets. The second ingredient is Landis type growth lemma for eigenfunctions. The lemma and the conjecture by Landis are independent statements. These two ingredients give two estimates of growth, which however compete with each other. Our local bound on the number of nodal domains on scale $1/\sqrt{\lambda_k}$ is $Ck^{\frac{n-1}{n}}$, which is better than Ck. The bound Ck follows from the idea of Donnelly and Fefferman and the sharp Remez type inequality for eigenfunctions. The improvement from Ck to $Ck^{\frac{n-1}{n}}(\log k)^{n-1}$ is obtained using the local volume bound for nodal domains proved in Section 3.1. The proof uses one more tool due to Landis, which is discussed in Section 2.2. This approach is similar to the one in [20], where estimates on the volume of a connected component of $B_{1/\sqrt{\lambda_k}} \setminus Z(u_k)$ are obtained under the additional assumption that the Riemannian metric is real-analytic. Finally, the sharp bound $Ck^{\frac{n-1}{n}}$ follows by adding the

argument of Fedor Nazarov in Section 3.2. This sharp bound is new for spherical harmonics.

We would like to mention that in dimension two there is an alternative approach due to the elegant idea of A. Eremenko (see Appendix), which however is two-dimensional only.

Our local bound on the number of nodal domains is sharp. When the dimension $n = 2$, one may consider the unit sphere in $\mathbb{R}^3$ and the restriction on S^2 of the homogeneous harmonic polynomial $u(x, y, z) = \Re(x + iy)^m$. It is a spherical harmonic with $\lambda_k \asymp k \asymp m^2$, whose nodal set consists of m circles intersecting at the poles. Let x be the North pole. Then the number of nodal domains intersecting a geodesic ball $B_r(x)$ on S^2 is $\asymp \sqrt{k}$. This example generalizes to higher dimensions, see [20, Proposition 7.1]. On the other hand, consider an eigenfunction on the standard n-dimensional flat torus, $T = (\mathbb{R}/\mathbb{Z})^n$, of the form

$$u(x_1, \ldots, x_n) = \prod_{j=1}^{n} \sin(2\pi m x_j).$$

The corresponding eigenvalue is $\lambda_k \asymp k^{2/n} \asymp m^2$. The nodal domains are m^n cubes on the torus. If B is a ball of radius $r > C/m$, then the number of nodal domains intersecting B is $\asymp (rm)^n \asymp kr^n$.

Notation. By $B_r(x)$ we will denote a ball with center at x and radius r in local coordinates on M. So $B_r(x)$ can be identified with a Euclidean ball and the symbol $|B_r(x)|$ will be used not for the volume with respect to the metric g, but for the Euclidean volume of $B_r(x)$ in local coordinates. Note that these two volumes are comparable. By B we will denote a Euclidean ball in local coordinates on M, whose radius and center are not specified, and B_r will be used for any ball of radius r. By $r_0, c, c_1, c_2, \ldots$ we will denote small constants and by $C, C_1, \ldots$ large constants that may depend on (M, g), but are independent of λ. In local coordinates we always assume that the operator Δ_g is uniformly elliptic with bounded derivatives of the coefficients g_{ij}.

It is more convenient to argue in terms of λ_k rather than in terms of k. The Weyl law states that $\lambda_k^{n/2} \sim ck$. Theorem 1.1 follows from the next theorem.

Theorem 1.2. *Let u be a Laplace eigenfunction with eigenvalue $\lambda > 2$ on a closed Riemannian manifold (M^n, g), where g is C^1-smooth. Given a ball B_{2r} in local coordinates, consider the connected components $\Omega_i \subset B_{2r}$ of B_{2r}/Z_u that intersect B_r. Then for all Ω_i, we have*

$$|\Omega_i| \geq c \min\left(\lambda^{-n/2}, \frac{r^n}{\left(\sqrt{\lambda} \log \lambda\right)^{n-1}} \right).$$

Furthermore the total number of Ω_i is not greater than

$$C\lambda^{n/2}r^n + C\lambda^{\frac{n-1}{2}}.$$

REMARK 1.3. The assumption $\lambda > 2$ is just to ensure $\log \lambda > 0$.

