

The Singular CR Yamabe Problem and Hausdorff Dimension

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Huber [7] in 1957 showed that a complete open surface whose negative part of the Gaussian curvature is integrable is conformally a closed surface with finitely many points deleted. This result has inspired a lot of extension in various geometric settings.. In conformal geometry, the celebrated work of Schoen-Yau [10] studied the developing map of a locally spherical manifold of positive Yamabe class. They showed that the developing map is one-to-one into the sphere, and the complement of the image has small Hausdorff dimension. As a consequence, the original manifold is a quotient of a Kleinian group. It is our purpose to consider further extensions to the CR setting.

We begin with a pseudo-hermitian manifold whose conformal Laplacian is a positive operator, a CR invariant notion. The CR conformal Laplacian

$$L_\theta = -\Delta_b + c_n R,$$

is paired with the Webster scalar curvature in the associated curvature equation for conformal change of contact form $\hat{\theta} = u^2\theta$,

$$L_\theta u = \tilde{R}u^{Q+2/Q-2},$$

where $Q = 2n+2$ denotes the homogeneous dimension. Similar to the conformal case, the analysis of this equation becomes non-trivial when the operator is positive. On the other hand, when the operator is positive, much is known about the geometry and analysis of the manifold. For example, in the case where the CR structure is locally spherical and the CR Yamabe invariant is positive, the development map can be shown to be injective and the complement of its image is small [3]. In the following, we localize the analysis to a neighborhood of infinity and derive an analytic result about the size at infinity, assuming the scalar curvature is largely positive, assuming suitable size control of the negative part of the scalar curvature.

Theorem 0.1. *Let (M^n, θ, J) be a compact pseudo-hermitian manifold. Let $Q = 2n + 2$ be the homogeneous dimension. Let $S \subseteq M$ be a closed set. Let D be a bounded open neighborhood of S . Let*

$$\hat{\theta} = u^{\frac{4}{Q-2}}\theta$$

be a conformal contact form on $D \setminus S$ such that the corresponding pseudo-Hermitian metric is complete near S . Assume

$$R^-(\hat{\theta}) \in L^{\frac{2Q}{Q+2}}(D \setminus S, dV_{\theta}) \cap L^p(D \setminus S, dV_{\theta})$$

for some $p > Q/2$, where $dV_{\theta} = \theta \wedge (d\theta)^n$; and $R^-(\hat{\theta}) = \max\{-R(\hat{\theta}), 0\}$, where $R(\hat{\theta})$ is the scalar curvature coming from the contact form $\hat{\theta}$. Then the Hausdorff dimension satisfies

$$\dim_{\mathcal{H}}(S) \leq \frac{Q-2}{2}.$$

The notion of Hausdorff measure and dimension is that in [1] (cf. Chapter 13).

Remark: In a recent paper, Guidi-Maalaoui-Martino [6] construct contact forms of constant positive scalar curvature on $S^{2n+1} \setminus S^{2k+1}$ if $k < (n-2)/2$. This is a partial converse to our result.

Lemma 0.2. *Let μ be a non-negative Radon measure on a complete pseudo-Hermitian manifold (M, θ, J) . Assume μ is absolutely continuous with respect to the measure $\theta \wedge (d\theta)^n$. Define,*

$$G_d^{\infty} = \{x \in M : \overline{\lim}_{r \rightarrow 0} \mu(B_r(x)) r^{-d} = +\infty\}$$

for any $d \in [0, Q)$. Then,

$$\mathcal{H}_d(G_d^{\infty}) = 0,$$

where \mathcal{H}_d is the Hausdorff measure of dimension d .

Proof. We adapt a proof from the book by W. Ziemer [11]. Assume for every $x \in A$, where A is some Borel set, we have

$$\overline{\lim}_{r \rightarrow 0} \frac{\mu(B_r(x))}{r^d} > \lambda > 0$$

Then $\exists C = C(d)$; such that

$$\mu(A) \geq C\lambda\mathcal{H}_d(A) \tag{0.1}$$

where $\mathcal{H}_d(A)$ is the d -dimensional Hausdorff measure of A . We may assume $\mu(A) < +\infty$ or else there is nothing to prove. Choose $\varepsilon > 0$. Let us choose U open, $U \supset A$, $\mu(U) < +\infty$, which we can as μ is a Radon measure, and A a Borel set. Let \mathcal{G} be a family of closed balls $B_r(x) \subset U$ such that for $x \in A$

$$\frac{\mu(B_r(x))}{r^d} > \lambda, 0 < r < \varepsilon/2.$$

We may extract from \mathcal{G} a disjoint sub-family $\mathcal{F} \subset \mathcal{G}$ such that if \mathcal{F}^* is a finite sub-family of \mathcal{F} , one has

$$A \subseteq \cup \{B, B \in \mathcal{F}^*\} \cup \{\widehat{B} : \widehat{B} \in \mathcal{F} \setminus \mathcal{F}^*\}$$

where \hat{B} is a ball concentric with B , and radius 5 times that of B . This is proved as Cor. 1.3.3 [11] and the proof is unchanged for us. Thus by definition

$$\mathcal{H}_d^{5\varepsilon}(A) \leq \sum_{B \in \mathcal{F}^*} \left(\frac{\delta(B)}{2} \right)^d + C5^d \sum_{B \in \mathcal{F} \setminus \mathcal{F}^*} \left(\frac{\delta(B)}{2} \right)^d$$

where $\delta(B)$ is the diameter of B . Since $\mathcal{F} \subset \mathcal{G}$ and \mathcal{F} is disjoint, we have

$$\begin{aligned} \sum_{B \in \mathcal{F}} \left(\frac{\delta(B)}{2} \right)^d &\leq C\lambda^{-1} \sum_{B \in \mathcal{F}} \mu(B) \\ &\leq C\lambda^{-1} \mu(U) < +\infty. \end{aligned}$$

Since the remainder

$$C5^d \sum_{B \in \mathcal{F} \setminus \mathcal{F}^*} \left(\frac{\delta(B)}{2} \right)^d$$

can be made arbitrarily small, we get

$$\mathcal{H}_d^{5\varepsilon}(A) \leq C\lambda^{-1} \mu(U).$$

By the definition of Hausdorff measure we obtain (0.1).

