Sharp bounds on the Nusselt number in Rayleigh-Bénard convection and a bilinear estimate by Coifman-Meyer

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Abstract

We prove a conjecture in fluid dynamics concerning optimal bounds for heat transportation in the infinite Prandtl number limit and for large Rayleigh number Ra, predicted in [20] and [24]. Due to a *maximum principle* property for the temperature exploited by Constantin-Doering and Otto-Seis, this amounts to showing a-priori bounds for horizontally-periodic solutions of a fourth-order equation in a strip of large width. While there have been recent nearly-optimal results up to logarithmic divergences in Ra, we prove here optimal bounds employing Fourier analysis, integral representations, and a bilinear estimate due to Coifman and Meyer which uses the Carleson measure characterization of BMO functions by Fefferman.

Key Words: Rayleigh-Bénard convection, Nusselt number, representation formulas, Carleson measures, Fourier Analysis.

1 Introduction

Thermal convection processes are important to understand, since they appear in many applications such as engineering, meteorology, geophysics, astrophysics and oceanography. Rayleigh-Bérnard's model is paradigmatic for natural convection: one considers a fluid layer between rigid plates which are heated from below and cooled from above.

We will use the convention from [26], presenting non-dimensionalized equation though the scaling

$$x \to \mathrm{Ra}^{-\frac{1}{3}} x; \qquad t \mapsto \mathrm{Ra}^{-\frac{2}{3}} t; \qquad \mathbf{u} \to \mathrm{Ra}^{-\frac{1}{3}} \mathbf{u}; \qquad p \mapsto \mathrm{Ra}^{\frac{2}{3}} p,$$

where Ra stands for the Rayleigh number, $\mathbf{u}$ for the fluid velocity and p for the pressure. The physical system is considered in the domain $[0, L]^2 \times [0, 1]$ with periodicity in the horizontal variables, and through the above scaling the new variables are defined in the set $[0, HL]^2 \times [0, H]$ where $H = \mathrm{Ra}^{\frac{1}{3}}$, the height of the container, becomes the relevant parameter. When H is small, the heat transfer happens almost entirely by conduction. However, as H increases, such a steady state becomes unstable and bifurcation of solutions occurs, see e.g. Chapter 6 in [2]. One then starts observing *convection rolls*, where the heated fluid, becoming lighter, moves upward and then returns near the bottom after cooling down driven by gravity, see some description in [6]. When H becomes larger, the formation of thin conducting layers near the boundary is observed, while in the bulk of the container heat transfer mostly happens via convection after a cascade of bifurcations, generating chaotic dynamics and fully developed turbulence ([22]).

We will consider in this paper the Boussinesq approximation in the infinite Prandtl-number limit, namely when the viscosity of the fluid is much bigger than the thermal diffusion. For the derivation of the governing equations of the model we still refer to [6]. Writing the velocity as $\mathbf{u} = u\,\mathbf{e}_1 + v\,\mathbf{e}_2 + w\,\mathbf{e}_3$,

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and denoting the temperature by T, the infinite Prandtl-number limit of the Boussinesq equations in a container of $\mathbb{R}^3$ is given by

$$\partial_t T + \mathbf{u} \cdot \nabla T - \Delta T = 0; \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0; \tag{2}$$

$$-\Delta \mathbf{u} + \nabla p = T\mathbf{e}_3. \tag{3}$$

We supplement these equations with periodicity in $(x_1, x_2) \in [-HL/2, HL/2] \times [-HL/2, HL/2]$, and Dirichlet boundary conditions, namely

$$T = \begin{cases} 1 & \text{for } x_3 = 0; \\ 0 & \text{for } x_3 = H, \end{cases} \qquad \mathbf{u} = 0 \quad \text{for } x_3 = 0, H. \tag{4}$$

Using the incompressibility condition (2), one can eliminate the pressure term and obtain for the $\mathbf{e}_3$-component of the velocity w the equations

$$\Delta^2 w = -\Delta_h T; \tag{5}$$

$$w = \partial_{x_3} w = 0 \qquad \text{for } x_3 = 0, H, \tag{6}$$

see [26], where Δ_h denotes the *horizontal Laplacian* in the variables x_1, x_2. In the sequel we will always assume $H \gg 1$.