REMARK 1.4. The local bound on the number of nodal domains is consistent (on the heuristic level) with the upper bound in Yau's conjecture (see [**17**, **25**]) on the volume of nodal sets:

$$H^{n-1}(Z_u) \leq C\sqrt{\lambda},$$

where H^{n-1} denotes $(n-1)$ dimensional Hausdorff measure. If Yau's conjecture is true, then one may expect that the average number of nodal domains intersecting a ball of radius $1/\sqrt{\lambda}$ (average with respect to shifting the center of the ball along the manifold) should be constant (see [**17**, **24**]). If we fix r and let λ be very large, then the number of nodal domains in B_r should be not more than $C(r\sqrt{\lambda})^n$. The heuristic obstacle in both questions is the same: why there cannot be many nodal domains that are long, twisting and narrow.

2. The main tools

2.1. Remez type inequality for eigenfunctions. We note that this section is not self-contained. The first main ingredient of this paper is the Remez type inequality for eigenfunctions. The complete proof of this inequality is contained in the lecture notes [**16**, **18**] and we decided not to include it here.

Let u be a Laplace eigenfunction with eigenvalue λ on a closed Riemannian manifold (M^n, g), where g is C^1-smooth in local coordinates. Donnelly and Fefferman [**6**] proved the following growth estimate for Laplace eigenfunctions (in [**6**] it is formulated for C^∞-smooth metrics, but only C^1-smoothness of g_{ij} is actually needed, see Appendix):

$$(1) \qquad \sup_{\frac{1}{2}B} |u| \geq ce^{-C\sqrt{\lambda}} \sup_B |u|$$

for any ball $B \subset M$ and the twice smaller ball $\frac{1}{2}B \subset M$ with the same center.

REMARK 2.1. The recent works [**2**, **8**, **9**] on the distribution of L^2 mass of eigenfunctions imply that for two-dimensional surfaces with negative curvature, one can improve $C\sqrt{\lambda}$ in (1) to $o(\sqrt{\lambda})$.

In [**7**] Donnelly and Fefferman raised a question on the distribution of values of the eigenfunctions. One of the simplest versions of their question is the following. Let C be a large constant and normalize the eigenfunction u so that $\sup_M |u| = 1$. Can $|u|$ be $e^{-C\sqrt{\lambda}}$ small on half of M (is it possible that the set where $|u| < e^{-C\sqrt{\lambda}}$ has measure at least half of M)?

The answer (see [**16**, **18**]) is that it cannot happen if C is large enough (depending on (M, g) and independent of λ). This question is related to the Landis conjecture [**14**, p. 169], which states that if h is a solution to divergence type equation

$$\operatorname{div}(A\nabla h) = 0$$

in a ball B, where A is an elliptic matrix-valued function with smooth coefficients, then the following version of three balls inequality for wild sets holds.

Let K be a closed subset of $\frac{1}{2}B$ with positive volume $|K| > 0$. There are constants $\alpha \in (0, 1)$ and $C > 0$, which depend only on A and $|K|$, such that if $|h| \leq \varepsilon$ in K and $|h| \leq 1$ in B, then

$$|h| \leq C\varepsilon^{\alpha} \quad \text{in } \frac{1}{2}B.$$

The proof of the Landis conjecture is presented in the lecture notes [**16**].

Finally, the Remez type inequality for eigenfunctions (see [**16**, **18**]) states that for any ball B in local coordinates on M and any set $E \subset B$ with positive volume, we have

$$(2) \qquad \sup_E |u| \geq c \left(\frac{c|E|}{|B|} \right)^{C\sqrt{\lambda}} \sup_B |u|,$$

where $c, C > 0$ depend on M.