For some fixed point $x_0 \in M$, define for $\rho(x, x_0)$ the pseudo-hermitian metric,

$$A_m = \left\{ x : \rho(x, x_0) < m, \overline{\lim}_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^d} > \frac{1}{m} \right\}.$$

Then A_m is bounded and so $\mu(A_m) < +\infty$. Using (0.1)

$$\mu(A_m) \geq \frac{c}{m} \mathcal{H}_d(A_m).$$

Thus, $\mathcal{H}_d(A_m) < +\infty$, and we conclude that $\mathcal{H}_Q(A_m) = 0$, since $d < Q$, that is the $\theta \wedge (\theta)^n$ measure of A_m is zero. As μ is a Radon measure, the absolute continuity of μ , with respect to $\theta \wedge (d\theta)^n$, implies $\mu(A_m) = 0$. Hence, by (0.1) again, we have $\mathcal{H}_d(A_m) = 0$. Since

$$A = \cup A_m$$

we conclude $\mathcal{H}_d(A) = 0$. Our lemma follows by setting $A = G_d^\infty$. \square

We next prove an intermediate result on the Hausdorff dimension preparatory to proving our main theorem. To do so, and for the main theorem, we need:

Lemma 0.3. *Let (M, θ, J) be a compact pseudo-hermitian manifold. Assume the background scalar curvature satisfies $R(\theta) \leq 0$. Let $S \subset M$ be a closed set and D an open neighborhood of S , which is small, i.e. $\text{vol}(D) < \varepsilon$. Let $\hat{\theta} = u^{\frac{4}{Q-2}} \theta$ be a conformal contact form in $D \setminus S$. Assume $R^-(\hat{\theta}) \in L^p(D \setminus S, dV_{\hat{\theta}})$, $p > \frac{Q}{2}$ and $R^-(\hat{\theta}) = \max\{-R(\hat{\theta}), 0\}$. Then if the pseudo-hermitian manifold $(M, \hat{\theta}, J)$ is complete, we have*

$$u(x) \rightarrow +\infty, \text{ as } x \rightarrow S.$$

Remark: The hypothesis $R(\theta) \leq 0$, can be dropped. However, in our application, we can arrange $R(\theta) \leq 0$ by taking connected sums of M with manifolds with very negative Yamabe constants, so we assume this as it simplifies the proof. We use the Moser iteration argument in [2] and ideas in [8].

Proof. We first observe for $f \in C_0^\infty(D \setminus S)$, we have

$$\left(\int_{D \setminus S} |f|^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} \right)^{\frac{Q-2}{Q}} \leq c \left\{ \int_{D \setminus S} \left(|\nabla_b^{\hat{\theta}} f|^2 + R(\hat{\theta})|f|^2 \right) dV_{\hat{\theta}} \right\} \quad (0.2)$$

for some $c > 0$. The expression above is a conformal invariant and so we may replace $\hat{\theta}$ by θ and we simply show (0.2) for θ . By the Sobolev inequality

$$\left(\int_{D \setminus S} |f|^{\frac{2Q}{Q-2}} dV_{\theta} \right)^{\frac{Q-2}{Q}} \leq c \int_{D \setminus S} |\nabla_b^{\theta} f|^2 dV_{\theta} \quad (0.3)$$

We write the right side as

$$\int_{D \setminus S} \left(|\nabla_b^{\theta} f|^2 + R(\theta)|f|^2 \right) dV_{\theta} - \int_{D \setminus S} R(\theta)|f|^2 dV_{\theta},$$

where

$$\begin{aligned} \left(\int_{D \setminus S} R(\theta)|f|^2 dV_{\theta} \right) &\leq \left(\int_{D \setminus S} |f|^{\frac{2Q}{Q-2}} dV_{\theta} \right)^{\frac{Q-2}{Q}} \left(\int_{D \setminus S} |R|^{\frac{Q}{2}} dV_{\theta} \right)^{2/Q} \\ &\leq \varepsilon \left(\int_{D \setminus S} |f|^{\frac{2Q}{Q-2}} dV_{\theta} \right)^{\frac{Q-2}{Q}}. \end{aligned}$$

Thus from (0.3), we get

$$\left(\int_{D \setminus S} |f|^{\frac{2Q}{Q-2}} dV_{\theta} \right)^{\frac{Q-2}{Q}} \leq c \int_{D \setminus S} \left(|\nabla_b^{\theta} f|^2 + R|f|^2 \right) dV_{\theta} + \varepsilon \left(\int_{D \setminus S} |f|^{\frac{2Q}{Q-2}} dV_{\theta} \right)^{\frac{Q-2}{Q}}$$

From the last inequality (0.2) follows. Next write $\hat{\theta} = e^{\frac{(Q-2)}{2}w}\theta$. We re-write this as $\theta = e^{-\frac{(Q-2)}{2}w}\hat{\theta} = v^{\frac{4}{Q-2}}\hat{\theta}$. Thus,

$$-c_n \Delta_b^{\hat{\theta}} v + R(\hat{\theta})v = R(\theta)v^{\frac{Q-2}{Q+2}} \leq 0.$$

This gives

$$-c_u \Delta_b^{\hat{\theta}} v + R^+(\hat{\theta})v \leq R^-(\hat{\theta})v \quad (0.4)$$

Our goal is to apply the Moser iteration method to (0.4) with (0.2), and conclude a local L^∞ bound on v , where $v^{\frac{4}{Q-2}} = e^{-\frac{Q-2}{2}w}$. The L^∞ bound we want to prove is

$$\|v\|_{L^\infty(B_{1/2}^{\hat{\theta}}(x_0))} \leq c \left(\int_{B_1^{\hat{\theta}}} |v|^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} \right)^{\frac{Q-2}{2Q}} \quad (0.5)$$

with the normalization

$$\int_{B_1^{\hat{\theta}}(x_0)} dV_{\hat{\theta}} = 1. \quad (0.6)$$

Here $B_1^{\hat{\theta}}$ is a unit ball in the metric of $\hat{\theta}$. Now

$$\begin{aligned} |v|^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} &= e^{-\frac{(Q-2)^2}{8} \cdot \frac{2Q}{Q-2} w} e^{\frac{(Q-2)}{2} \cdot \frac{Q}{2} w} \theta \wedge (d\theta)^n \\ &= \theta \wedge (d\theta)^n = dV_{\theta}. \end{aligned}$$