The following quantity measures the average vertical heat flux in terms of the steady state conduction heat flux, see formula (7) in [26].

Definition 1.1 *The* Nusselt number Nu *is defined as*

$$\mathrm{Nu} = \frac{1}{H} \int_0^H \langle (\mathbf{u}T - \nabla T) \cdot \mathbf{e}_3 \rangle \, dx_3,$$

where

$$\langle h \rangle (x_3) = \limsup_{t_0 \to +\infty} \frac{1}{t_0} \int_0^{t_0} \frac{1}{(HL)^2} \int_{[-HL/2, HL/2]^2} h(t, x_1, x_2, x_3) \, dx_1 dx_2 dt.$$

Thus the angle brackets denote the horizontal space and time averages. Our main result in this paper is the following theorem.

Theorem 1.2 *There exists a universal constant $C_0 > 0$ such that as $H \to +\infty$*

$$\mathrm{Nu} \leq C_0.$$

In most of the physics literature the dimensionless parameter appears as a coefficient in (3):

$$-\Delta \mathbf{u} + \nabla p = \mathrm{Ra}\, T\mathbf{e}_3,$$

Formula (2.6) in [14] defines the Nusselt number using this scaling, which we refer to as $\mathrm{Nu}_{\mathrm{phys}}$. The theorem above corresponds in the physics language to the following result.

Corollary 1.3 *In the limit $\mathrm{Ra} \to +\infty$ the following bound holds for the Nusselt number*

$$\mathrm{Nu}_{\mathrm{phys}} \leq C_0 (\mathrm{Ra})^{\frac{1}{3}}.$$

We now provide a brief history of this problem. Malkus ([24]) and Howard ([20]) gave compelling boundary-layer arguments predicting the scaling $\mathrm{Nu}_{\mathrm{phys}} \simeq (\mathrm{Ra})^{\frac{1}{3}}$ (we are back to the notation in Definition 1.1). The derivation of upper bounds in convective heat transport begins with seminal papers by Howard ([19], [21]) and further work by Busse ([3], [4]). Some sub-optimal bounds, in terms of higher powers of Ra, were proven in [19], [13]. A powerful and beautiful new idea was introduced to the subject by Constantin and Doering, [9], called the *background field method*. To a background temperature one associates a quadratic form that needs to be non-negative definite: one then tries to minimize a suitable integral on such backgrounds (see also [11], [13], [12]). This method allowed the authors to obtain an optimal bound up to a logarithmic factor in Ra, and was carried forward in another paper with a delicate analysis of singular integrals by Doering, Otto and Reznikoff, [14]. More recently, in [26] another improvement of the estimate from the background field method was derived, showing at the same time some of the limitations of such a method, see also [25]. In the same paper, a sharper estimate was derived using a maximum principle for the temperature, which always lies in the range $[0,1]$ (see also [10]). More precisely, in [26] it is shown that

$$\mathrm{Nu} \leq C(\mathrm{Ra})^{\frac{1}{3}} \log^{\frac{1}{3}} \log \mathrm{Ra}. \tag{7}$$

For further reading about this problem there is an extensive review article by Ahlers et al. ([1]).

It is perhaps worthwhile to point out that numerical studies by Ierley, Kerswell and Plasting ([23]) predicted in the high Rayleigh number regime, the more precise bound

$$\mathrm{Nu}_{\mathrm{phys}} \leq 0.139\, (\mathrm{Ra})^{\frac{1}{3}}.$$

Our method does not give any information about the above numerical constant.

We also refer to [7] and [28] for related results on upper bounds in the finite-Prandtl number regime.