2.2. Weak maximum principle and a version of Landis' growth lemma.

Lemma 2.2 ([**15**, p. 24]). *Let h be a solution to*

$$\operatorname{div}(A\nabla h) = 0 \quad in \ B_1(0) \subset \mathbb{R}^n,$$

where A is an elliptic matrix valued function. Let $r \in (0, 1)$. Assume that a domain $\Omega \subset B_r(0)$ contains the origin and u is zero on $\partial\Omega \cap B_r(0)$. If

$$\frac{|\Omega|}{|B_r(0)|} \leq c_0,$$

where $c_0 > 0$ is a sufficiently small constant depending on the ellipticity constant of A, then

$$|u(0)| \leq \frac{1}{100} \sup_{\Omega \cap B_r(0)} |u|.$$

We note that the maximum principle does not hold for Laplace eigenfunctions and it creates some obstacles. However for narrow domains a version of the maximum principle holds.

Corollary 2.3. *Let u be the Laplace eigenfunction on M with eigenvalue λ. Consider a ball B_ρ (in local coordinates) with center at x and radius $\rho \leq 1/\sqrt{\lambda}$. Let Ω be a connected component of $B_\rho \setminus Z_u$ that contains x. If*

$$\frac{|\Omega|}{|B_\rho|} \leq c_0,$$

where $c_0 > 0$ is sufficiently small, then

$$|u(x)| \leq \frac{1}{10} \sup_\Omega |u|.$$

Proof. We will use the harmonic extension of eigenfunctions. Consider the function $h(x,t) = u(x) \exp(\sqrt{\lambda} t)$ on $M \times \mathbb{R}$ equipped with the Riemannian metric of the product. A direct computation shows that h is harmonic in $M \times \mathbb{R}$. Consider a ball B on $M \times \mathbb{R}$ of radius ρ with center at $(x, 0)$. Denote $\Omega \times (-\rho, \rho)$ by $\widetilde{\Omega}$. Note that

$$\frac{|\widetilde{\Omega} \cap B|}{|B|} \leq C c_0.$$

If c_0 is small, then by Lemma 2.2

$$|u(x)| = |h(x,0)| \leq \frac{1}{100} \sup_{\widetilde{\Omega}} |h| = \frac{e}{100} \sup_\Omega |u|. \qquad \text{q.e.d.}$$

Definition. We will say that an open set Ω is c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$ if

$$\frac{|\Omega \cap B_{1/\sqrt{\lambda}}|}{|B_{1/\sqrt{\lambda}}|} \leq c_0$$

for any ball $B_{1/\sqrt{\lambda}}$ of radius $1/\sqrt{\lambda}$ with center in $\overline{\Omega}$.

Lemma 2.4 (A version of weak maximum principle)**.** *Let B be an open ball in local coordinates on M with radius $r \leq r_0$, and let u be the Laplace eigenfunction on M with eigenvalue λ. Suppose that an open set $\Omega \subset B$ is c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$ and $u = 0$ on $\partial\Omega \cap B$. If c_0 is sufficiently small (smallness depends on M and is independent of λ), then*

$$\sup_\Omega |u| \leq 3 \sup_{\partial\Omega \cap \partial B} |u|.$$

Proof. WLOG, assume $\sup_{\partial\Omega\cap\partial B}|u| = 1$. Consider the harmonic function $h(x,t) = u(x)\exp(\sqrt{\lambda}t)$ on $M \times \mathbb{R}$. Let $\widetilde{\Omega} = \Omega \times (-\frac{1}{\sqrt{\lambda}}, \frac{1}{\sqrt{\lambda}})$ and let $\eta : \mathbb{R} \to [0,1]$ be a positive continuous function such that

- $\eta = 1$ on $[-\frac{1}{2\sqrt{\lambda}}, \frac{1}{2\sqrt{\lambda}}]$,
- $\eta(t) = 0$ for t with $|t| > \frac{3}{4}\frac{1}{\sqrt{\lambda}}$.