So from (0.5)

$$\|v\|_{L^\infty(B_{1/2}^{\hat{\theta}}(x_0))} \leq c \left(\int_{B_1^{\hat{\theta}}(x_0)} dV_{\theta} \right)^{\frac{Q-2}{2Q}}. \quad (0.7)$$

But $\hat{\theta}$ gives rise to a complete metric. So if $x_0 \rightarrow S$, the volume of the ball $B_1^{\hat{\theta}}(x_0)$, as measured in the background metric θ , must go to zero. So as $x \rightarrow S$, the right hand side of (0.7) must go to zero. Thus $v(x) \rightarrow 0$ as $x \rightarrow S$. This means $w(x) \rightarrow +\infty$ as $x \rightarrow S$. This proves our lemma if (0.5) holds. To prove (0.5), we multiply (0.4) by $\eta^2 v^\beta$ where $0 \leq \eta \leq 1$, is a cutoff function supported in an open set $U \subset D \setminus S$, which we can ultimately take to be a unit ball. Then we integrate with respect to $dV_{\hat{\theta}}$ with the normalization (0.6). We get after integration by parts, and Cauchy-Schwartz:

$$\begin{aligned} &c_n \beta \int_U \eta^2 |\nabla_b^\theta v|^2 v^{\beta-1} dV_{\hat{\theta}} + \int_U R^+(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} \\ &\leq \int_U R^-(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} + \frac{c}{\beta} \int_U |\nabla \eta|^2 v^{\beta+1} dV_{\hat{\theta}} \end{aligned}$$

This yields,

$$\begin{aligned} &\frac{4c_n \beta}{(\beta+1)^2} \int_U \eta^2 \left| \nabla_b^{\hat{\theta}} v^{\frac{\beta+1}{2}} \right|^2 dV_{\hat{\theta}} + \int_U R^+(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} \\ &\leq \int_U R^-(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} + \frac{c}{\beta} \int_U |\nabla \eta|^2 v^{\beta+1} dV_{\hat{\theta}}. \quad (0.8) \end{aligned}$$

Now

$$\nabla_b^{\hat{\theta}} \left(\eta v^{\frac{\beta+1}{2}} \right) = \eta \nabla_b^{\hat{\theta}} v^{\frac{\beta+1}{2}} + v^{\frac{\beta+1}{2}} \nabla_b^{\hat{\theta}} \eta.$$

Using above in (0.8) we get after rearranging terms,

$$\begin{aligned} &\frac{c_n \beta}{(\beta+1)^2} \int_U \left| \nabla_b^{\hat{\theta}} \left(\eta v^{\frac{\beta+1}{2}} \right) \right|^2 dV_{\hat{\theta}} + \int_U R^+(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} \\ &\leq \int_U R^-(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} + \frac{c\beta}{(\beta+1)^2} \int_U \left| \nabla_b^{\hat{\theta}} \eta \right|^2 v^{\beta+1} dV_{\hat{\theta}} \end{aligned}$$

Multiplying by $\frac{(\beta+1)^2}{\beta}$, we get using $\beta \geq 1$,

$$\begin{aligned} & \int_U \left| \nabla_b^{\hat{\theta}} \left(\eta v^{\frac{\beta+1}{2}} \right) \right|^2 dV_{\hat{\theta}} + \frac{(\beta+1)^2}{\beta^2} \int_U R^+(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} \leq \\ & \frac{(\beta+1)^2}{\beta} \int_U R^-(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} + \frac{(\beta+1)^2}{\beta^2} \int_U \left| \nabla_b^{\hat{\theta}} \eta \right|^2 v^{\beta+1} dV_{\hat{\theta}} \end{aligned} \quad (0.9)$$

Since

$$\begin{aligned} & \int_U \left| \nabla_b^{\hat{\theta}} \left(\eta v^{\frac{\beta+1}{2}} \right) \right|^2 dV_{\hat{\theta}} + \int_U R(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}} \\ & \leq \int_U \left| \nabla_b^{\hat{\theta}} \left(\eta v^{\frac{\beta+1}{2}} \right) \right|^2 dV_{\hat{\theta}} + \int_U R^+(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}}, \end{aligned}$$

on applying (0.2) to (0.9), we have

$$\begin{aligned} \left(\int_U \left| \eta v^{\frac{\beta+1}{2}} \right|^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} \right)^{\frac{Q-2}{Q}} & \leq c \int_U \left| \nabla_b^{\hat{\theta}} \eta \right|^2 v^{\beta+1} dV_{\hat{\theta}} \\ & + \frac{c(\beta+1)^2}{\beta} \int_U R^-(\hat{\theta}) \eta^2 v^{\beta+1} dV_{\hat{\theta}}. \end{aligned} \quad (0.10)$$

We need a gain in L^p norms to iterate via Moser, and for this we now use our hypothesis that $R^-(\hat{\theta}) \in L^p(D \setminus S, dV_{\hat{\theta}})$, for some $p > Q/2$. Using Hölder on the second term on the right in (0.10)

$$\leq c\beta \left(\int_U R^-(\hat{\theta})^p dV_{\hat{\theta}} \right)^{\frac{1}{p}} \left(\int_U (\eta^2 v^{\beta+1})^q dV_{\hat{\theta}} \right)^{1/q}.$$

Since $p > \frac{Q}{2}$, and $\frac{1}{p} + \frac{1}{q} = 1$, it follows that $q < \frac{Q}{Q-2}$. Thus by hypothesis, we have

$$\begin{aligned} \left[\int_U \left(\eta v^{\frac{\beta+1}{2}} \right)^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} \right]^{\frac{Q-2}{Q}} & \leq c\beta \left[\int_U \left(\eta v^{\frac{\beta+1}{2}} \right)^{2q} dV_{\hat{\theta}} \right]^{\frac{1}{q}} \\ & + \int_U \left| \nabla_b^{\hat{\theta}} \eta \right|^2 v^{\beta+1} dV_{\hat{\theta}}. \end{aligned} \quad (0.11)$$