In the following we will consider the quantity

$$\theta = T - \frac{1}{(LH)^2} \int_{[-LH/2, LH/2]^2} T dx',$$

which by (4) satisfies the uniform bound

$$|\theta| \leq 1 \qquad \text{in} \quad \{0 \leq x_3 \leq H\}. \tag{8}$$

We note that (5)-(6) can be rewritten as

$$\begin{cases} \Delta^2 w = \Delta_h \theta & \text{in } \mathbb{R}^2 \times [0,H]; \\ w = \partial_{x_3} w = 0 & \text{on } \{x_3 = 0, H\}, \end{cases} \tag{9}$$

with all quantities periodically extended in (x_1, x_2).

Taking the horizontal space- and time-average to (1), using the boundary conditions and the constancy of vertical heat flux one finds that for all x_3

$$\mathrm{Nu} = \langle T\, w \rangle - \partial_{x_3} \langle T \rangle,$$

where w is as in (9). This fact and the bounds on T clearly imply that for all $\delta \in (0, H)$

$$\mathrm{Nu} = \frac{1}{\delta} \left(\int_0^\delta \langle T\, w \rangle \, dx_3 + 1 - \langle T|_{x_3 = \delta} \rangle \right) \leq \frac{1}{\delta} \left(\int_0^\delta \langle \theta\, w \rangle \, dx_3 + 1 \right),$$

where in the first equality we used the fact that w has null horizontal average.

In [26] (Section C, proof of Theorem 2) it was noticed that since $\sup_{x_3}\langle\theta^2\rangle \leq \langle\sup_x \theta^2\rangle$, one gets

$$\mathrm{Nu} \leq \sup_{x_3\in(0,\delta)} \langle w^2\rangle^{\frac{1}{2}} + \frac{1}{\delta} \qquad \text{for } \delta\in(0,1), \tag{10}$$

and the quantity $\langle w^2\rangle$ was controlled for δ small via a bound on $\sup_{x_3\in(0,1)}\langle(\partial^2_{x_3} w)^2\rangle$, leading to (7) mainly via Fourier analysis, Caccioppoli-type estimates and the maximum principle.

Here we choose instead $\delta = 1$, to write

$$\mathrm{Nu} \leq \sup_{x_3\in(0,1)} \langle \theta\, w\rangle + 1,$$

getting then uniform bounds on $\langle\theta\, w\rangle$ from $\theta\in L^\infty$ by controlling $\langle|w|\rangle$ uniformly in $x_3\in[0,1]$, without passing through L^∞-estimates on second-order derivatives of w. We use instead a variant of a theorem by Coifman and Meyer from [8] exploiting *Carleson measures*. Carleson measures, recalled in the Appendix, were introduced in [5] for the resolution of the Corona problem. Later, in the seminal paper [16] a characterization of BMO functions was obtained via Carleson measures, see Theorem 2.1, used in [15] and [16] to show that the dual of Hardy's space is BMO. Recall that $f\in \mathrm{BMO}(\mathbb{R}^n)$ if one has a uniform control of a suitable integral involving f and its average for all balls $B\subseteq\mathbb{R}^n$ (see Chapter VI in [27]), and that clearly one has

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq 2\|f\|_{L^\infty(\mathbb{R}^n)}. \tag{11}$$

We explain next the main steps in the strategy to achieve our goal. First, since we are dealing with a bi-Laplace equation on a large strip (with both Dirichlet and Neumann boundary conditions), it is useful to understand the solution of such a boundary-value problem in a half-space, which represents a prototype situation when we are close to the bottom of the strip. In this case, the expression of the Green's kernel for the counterpart of (9) in $\mathbb{R}^3_+$ is explicitly known, and we prove for it suitable decay properties, as well as for some if its derivatives, see Lemma 2.5.