Consider a harmonic function $\tilde{h}$ in $\widetilde{\Omega}$ such that $\tilde{h}(x,t) = h(x,t)\eta(t)$ on $\partial\widetilde{\Omega}$. We note that $\tilde{h}$ vanishes on the top and on the bottom of the boundary of the cylinder $\widetilde{\Omega}$. The zero sets of eigenfunctions satisfy the exterior cone condition (see Appendix), so Ω and $\widetilde{\Omega}$ also satisfy it. Hence the Dirichlet problem in $\widetilde{\Omega}$ is solvable, $\tilde{h}$ exists and by the maximum principle

$$\sup_{\widetilde{\Omega}}|\tilde{h}| = \sup_{\partial\widetilde{\Omega}}|\tilde{h}| = \sup_{\partial\Omega}|h| \sup_{(-\frac{1}{\sqrt{\lambda}}, \frac{1}{\sqrt{\lambda}})} \exp(\sqrt{\lambda}t) = e.$$

Let x_{max} be a point in $\overline{\Omega}$ such that $|u(x_{max})| = \sup_{\Omega}|u|$. We may assume $x_{max} \in \Omega$. Denote $(x_{max}, 0)$ by $\tilde{x}_{max}$. Note that

$$\frac{|\widetilde{\Omega} \cap B_{\frac{1}{2\sqrt{\lambda}}}(\tilde{x}_{max})|}{|B_{\frac{1}{2\sqrt{\lambda}}}(\tilde{x}_{max})|} \leq Cc_0$$

and $h - \tilde{h}$ is zero on $\partial\widetilde{\Omega} \cap \{-\frac{1}{2\sqrt{\lambda}} < t < \frac{1}{2\sqrt{\lambda}}\}$. If c_0 is sufficiently small, then by Lemma 2.2 applied to $h - \tilde{h}$, we have

$$|h(\tilde{x}_{max}) - \tilde{h}(\tilde{x}_{max})| \leq \frac{1}{100} \sup_{\Omega\times\{-\frac{1}{2\sqrt{\lambda}}, \frac{1}{2\sqrt{\lambda}}\}} |h - \tilde{h}|.$$

Since $|\tilde{h}| \leq e$ in $\widetilde{\Omega}$, we have

$$|h(\tilde{x}_{max})| \leq e + \frac{e}{100} + \frac{1}{100} \sup_{\Omega\times\{-\frac{1}{2\sqrt{\lambda}}, \frac{1}{2\sqrt{\lambda}}\}} |h| = e + \frac{e}{100} + \frac{1}{100}e^{1/2}|h(\tilde{x}_{max})|.$$

Thus $|u(x_{max})| = |h(\tilde{x}_{max})| \leq 3$.   q.e.d.

Lemma 2.5 (A version of Landis growth lemma)**.** *Let u be the Laplace eigenfunction on M with eigenvalue λ. Suppose that an open set $\Omega \subset B_r$ is c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$ and $u = 0$ on $\partial\Omega \cap B_r$. If ε is sufficiently small and*

$$\frac{|\Omega|}{|B_r|} \leq \varepsilon^{n-1},$$

then

$$\sup_{B_{r/2}\cap\Omega} |u| \leq e^{-c/\varepsilon} \sup_{\Omega\cap B_r} |u|.$$