Now by Hölder's inequality and the normalization in (0.6),

$$\int_U \left| \nabla_b^{\hat{\theta}} \eta \right|^2 v^{\beta+1} dV_{\hat{\theta}} \leq \left[\int_U \left(\left| \nabla_b^{\hat{\theta}} \eta \right| v^{\frac{\beta+1}{2}} \right)^{2q} dV_{\hat{\theta}} \right]^{1/q}.$$

Using the above in (0.11) and taking the square root we obtain

$$\left[\int_U \left(\eta v^{\frac{\beta+1}{2}} \right)^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} \right]^{\frac{Q-2}{2Q}} \leq c\beta^{1/2} \left[\int_U \left[\left(\eta + \left| \nabla_b^{\hat{\theta}} \eta \right| \right) v^{\frac{\beta+1}{2}} \right]^{2q} dV_{\hat{\theta}} \right]^{\frac{1}{2q}}$$

with $2q < \frac{2Q}{Q-2}$. The gain in exponent allows the iteration to proceed to conclude (0.5). \square

We use the previous lemma to prove:

Lemma 0.4. *Let (M, θ, J) be a compact pseudo-hermitian manifold with scalar curvature $R \leq 0$. Let $S \subseteq M$ be a closed set and D an open neighborhood of S . Let $\hat{\theta} = u^{\frac{4}{Q-2}} \theta$ be geodesically complete in $D \setminus S$. Assume that*

$$R^-(\hat{\theta}) \in L^{\frac{2Q}{Q+2}}(D \setminus S, dV_{\hat{\theta}}) \cap L^p(D \setminus S, dV_{\hat{\theta}})$$

$$\text{for some } p > \frac{Q}{2}; R^-(\hat{\theta}) = \max\{-R(\hat{\theta}), 0\}.$$

Then, the 2-capacity of S is zero and the Hausdorff dimension of S satisfies

$$\dim_{\mathcal{H}}(S) \leq Q - 2.$$

Here the Hausdorff dimension and capacity is in the sense of [1] (also see [5]).

Proof. Recall the scalar curvature equation

$$-c_u \Delta_b u = -Ru + R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} - R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} \quad (0.12)$$

in $D \setminus S$. Now by Hölder,

$$\int_{D \setminus S} R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} dV_{\hat{\theta}} \leq \left(\int_{D \setminus S} R^-(\hat{\theta})^{\frac{2Q}{Q+2}} u^{\frac{2Q}{Q-2}} dV_{\hat{\theta}} \right)^{\frac{Q+2}{2Q}} \text{Vol}(D)^{\frac{Q-2}{2Q}} < +\infty, \quad (0.13)$$

where $\text{Vol}(D)$ is with respect to θ . The expression in (0.13) is finite by hypothesis. Next, under the hypothesis of our Lemma 0.3, we have

$$u(x) \rightarrow +\infty, x \rightarrow S.$$

Using this fact above, define the test function

$$u_{\alpha, \beta} = \begin{cases} \beta, & u \geq \alpha + \beta. \\ u - \alpha, & u < \alpha + \beta. \end{cases}$$

and set

$$\phi_{\alpha, \beta} = u_{\alpha, \beta} - \beta + \beta(1 - \eta)$$

where η is a cut-off function that is equal to 1 in a neighborhood of S . Define

$$\Sigma_{\alpha} = \{x \in D : u(x) > \alpha\}.$$

We note that if β is sufficiently large, then $u_{\alpha, \beta} \in (0, \beta]$ in Σ_{α} , and

$$\phi_{\alpha, \beta} = 0 \text{ on } \{x \in D : u(x) = \alpha\} \cup \{x \in D : u \geq \alpha + \beta\}.$$

Note that

$$\nabla_b \phi_{\alpha, \beta} = \nabla_b u_{\alpha, \beta} - \beta \nabla_b \eta, \quad \nabla_b u = \nabla_b u_{\alpha, \beta} \text{ where } \nabla_b u_{\alpha, \beta} \neq 0.$$

We multiply (0.12) by $\phi_{\alpha,\beta}$ and get after integration by parts, using as dV_θ the volume element $\theta \wedge (d\theta)^n$,

$$\begin{aligned} c_n \int_{\Sigma_\alpha} \nabla_b u \cdot \nabla_b \phi_{\alpha,\beta} dV_\theta &= \int_{\Sigma_\alpha} \phi_{\alpha,\beta} \left(-Ru + R(\hat{\theta})u^{\frac{Q+2}{Q-2}} \right) dV_\theta. \\ c_n \int_{\Sigma_\alpha} |\nabla_b u_{\alpha,\beta}|^2 dV_\theta &= \beta \int_{\Sigma_\alpha} \left(c_n \nabla_b u \cdot \nabla_b \eta + \left(-Ru + R(\hat{\theta})u^{\frac{Q+2}{Q-2}} \right) (1 - \eta) \right) dV_\theta \\ &\quad - \int_{\Sigma_\alpha} \left(-Ru + R(\hat{\theta})u^{\frac{Q+2}{Q-2}} \right) (\beta - u_{\alpha,\beta}) dV_\theta + \int_{\Sigma_\alpha} R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} (\beta - u_{\alpha,\beta}) dV_\theta. \end{aligned} \quad (0.14)$$

Using (0.13), $R \leq 0$, $u_{\alpha,\beta} \in (0, \beta]$ and the location of the support of $1 - \eta$, we easily see the right hand side of (0.14) is bounded by $C\beta$, where C depends on α, η but not on β . Thus by Lemma 0.3,

$$\int_{\Sigma_\alpha} \left| \nabla_b \left(\frac{u_{\alpha,\beta}}{\beta} \right) \right|^2 \leq c/\beta \rightarrow 0,$$

if $\beta \rightarrow \infty$. Thus $\text{Cap}_2(S, D) = 0$. Using Cor. 4.6 in [5], we conclude the Hausdorff dimension of S satisfies $\dim_{\mathcal{H}} S \leq Q - 2$. \square

Our goal is now to improve Lemma 0.4. To do so we first prove:

Lemma 0.5. *Given that u satisfies (0.12) the scalar curvature equation and the hypothesis on $R^-(\hat{\theta})$ of Lemma 0.4 is satisfied and $R \leq 0$, where R is the scalar curvature of the background contact form θ . Then if $\hat{\theta} = u^{4/(Q-2)}\theta$ is geodesically complete near $S \subseteq M, S$ closed, we have that $-\Delta_b u$ is a Radon measure and $-\Delta_b u|_S \geq 0$. We restrict our attention in this lemma to D , an open neighborhood of S .*

Proof. We begin by proving that if

$$-c_n \Delta_b u = -Ru + R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} - R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} = f$$

in $D \setminus S$, then $f \in L^1(D)$. Consider the function.

$$\alpha_s(t) = \begin{cases} t, & t \leq s, \\ \text{increasing}, & t \in [s, 10s], \\ 2s, & t \geq 10s. \end{cases}$$

We observe that $\alpha'_s \in [0, 1]$, $\alpha''_s \leq 0$, since $\alpha(s)$ can be selected to be concave. Then

$$-\Delta_b \alpha_s(u) = -\alpha''_s(u) |\nabla u|^2 + \alpha'_s(u) (-\Delta_b u) \quad (0.15)$$

Then for $s > \max\{u(x) : x \in \partial D\}$, and by recalling that $\Delta_b = \sum_{i=1}^{2n} X_i^2$, we have by Green's theorem and where ν is the outward unit normal,

$$\int_D -\Delta_b \alpha_s(u) dV_\theta = -\sum_{i=1}^{2n} \int_{\partial D} \langle \nu, X_i \rangle \alpha_s(u) d\sigma$$

On ∂D , $\alpha_s(u) = u$ by construction of $\alpha_s(t)$ and choice of s . Using (0.15) above, we get

$$\begin{aligned} -\sum_{i=1}^{2n} c_n \int_{\partial D} \langle \nu, X_i \rangle u d\sigma &= -\int_D c_n \alpha_s''(u) |\nabla_b u|^2 \\ &\quad + \int_D \alpha_s'(u) \left[-Ru - R(\hat{\theta})u^{\frac{Q+2}{Q-2}} + R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} \right] dV_\theta. \end{aligned} \quad (0.16)$$

Note that $\alpha_s'' \leq 0$ by construction, and $-Ru \geq 0$ by hypothesis. So from (0.16),

$$\int_D \alpha_s'(u) \left(R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} - Ru \right) \leq -c_n \sum_{i=1}^{2n} \int_{\partial D} \langle \nu, X_i \rangle u d\sigma + \int_D R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} dV_\theta.$$

By hypothesis we saw using the Hölder inequality in the previous lemma,

$$\int_D R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} dV_\theta < +\infty.$$

Thus, from Fatou's lemma, letting $s \rightarrow +\infty$,

$$\int_D \left(R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} - Ru \right) dV_\theta < +\infty. \quad (0.17)$$

This proves $f \in L^1(D)$. Next, from (0.15) again, since $\alpha_s'' \leq 0$,

$$\begin{aligned} \int_D |\Delta_b \alpha_s(u)| &\leq -c_n \int_D \alpha_s''(u) |\nabla_b u|^2 \\ &\quad + \int_D \alpha_s' \left[-Ru + R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} + R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} \right] dV_\theta \end{aligned}$$

Using the identity (0.16) again and replacing the integral $c_n \int_D -\alpha_s''(u) |\nabla_b u|^2$ above via (0.16),

$$\begin{aligned} \int_D |\Delta_b \alpha_s(u)| &\leq c_n \sum_{i=1}^{2n} \int_{\partial D} \langle \nu, X_i \rangle u d\sigma \\ &\quad + 2 \int_D \alpha_s'(u) \left(R^+(\hat{\theta})u^{\frac{Q+2}{Q-2}} - Ru \right) dV_\theta + 2 \int_D R^-(\hat{\theta})u^{\frac{Q+2}{Q-2}} dV_\theta \end{aligned}$$

Using (0.17) and (0.13) again, we get

$$\int_D |\Delta_b \alpha_s(u)| < +\infty.$$

This for $\phi \in C^0(D)$,

$$\begin{aligned} |\langle -\Delta_b \alpha_s(u), \phi \rangle| &= \left| \int_D \Delta \alpha_s(u) \phi \right| \\ &\leq \|\phi\|_{C^0(D)} \int_D |\Delta_b \alpha_s(u)|. \end{aligned}$$

We now have,

$$\begin{aligned} \langle \Delta_b u, \phi \rangle &= \int_D \nabla_b u \cdot \nabla_b \phi \, dV_\theta, \quad \phi \in C'_0(D) \\ &= \lim_{s \rightarrow \infty} \int_D \alpha'_s(u) \nabla_b \cdot \nabla_b \phi \, dV_\theta \\ &= \lim_{s \rightarrow \infty} \langle -\Delta_b \alpha_s(u), \phi \rangle. \end{aligned}$$

Thus

$$|\langle -\Delta_b u, \phi \rangle| \leq c \left\{ \sum_{i=1}^{2n} \left| \int_{\partial D} \langle \nu, X_i \rangle u \, d\sigma \right| + \int_D R^-(\hat{\theta})^{\frac{Q+2}{Q-2}} u^{\frac{Q+2}{Q-2}} \, dV_\theta \right\} \|\phi\|_{C^0(D)}.$$

Thus $-\Delta_b u$ is a Radon measure. We now show that it is a non-negative Radon measure. Let $\phi \geq 0, \phi \in C_0^\infty(D)$, then we have from above,

$$\begin{aligned} c_n \langle -\Delta u, \phi \rangle &= \lim_{s \rightarrow \infty} c_n \int_D (-\Delta \alpha_s(u)) \phi \, dV_\theta \\ &= \lim_{s \rightarrow \infty} \int_D -c_n \alpha''_s(u) |\nabla_b u|^2 \phi \, dV_\theta \\ &\quad + \lim_{s \rightarrow \infty} \int_D \alpha'_s(u) (-Ru) \phi \, dV_\theta \\ &\quad + \lim_{s \rightarrow \infty} \int_D \alpha'_s(u) R(\hat{\theta}) u^{\frac{Q+2}{Q-2}} \phi \, dV_\theta \end{aligned}$$