We then study the solution of the problem

$$\begin{cases} \Delta^2\tilde{w}_1 = \chi_{\{0\leq x_3\leq H/2\}}\Delta_h\theta & \text{in } \mathbb{R}^3_+; \\ \tilde{w}_1 = 0 & \text{on } \{x_3=0\}; \\ \partial_{x_3}\tilde{w}_1 = 0 & \text{on } \{x_3=0\}, \end{cases} \tag{12}$$

forgetting about the boundary conditions on the upper side of the strip, that can be represented convoluting with a kernel $K(x,y)$, i.e.

$$\tilde{w}_1(x) = \int_{\{0\leq x_3\leq H/2\}} K(x,y)\theta(y)dy.$$

Since we only care about the behavior of $\tilde{w}_1$ for $x_3\in[0,1]$, understanding solutions to the latter problem will be a key point in the argument. It turns out that the kernel K has cubic decay, which would lead to logarithmic divergences in H after integration. However, a more precise asymptotics of K yields, see Remark 2.7 and Lemma 3.1 (setting here $L=1$ for the purpose of brevity)

$$\langle\theta\, w\rangle \simeq \langle\theta\,\tilde{w}_1\rangle \simeq \frac{1}{H^2}\int_{[-H,H]^2} dx'\theta_1^{x_3}(x')\, x_3^2 \int_{\{4x_3<y_3\leq H/2\}} y_3\, dy_3((\Delta_h P_{y_3}) * \theta_1^{y_3})(x'),$$

where we used the notation $\theta^t(x') := \theta(x',t)$, $t\geq 0$. We then *symmetrize* the latter expression integrating by parts in x' and using the semigroup property of Poisson's kernel, to write

$$\langle\theta\, w\rangle \simeq ⨍_{[-H,H]^2}\int_{\{4x_3\leq y_3\leq H/2\}} x_3^2 y_3((\nabla_h P_{y_3/2}) * \theta_1^{x_3})(x')\cdot((\nabla_h P_{y_3/2}) * (\theta^{y_3}\chi_{[-H,H]^2})(x')dy_3 dx,$$

see the proof of Proposition 3.2 for precise estimates. By Theorem 2.1 one has that (for x_3 fixed)

$$|(t\nabla_h P_{t/2} * \theta_1^{x_3})(x')|^2 \frac{dx'dt}{t}$$

is a Carleson measure, see Definition 5.2, since $\tilde{\theta}_1^{x_3}(\cdot)$ is bounded (once x_3 is fixed). In order to deal with the y_3-dependence in θ^{y_3}, see Proposition 3.3, we adapt a bilinear estimate from [8], for which we present a variant in Section 2, see Proposition 2.3 for the specific statement. Such estimates have the advantage to exploit cancellations via paraproducts (see the Appendix) from the L^2-integrability of one of the factors. This property allows us to localize the integration in a suitable box of size H and instead of using the Hardy-Littlewood maximal function as done in [8], we directly estimate the *non-tangential maximal function*, in order to avoid logarithmic divergences. We notice that even in two dimensions our strategy does not seem to simplify, see Remark 2.8.

We then subtract from $\tilde{w}_1$ a correction $\hat{w}_1$ that is bi-harmonic in the strip and fixes the boundary conditions on $\{x_3 = H\}$, which were not satisfied by $\tilde{w}_1$ on $\{x_3 = H\}$: this step is mostly worked out via Fourier analysis, similar to [26], keeping track of the uniformity of the estimates when H is large. This gives control on a *surrogate* of the function w in (9), for which the right hand-side is set equal to zero on the upper half-strip. However, the same argument works when replacing in (12) $\chi_{\{0 \le x_3 \le H/2\}}$ by $\chi_{\{H/2 \le x_3 \le H\}}$: for the latter we can indeed control the L^∞ norm near the lower boundary due to bi-harmonicity and the local boundary conditions. The estimates obtained in this way allow one to obtain the conclusion of Theorem 1.2.

The plan of the paper is as follows. In Section 2 we recall some basic facts about Poisson's kernel and state a variant of a bilinear estimate from [8]. We then study the Green's function for the bi-Laplacian in the half-space, providing some asymptotic behaviour on its horizontal Laplacian via scaling and invariance properties. In Section 3 we derive estimates on solutions to the counterpart of (9) in the half-space, imposing Dirichlet and Neumann boundary conditions on $\{x_3 = 0\}$ only, obtaining a key estimate on averaged integrals over horizontal periodic squares. In Section 4 the boundary conditions are then fixed on the upper layer $\{x_3 = H\}$, and uniform bounds on Nusselt's number are derived. An appendix describes the proof of Proposition 2.3, adapting the arguments for Theorem 33 in [8].