Proof. Let x_0 be the center of B_r. By B_ρ we will denote the ball with center at x_0 and radius ρ (in local coordinates on M). Define the monotone function

$$M(\rho) = \sup_{\Omega \cap B_\rho} |u|.$$

Let $x_\rho \in \partial B_\rho \cap \Omega$ be a point such that

$$|u(x_\rho)| = \sup_{\partial B_\rho \cap \Omega} |u|.$$

By the weak maximum principle

$$M(\rho) \le 3|u(x_\rho)|.$$

Define $\tilde{\varepsilon} = A\varepsilon$, where A is a large constant to be specified later. We will show that if $\rho + \tilde{\varepsilon} r < r$ and if

$$\frac{|\Omega \cap B_{\rho + \tilde{\varepsilon} r} \setminus B_{\rho - \tilde{\varepsilon} r}|}{|B_{\tilde{\varepsilon} r}|} \le c_0,$$

then

$$|u(x_\rho)| \le \frac{1}{10} M(\rho + \tilde{\varepsilon} r).$$

Indeed, if $\tilde{\varepsilon} r < 1/\sqrt{\lambda}$ we may apply Corollary 2.3 to $B_{\tilde{\varepsilon} r}(x_\rho)$, and if $\tilde{\varepsilon} r \ge 1/\sqrt{\lambda}$ we may use the fact that Ω is c_0-narrow on scale $1/\sqrt{\lambda}$ and apply Corollary 2.3 to $B_{1/\sqrt{\lambda}}(x_\rho)$. Hence

$$M(\rho + \tilde{\varepsilon} r) \ge 10|u(x_\rho)| \ge 3M(\rho).$$

Consider the numbers $\rho_k = r/2 + k\tilde{\varepsilon} r$ such that $\rho_k \in (r/2, r)$. The total number of such k is $K \asymp 1/\tilde{\varepsilon}$. If at least half of ρ_k satisfy $M(\rho_{k+1}) \ge 3M(\rho_k)$, then we are done. If for at least half of k we have $M(\rho_{k+1}) < 3M(\rho_k)$ (and therefore $|\Omega \cap B_{\rho_{k+1}} \setminus B_{\rho_{k-1}}| \ge c_0 \tilde{\varepsilon}^n |B_r|$), then

$$|\Omega| \ge c_1 \tilde{\varepsilon}^n K |B_r| \ge c_2 \tilde{\varepsilon}^{n-1} |B_r| > \varepsilon^{n-1} |B_r| \quad \text{if } A = \frac{\tilde{\varepsilon}}{\varepsilon} \gg 1.$$

Thus it cannot happen that for at least half of k we have $M(\rho_{k+1}) < 3M(\rho_k)$.   q.e.d.

3. Proof of Theorem 1.2

3.1. Local bound for the volume of nodal domain. Consider any connected component Ω of B_{2r}/Z_u that intersects B_r. We may assume that Ω is c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$, otherwise $|\Omega| > c_1 \lambda^{-n/2}$ and there is nothing to prove. Let $x \in B_r \cap \Omega$ and define $\varepsilon > 0$ by

$$(3) \qquad\qquad |\Omega| = \varepsilon^{n-1} |B_{r/4}(x)|.$$

Our goal is to show that ε cannot be too small in terms of λ. For any $\rho \in (r/4, r]$, we have

$$\frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} < \varepsilon^{n-1}.$$

Since $x \in \Omega$ and Ω is open

$$\frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} = 1 \text{ for sufficiently small } \rho.$$

Consider the interval (r_0, r) such that

$$\frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} < \varepsilon^{n-1}$$

for all $\rho \in (r_0, r)$ and

$$(4) \qquad \frac{|\Omega \cap B_{r_0}(x)|}{|B_{r_0}(x)|} = \varepsilon^{n-1}.$$