We can drop the first term on the right above since $\alpha''_s \leq 0$ and so

$$c_n \langle -\Delta u, \phi \rangle \geq - \left(\int_{\text{supp } \phi \setminus S} | -Ru + R(\hat{\theta}) u^{\frac{Q+2}{Q-2}} | \, dV_\theta \right)$$

The integral on the right hand side is finite from above, and so since

$$\int_{\text{supp } \phi \setminus S} dV_\theta \rightarrow 0,$$

with $\|\phi\|_{C^0(D)} = 1$, we get

$$- \int_{\text{supp } \phi \setminus S} | -Ru + R(\hat{\theta}) u^{\frac{Q+2}{Q-2}} | \, dV_\theta \rightarrow 0$$

and so $-\Delta_b u|_S \geq 0$. □

Remark: We note that $-\Delta u = f, f \in L^1(D)$, and so our Radon measure is non-negative and absolutely continues with respect to the volume measure dV_θ . Next from Lemma 0.4, $\dim_{\mathcal{H}}(S) \leq Q - 2 < Q$. Thus Lemma 0.2 applies.

Let $\Omega \subseteq M$ be a bounded, open set. Let A be a non-negative Radon measure on M . We define for ρ the metric distance,

$$R_\mu^{\alpha, \Omega}(x) = \int_\Omega \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}}; 0 < \alpha \leq Q.$$

Let $E \subseteq \Omega$. We define

Definition 0.6. The capacity

$$C^\alpha(E, \Omega) = \inf \{ \mu(\Omega) \mid R_\mu^{\alpha, \Omega}(x) \geq 1, \forall x \in E \}.$$

Let us look at a coordinate patch U of M , and there we may assume that $U \cap M$ is Heisenberg with a dilation structure δ_λ . We assume $E \subset \Omega \subset U$. Then

Lemma 0.7. Set $A_\lambda = \{\delta_\lambda(x) : x \in A\}$. Then

$$C^\alpha(E_\lambda, \Omega_\lambda) = \lambda^{Q-\alpha} C^\alpha(E, \Omega), \alpha \in (1, Q].$$

Proof. For a non-negative Radon measure μ , define a Radon measure μ^* associated to μ for all $A_\lambda \subset \Omega_\lambda$ by setting:

$$\mu^*(A_\lambda) = \mu(A).$$

Then

$$\begin{aligned} R_{\mu^*}^{\alpha, \Omega_\lambda}(\delta_\lambda(x)) &= \int_{\Omega_\lambda} \frac{d\mu^*(y)}{\rho(\delta_\lambda(x), y)^{Q-\alpha}} \\ &= \int_\Omega \frac{d\mu(y)}{\rho(\delta_\lambda(x), \delta_\lambda(y))^{Q-\alpha}} = \lambda^{\alpha-Q} \int_\Omega \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} \\ &= \lambda^{\alpha-Q} R_\mu^{\alpha, \Omega}(x). \end{aligned}$$

So,

$$\begin{aligned} C^\alpha(E_\lambda, \Omega_\lambda) &= \inf \{ \mu^*(\Omega_\lambda) : R_{\mu^*}^{\alpha, \Omega_\lambda}(\lambda x) \geq 1, \forall x \in E \} \\ &= \inf \{ \mu(\Omega) : \lambda^{\alpha-Q} R_\mu^{\alpha, \Omega}(x) \geq 1, \forall x \in E \} \\ &= \lambda^{Q-\alpha} \inf \{ \lambda^{\alpha-Q} \mu(\Omega) : \lambda^{\alpha-Q} R_\mu^{\alpha, \Omega}(x) \geq 1, \forall x \in E \} \\ &= \lambda^{Q-\alpha} C^\alpha(E, \Omega). \quad \square \end{aligned}$$

Lemma 0.8. Let $x_0 \in M$ be a fixed point. Consider a metric ball $B_r(x_0)$. In this metric ball we can assume M is Heisenberg if $r \leq r_0$ for r_0 small, and so we have scaling like in the previous lemma. Then for $\alpha > 0$,

$$C^\alpha(\partial B_2(x_0), B_{5r}(x_0)) = c_0 > 0.$$

Proof. Let us take the surface measure on $\partial B_r(x_0)$, given by $d\sigma = \theta \wedge e^1 \wedge \dots \wedge e^{2n-1}$, see [4]. Then for $x \in \partial B_r(x_0)$

$$\begin{aligned} R_\sigma^{\alpha, B}(x) &\geq m_0 > 0, \quad B = B_{5r}(x_0). \\ &= \int_{\partial B_r(x_0)} \frac{d\sigma(y)}{\rho(x, y)^{Q-\alpha}} \geq m_0 > 0. \end{aligned}$$

Thus by definition:

$$C^\alpha(\partial B_r, B) \leq \frac{1}{m_0} < +\infty.$$

Thus our capacity is finite. We want to prove that it is non-zero. We claim that we can find $p \in \partial B_r(x_0)$ such that for any finite non-negative measure μ on $B_{5r}(x_0)$, we have

$$\mu(B_s(p) \cap B_{5r}(x_0)) \leq c_n \mu(B_{5r}(x_0)) s^{Q-1} \quad (0.18)$$

for all $s > 0$. Note that if s is sufficiently large $s \geq 100$, then (0.18) is obvious. Let us normalize $\mu(B_{5r}(x_0)) = 1$. Assume (0.18) is false. Then for each $q \in \partial B_r(x_0)$, $\exists r_q > 0$ such that

$$\mu(B_{r_q}(q) \cap B_{5r}(x_0)) \geq c_0 r_q^{Q-1}. \quad (0.19)$$

Then $\{B_{r_q}(q)\}$ is a covering of $\partial B_r(x_0)$. Using Vitali's lemma, we may find a disjoint sub-family of balls

$$\{B_{r_{q_1}}(q_1), \dots, B_{r_{q_k}}(q_k)\}, q_j \in \partial B_r(x_0),$$

such that

$$\{B_{3r_{q_1}}(q_1), \dots, B_{3r_{q_k}}(q_k)\}$$

covers $\partial B_r(x_0)$. The metric family of balls satisfy the doubling property (cf. [9]). By disjointness and by (0.19),

$$c_0 \sum_{j=1}^k r_{q_j}^{Q-1} \leq \sum_{j=1}^k \mu(B_{r_{q_j}}(q_j) \cap B_{5r}(x_0)) \leq \mu(B_{5r}(x_0)) = 1. \quad (0.20)$$