Note that $0 < r_0 \le r/4$. We may assume that

$$2^{-k_0-1}r < r_0 \le 2^{-k_0}r$$

for some integer $k_0 \ge 2$ and $\varepsilon < 1/2$. For every integer $k \in [0, k_0]$, we can apply growth Lemma 2.5 in $B_{2^{-k}r}(x)$ to obtain

$$(5) \qquad \sup_{\Omega \cap B_{2^{-j-1}r}(x)} |u| \le e^{-\frac{c_2}{\varepsilon}} \sup_{\Omega \cap B_{2^{-j}r}(x)} |u|,$$

and by multiplying (5) for $j = 0, 1, ..., k_0 - 1$, we have

$$\sup_{\Omega \cap B_{r_0}(x)} |u| \le \left(\frac{r_0}{r}\right)^{\frac{c_3}{\varepsilon}} \sup_{\Omega \cap B_r(x)} |u|$$

and therefore

$$(6) \qquad \sup_{\Omega \cap B_{r_0}(x)} |u| \le \left(\frac{r_0}{r}\right)^{\frac{c_3}{\varepsilon}} \sup_{B_r(x)} |u|.$$

By the Remez-type inequality (2) and by the choice of r_0 described by (4), we have

$$\sup_{\Omega \cap B_{r_0}(x)} |u| \ge c \left(c \frac{|\Omega \cap B_{r_0}(x)|}{|B_r(x)|} \right)^{C\sqrt{\lambda}} \sup_{B_r(x)} |u| =$$

$$= c^{C\sqrt{\lambda}+1} \left(\frac{|\Omega \cap B_{r_0}(x)|}{|B_{r_0}(x)|} \right)^{C\sqrt{\lambda}} \left(\frac{|B_{r_0}(x)|}{|B_r(x)|} \right)^{C\sqrt{\lambda}} \sup_{B_r(x)} |u| \ge$$

$$\ge c_4^{C\sqrt{\lambda}+1} \varepsilon^{(n-1)C\sqrt{\lambda}} \left(\frac{r_0}{r}\right)^{C\sqrt{\lambda}} \sup_{B_r(x)} |u|.$$

So

$$\left(\frac{r_0}{r}\right)^{\frac{c_3}{\varepsilon}} \ge c_4^{C\sqrt{\lambda}+1} \varepsilon^{(n-1)C\sqrt{\lambda}} \left(\frac{r_0}{r}\right)^{C\sqrt{\lambda}}.$$

Recall that $\frac{r_0}{r} \le \frac{1}{4}$. If $\frac{1}{\varepsilon} \gg \sqrt{\lambda}$, then

$$\left(\frac{r_0}{r}\right)^{\frac{c_3}{2\varepsilon}} < c_4^{C\sqrt{\lambda}+1} \left(\frac{r_0}{r}\right)^{C\sqrt{\lambda}}$$

and therefore

$$\left(\frac{1}{4}\right)^{\frac{c_3}{2\varepsilon}} \geq \varepsilon^{(n-1)C\sqrt{\lambda}},$$

which yields

$$\varepsilon \log \frac{1}{\varepsilon} \geq \frac{c_4}{\sqrt{\lambda}}.$$

The last inequality implies

$$\varepsilon \geq c_5 \frac{1}{\sqrt{\lambda} \log \lambda} \quad \text{if } \lambda > 2.$$

Thus by (3)

$$|\Omega| \geq c_6 \frac{r^n}{\left(\sqrt{\lambda} \log \lambda\right)^{n-1}}.$$

3.2. Local bound for the number of nodal domains. We follow the argument suggested by Fedor Nazarov. All misprints, bad notation and presentation are on the authors.

Denote by K the number of connected components Ω_i of $B_{2r} \setminus Z_u$ that intersect a twice smaller ball B_r with the same center and are c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$. The number of Ω_i, which are not c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$ (and therefore have volume at least $c_1 \lambda^{-n/2}$) is bounded by $C\lambda^{n/2}r^n$. We want to show that K cannot be too large and we argue by assuming the contrary and arrive to a contradiction. Let $\varepsilon = a/\sqrt{\lambda}$ and assume that $K > 2^{n+2}/\varepsilon^{n-1}$. The number $a > 0$ will be a small constant depending only on (M, g). Each Ω_i intersects B_r, so there is a point x_i in $\Omega_i \cap B_r$. Since

$$\sum |\Omega_i \cap B_r(x_i)| \leq |B_{2r}|,$$

there are at least $\frac{3}{4}K$ domains Ω_i such that $\frac{|\Omega_i \cap B_r(x_i)|}{|B_{2r}|} \leq \frac{4}{K}$ and therefore

$$(7) \qquad \frac{|\Omega_i \cap B_r(x_i)|}{|B_r(x_i)|} \leq \varepsilon^{n-1}.$$

Denote by S_m the set of domains Ω_i, which are c_0-narrow on scale $\frac{1}{\sqrt{\lambda}}$ and

$$(8) \qquad \frac{|\Omega_i \cap B_{2^{-j}r}(x_i)|}{|B_{2^{-j}r}(x_i)|} \leq \varepsilon^{n-1} \quad \text{for } j \in \{0, 1, \ldots, m\}.$$

The number of elements in S_0 satisfies $|S_0| > K - K/4$ by (7). Fix $m \geq 0$, and consider only $\Omega_i \in S_m$ and forget about the rest of Ω_i. Let $\rho = r2^{-m}$. Using (8) and applying growth Lemma 2.5 for $B_{2^{-j}r}(x_i)$ for $j \in \{0, 1, \ldots, m\}$, we have

$$(9) \qquad \frac{\sup_{\Omega_i \cap B_{\rho/2}(x_i)} |u|}{\sup_{B_{2r}} |u|} \leq \left(\frac{\rho/2}{2r}\right)^{c/\varepsilon}.$$

By Remez type inequality ([2]) applied to $\cup(\Omega_i \cap B_\rho(x_i))$, where the union is taken over $\Omega_i \in S_m$, we have

$$c\left(c\frac{\sum|\Omega_i \cap B_{\rho/2}(x_i)|}{|B_{2r}|}\right)^{C\sqrt{\lambda}} \leq \frac{\sup_{\cup(\Omega_i \cap B_{\rho/2}(x_i))}|u|}{\sup_{B_{2r}}|u|}.$$

So

$$c\left(c\frac{\sum|\Omega_i \cap B_{\rho/2}(x_i)|}{|B_{2r}|}\right)^{C\sqrt{\lambda}} \leq \left(\frac{\rho}{4r}\right)^{c/\varepsilon}$$

and therefore

$$c\left(c\frac{\sum|\Omega_i \cap B_{\rho/2}(x_i)|}{|B_{\rho/2}|}\right)^{C\sqrt{\lambda}} \leq \left(\frac{\rho}{4r}\right)^{c/\varepsilon - Cn\sqrt{\lambda}}.$$

Since $\varepsilon = a/\sqrt{\lambda} \ll 1/\sqrt{\lambda}$, we have

$$\frac{\sum|\Omega_i \cap B_{\rho/2}(x_i)|}{|B_{\rho/2}|} \leq C_2\left(\frac{\rho}{4r}\right)^{c_3/a} \leq 4^{-m-2}.$$

So the number of $\Omega_i \in S_m$ with

$$\frac{|\Omega_i \cap B_{\rho/2}(x_i)|}{|B_{\rho/2}(x_i)|} > \varepsilon^{n-1}$$

is not greater than $K4^{-m-2}$. So $|S_m \setminus S_{m+1}| \leq K4^{-m-2}$. Note that $S_{m+1} \subset S_m$. Thus for each S_m the number of $\Omega_i \in S_m$ is at least

$$K - \sum_j K4^{-j} \geq K/2.$$

However

$$\frac{|\Omega_i \cap B_\rho(x_i)|}{|B_\rho(x_i)|} = 1 \text{ for sufficiently small } \rho.$$

The number of Ω_i is finite by the volume bound proved in Section 3.1. So S_m should be empty for m sufficiently large and the contradiction is obtained.