Now consider the surface measure introduced in [4]. As the situation is locally Heisenberg, we observe $\theta \wedge e^1 \wedge \dots \wedge e^{2n-1} = d\sigma$ scales under Heisenberg scaling δ_λ ,

$$\delta_\lambda(\theta \wedge e^1 \wedge \dots \wedge e^{2n-1}) = \lambda^{Q-1} \theta \wedge e^1 \wedge \dots \wedge e^{2n-1}. \quad (0.21)$$

So

$$\begin{aligned} |\partial B_r(x_0)| &= \int_{\partial B_r(x_0)} d\sigma \\ &\leq \sum_{j=1}^k |B_{3r_{q_j}}(q_j) \cap \partial B_r(x_0)| \end{aligned}$$

Using (0.21), we get

$$|\partial B_r(x_0)| \leq C \sum r_{q_j}^{Q-1} |\partial B_r(x_0)|.$$

We have a contradiction to (0.20) if c_0 is large enough. So our claim is proved. We can now prove our lemma. For $\alpha \in (1, Q)$, and p chosen to satisfy (0.18),

$$\begin{aligned} R_\mu^{\alpha, B_{5r}(x_0)}(p) &= \int \frac{d\mu(y)}{\rho(p, y)^{Q-\alpha}} \\ &= \sum_{j=-\infty}^4 \int_{\{2^j < \rho(p, y) \leq 2^{j+1}\} \cap B_{5r}(x_0)} \frac{d\mu}{\rho(p, y)^{Q-\alpha}} + c_0 \mu(B_{5r}) \\ &= \sum_{j=-\infty}^4 2^{-j(Q-\alpha)} \mu(B_{2^{j+1}}(p) \cap B_{5r}(x_0)) + c_0 \mu(B_{5r}(x_0)). \end{aligned}$$

Using (0.18), we get

$$R_\mu^{\alpha, B_{5r}(x_0)}(p) \leq c \mu(B_{5r}(x_0)) \sum_{j=-\infty}^4 2^{j(\alpha-1)} + c_0 \mu(B_{5r}(x_0)).$$

Since $\alpha > 1$,

$$R_\mu^{\alpha, B_{5r}(x_0)}(p) \leq c_1 \mu(B_{5r}(x_0)), \quad c_1 > 0.$$

Thus by definition, and recalling $\mu(B_{5r}(x_0)) = 1$, we have

$$C^\alpha(\partial B_r, B_{5r}(x_0)) \geq \frac{1}{c_1} > 0. \quad \square$$

Definition 0.9. Let $E \subset M, p \in M$. We say E is α -thin at p for $\alpha \in (1, Q)$ if

$$\sum_{i \geq 1} \frac{C^\alpha(E \cap \omega_i^\delta(p), \Omega_i^\delta(p))}{C^\alpha(\partial B_{2^{-i}\delta}(p), B_{2^{-i+1}\delta}(p))} < +\infty,$$

where $\delta > 0$ is small and

$$\begin{aligned} \omega_i^\delta(p) &= \{x : \rho(x, p) \in [2^{-i}\delta, 2^{-i+1}\delta]\}, \\ \Omega_i^\delta(p) &= \{x : \rho(x, p) \in (2^{-i-1}\delta, 2^{-i+2}\delta)\}. \end{aligned}$$

Thus $\Omega_i^\delta(p) \supset \omega_i^\delta(p)$ is the doubled annuli.

The last lemma that we need is

Lemma 0.10. Let (M^n, θ, J) be a complete pseudo-hermitian manifold, and let μ be a finite Radon measure absolutely continuous with respect to $\theta \wedge (d\theta)^n$ on a bounded domain $G \subset M^n$. Let $S \subset G$, with S compact. Assume that the Hausdorff dimension satisfies the condition $\dim_{\mathcal{H}} S > d$. Let $\alpha \in (1, Q)$ and $\alpha < Q - \alpha$. Then there exists $p \in S$ and $E \subset S$, that is α -thin at p , such that

$$\int_G \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} \leq \frac{C}{\rho(x, p)^{Q-\alpha-d}}$$

for some constant C , and for all $x \in B_\delta(p) \setminus E$, for some $\delta > 0$.

Remark: In Lemma 0.5, we showed that under the hypotheses, we have

$$-\Delta_b u = f = \mu,$$

and $f \in L^1(D)$, so in our application μ is indeed a finite Radon measure that is absolutely continuous with respect to $\theta \wedge (d\theta)^n$.

Proof. By hypothesis, there is some $\varepsilon > 0$, such

$$\mathcal{H}_{d+\varepsilon}(S) = +\infty,$$

since the Hausdorff dimension of S is greater than d . Thus, by Lemma 0.2, there is a point $p \in S$ such that

$$\lim_{r \rightarrow 0} \frac{\mu(B_r(p))}{r^{d+\varepsilon}} \leq C < +\infty.$$

Thus,

$$\mu(B_r(p)) \leq Cr^{d+\varepsilon}, \quad (0.22)$$

where r is small. Next

$$R_\mu^{\alpha, \Omega}(x) = \int_\Omega \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}}. \quad (0.23)$$

Let i_0 be fixed and $i \geq i_0 + 2$. We decompose (0.23)

$$\begin{aligned} &= \int_{\Omega \setminus B_{2^{-i_0+2}\delta}} \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} + \int_{B_{2^{-i_0+2}\delta} \setminus \Omega} \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} + \int_{\Omega_\delta^i} \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}}. \quad (0.24) \\ &= \text{I} + \text{II} + \text{III} \end{aligned}$$

where we recall

$$\begin{aligned} \Omega_\delta^i &= \{y : \rho(p, y) \in (2^{-i-1}\delta, 2^{-i+2}\delta)\}, \\ \omega_\delta^i &= \{x : \rho(p, x) \in (2^{-i}\delta, 2^{-i+1}\delta)\}. \end{aligned}$$