Appendix

In this section we gather several facts that we used in the proof, and which are well-known to specialists, but the assumption on the metric in the references is C^∞-smoothness in place of C^1-smoothness (or it is not clear whether it is C^1 or C^∞). Unfortunately, the exposition is not self-contained, and the complete proof uses several exterior nontrivial facts.

Let u be a non-zero solution in $B_1(0)$ to the second order linear elliptic equation:

$$\sum_{i,j} \partial_i(a_{ij}\partial_j u) + \sum_j b_j \partial_j u + cu = 0,$$

where (a_{ij}) is an elliptic matrix with Lipschitz coefficients, and b_j, c are bounded.

Garofalo and Lin [**11**, **12**] proved a powerful monotonicity formula for solutions of such equations, which implies the three balls inequality:

$$(10) \qquad \sup_{B} |u| \le C \left(\sup_{\frac{1}{2}B} |u| \right)^{\alpha} \left(\sup_{2B} |u| \right)^{1-\alpha},$$

where B is ball such that $2B \subset B_1(0)$, and $C > 0$, $\alpha \in (0,1)$ depend only on the ellipticity constants, on the Lipschitz constant, and on the L^∞ norm of the lower order terms.

Cone condition for nodal sets. Assume that (a_{ij}) is the identity matrix at the origin. The monotonicity formula of Garofalo and Lin implies (after some work, see [**21**]) that u can be well approximated by a harmonic polynomial near the origin. The story of approximation properties of elliptic equations at a point with higher order zero goes back to the works of Bers [**1**], and Caffarelli and Friedman [**3**]. The formal statement is that there is a non-zero homogeneous harmonic polynomial P of some degree n, such that

$$u(x) = P(x) + o(|x|^n).$$

For homogeneous polynomials one can inscribe a cone with vertex at 0 in every nodal domain so that $|P(x)| \ge c|x|^n$, $c > 0$, in this cone. As the corollary of this fact one may conclude the following.

Let u be the Laplace eigenfunction on (M, g), where $g = (g_{ij})$ is C^1-smooth. Let Ω be a nodal domain of u and a point $x_0 \in \partial \Omega$. Then there is an open truncated cone in Ω with vertex at x_0. It implies the exterior cone condition as well. We refer to [**10**] for the question and estimates (in terms of the eigenvalue) on how large the spatial angle of the cone is.

Doubling condition for eigenfunctions due to Donnelly and Fefferman. Consider a ball $\widetilde{B}$ in local coordinates on M. Then a version of three balls inequality holds for eigenfunctions:

$$(11) \qquad \sup_{B} |u| \le C e^{C\sqrt{\lambda}} \left(\sup_{\frac{1}{2}B} |u| \right)^{\alpha} \left(\sup_{2B} |u| \right)^{1-\alpha} \qquad \text{if } 2B \subset \widetilde{B}.$$

Inequality (11) can be reduced to (10) by the harmonic extension, i.e. considering a harmonic function $h(x,t) = u(x)\exp(\sqrt{\lambda}t)$ on $M \times R$. Donnelly and Fefferman proved (11) in a different way by Carleman inequalities. They used compactness of M to show that iterations of (11) (see the argument in [**6**], page 162, after formula (1.5)) imply

$$(12) \qquad \sup_{\frac{1}{2}B} |u| \ge c e^{-C\sqrt{\lambda}} \sup_{B} |u|$$

for any ball B on M.

Eremenko's lemma ([**20**])**.** *Let u be a harmonic function on the plane. The doubling index of u in a ball B is defined by*

$$N = \log \frac{\sup_{2B} |u|}{\sup_B |u|}.$$

Then every nodal domain of u in $2B$ that intersects B has area at least $\frac{c}{N+C}$ and therefore the number of nodal domains that intersect B is not greater than $CN + C$.

The proof of Eremenko's lemma is an elegant two-dimensional argument that is using the fact that $\log |\nabla u|$ is subharmonic. The log-subharmonicity property is not true in higher dimensions.

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