Clearly for $x \in \omega_i^\delta \subseteq \Omega$,

$$\begin{aligned} I &\leq \frac{C\mu(\Omega)}{(2^{-i_0}\delta)^{Q-\alpha}} \frac{C(2^{-i}\delta)^{Q-\alpha-d}}{(2^{-i_0}\delta)^{Q-\alpha}} \frac{\mu(\Omega)}{\rho(p, x)^{Q-\alpha-d}} \\ &\leq \frac{C(\alpha, \delta, i_0, Q)}{\rho(p, x)^{Q-\alpha-d}}, \end{aligned} \quad (0.25)$$

since $d < Q - \alpha$. We estimate II by cutting in annuli. We sub-divide II further into two pieces; an inner piece and an outer piece. The inner piece of II is estimated by

$$\int_{B_{2^{-i-1}\delta}(p)} \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} \leq c \frac{\mu(B_{2^{-i-1}\delta})}{(2^{-i-1}\delta)^{Q-\alpha}}. \quad (0.26)$$

The outer piece of II we cut into annuli

$$\begin{aligned} & \sum_{k=i_0}^{i-1} \int_{B_{2^{-k+2}\delta}(p) \setminus B_{2^{-k+1}\delta}(p)} \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} \\ & \leq c \sum_{k=i_0}^{i-1} \frac{\mu(B_{2^{-k+2}\delta}(p))}{(2^{-k}\delta)^{Q-\alpha}}. \end{aligned}$$

Using (0.22) with $\varepsilon = 0$ and (0.26).

$$\begin{aligned} \text{II} & \leq c \sum_{k=i_0}^{i-1} \frac{(2^{-k}\delta)^d}{(2^{-k}\delta)^{Q-\alpha}} + \frac{c}{(2^{-i-1}\delta)^{Q-\alpha-d}}. \\ & \leq \frac{c}{\rho(x, p)^{Q-\alpha-d}}. \end{aligned} \tag{0.27}$$

We estimate III. Define

$$E_i^\lambda = \left\{ x \in \omega_\delta^i : \int_{\Omega_i^\delta} \frac{d\mu(y)}{\rho(x, y)^{Q-\alpha}} \geq \lambda 2^{i(Q-\alpha-d)} \right\},$$

where $\lambda > 0$ is fixed. By the definition of capacity (μ is a “trial” measure)

$$C^\alpha(E_i^\lambda, \Omega_i^\lambda) \leq \frac{\mu(\Omega_i^\delta)}{\lambda 2^{i(Q-\alpha-d)}}.$$

By (0.22),

$$C^\alpha(E_i^\lambda, \Omega_i^\lambda) \leq c \frac{(2^{-i+2}\delta)^{d+\varepsilon}}{\lambda 2^{i(Q-\alpha-d)}}.$$

By Lemmas 0.7 and 0.8, we have

$$C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta}) = c (2^{-i}\delta)^{Q-\alpha}, c > 0.$$

Thus

$$\sum_{i \geq i_0} \frac{C^\alpha(E_i^\lambda, \Omega_i^\delta)}{C^\alpha(\partial B_{2^{-i}\delta}, B_{2^{-i+1}\delta})} \leq \frac{c}{\lambda} \sum_{i \geq i_0} 2^{-i\varepsilon} < \infty.$$

Thus if $x \notin \bigcup_i E_i^\lambda = E$, and $x \in \omega_\delta^i$

$$\text{III} \leq \lambda \cdot 2^{i(Q-\alpha-d)} \leq \frac{c}{\rho(x, p)^{Q-\alpha-d}}. \tag{0.28}$$

Collecting the estimates for I, II, III, ie. Eqs. (0.25), (0.27) and (0.28)

$$\int_\Omega \frac{d\mu}{\rho(x, y)^{Q-\alpha}} \leq \frac{c}{\rho(x, p)^{Q-\alpha-d}},$$

if $x \notin E = \bigcup_i E_i^\lambda$. □

We now prove the main theorem.

Proof of Theorem 0.1. By taking a connected sum of M with a manifold of very negative Yamabe constant M_1 , $M\#M_1$ can be assumed to have background scalar curvature $R \leq 0$. By Lemma 0.5, for $S \subseteq M$ and S closed,

$$-\Delta_b u = \mu \text{ in } D,$$

where D is some neighborhood of S , and μ is a finite, Radon measure that is absolutely continuous with respect to $\theta \wedge (d\theta)^n$. Thus we have, via Green's representation,

$$u = \int_D G(x, y) d\mu(y) + h(x), \quad (0.29)$$

where h is harmonic with respect to Δ_b . Now

$$0 \leq G(x, y) \leq \frac{C}{\rho(x, y)^{Q-2}}.$$

(cf. [1, 9]). Thus from (0.29),

$$\begin{aligned} u(x) &\leq \int_D \frac{d\mu(y)}{\rho(x, y)^{Q-2}} + h(x) \\ &= R_\mu^{2,D}(x) + h(x). \end{aligned} \quad (0.30)$$

By contradiction assume Hausdorff $\dim_{\mathcal{H}}(S) = d > \frac{Q-2}{2}$. We know that $d \leq Q-2$. Hence, from Lemma 0.10 and (0.30), there exists $p \in S$ such that

$$u(x) \leq \frac{c}{\rho(x, p)^{Q-2-d}}.$$

We obtain,

$$u(x)^{\frac{2}{Q-2}} \leq \frac{c}{\rho(x, p)^{2\frac{Q-2-d}{Q-2}}}. \quad (0.31)$$

Set

$$\tau = \frac{2(Q-2-d)}{Q-2} = 2 - \frac{2d}{Q-2} < 1.$$

Thus the curve length can be computed using (0.31) and using the metric

$$\hat{\theta} = u^{\frac{4}{Q-2}} \theta$$

we obtain the metric distance to the singular set is bounded by

$$\leq \int_\gamma \frac{ds}{s^\tau} < +\infty.$$

This shows $M \setminus S$ is not complete, which is a contradiction. This ends the proof. \square

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