Notation. Throughout the paper, the letter C will denote a large, but fixed, positive constant which is allowed to vary from one formula to another, and even within the same formula. We also use the standard symbol $O(\cdot)$ for a quantity that is upper bounded by a fixed constant times its argument.

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2 Some preliminary results

In this section we collect some useful preliminary facts. We recall some properties of Poisson's kernel and a result from [16] concerning harmonic extensions from $\mathbb{R}^2$ to the half three-space, stating then a variant of a bilinear estimate from [8]. Moreover, we analyze the asymptotic behavior of another kernel in $\mathbb{R}^3_+$, useful to represent the solutions of (12).

2.1 Poisson's kernel, a biharmonic kernel and a bilinear estimate

We first recall the definition and some properties of Poisson's kernel in the upper half-space, which in three dimensions is given by the formula

$$P_s(x') = \frac{1}{2\pi}\frac{s}{(s^2+|x'|^2)^{\frac{3}{2}}}; \qquad s>0,\ x'\in\mathbb{R}^2. \tag{13}$$

If $f(x')$ is a bounded function on $\mathbb{R}^2$, then

$$F(x',s) := (P_s * f)(x') \tag{14}$$

is the harmonic extension of f to $\mathbb{R}^3_+$. For later convenience, we notice that

$$\Delta_h P_s(x') = \frac{3s\left(3|x'|^2-2s^2\right)}{2\pi\,(s^2+|x'|^2)^{\frac{7}{2}}}. \tag{15}$$

We also recall the following result (in a particular case), giving a characterization of BMO functions.

Theorem 2.1 ([16], Theorem 3) *There exists a constant $C_0>0$ such that, for $f(\cdot)\in \mathrm{BMO}(\mathbb{R}^2)$ and letting*

$$T(x^0,h) := \left\{(x,s)\in\mathbb{R}^3_+ \ :\ |x-x^0|\le h, s\in(0,h)\right\}; \qquad x^0\in\mathbb{R}^2,\ h>0,$$

one has

$$\sup_{x^0\in\mathbb{R}^2}\int_{T(x^0,h)} |t\nabla_h F(x',t)|^2\,dx'\,\frac{dt}{t} \le C_0\,h^2\|f\|^2_{\mathrm{BMO}(\mathbb{R}^2)}, \tag{16}$$

where F is as in (14) *amd ∇_h stands for the horizontal gradient.*

Remark 2.2 *(i) Even though the quadratic dependence in the* BMO *norm is not explicitly stated in Theorem 3 of [16], it appears in the second formula on page 147.*

(ii) Provided $(P_1*f)(0)\in\mathbb{R}$, finiteness of the integral in (16) *implies in turn that f is of class* BMO*.*

For a measurable and bounded function $F(z,t)$ defined on $\mathbb{R}^3_+$, the *non-tangential maximal function* (see Figure 1) $NF:\mathbb{R}^2\to\mathbb{R}$ is given by

$$NF(x) = \sup\left\{|F(z,t)|\ :\ |x-z|<t\right\}. \tag{17}$$

For F as before and $t>0$, we define $F^t:\mathbb{R}^2\to\mathbb{R}$ as

$$F^t(x') := F(x',t).$$

The above function is useful in the proof of [8, Theorem 33], for which we have the following variant.

Proposition 2.3 *Let ψ be the vectorial convolution kernel in $\mathbb{R}^2$ given by*

$$\psi(x') = \nabla_{x'}P_1(x').$$

For $t>0$ let $\psi_t(\cdot)=t^{-2}\psi(t^{-1}\cdot)$. Let ρ be a BMO *function in $\mathbb{R}^2$ and let F be a bounded function on $\mathbb{R}^3_+$ with compact support. Let also $m:(0,+\infty)\to\mathbb{R}$ be measurable with $|m(t)|\le 1$ for all t. Then there exists a constant $C>0$, independent of ρ, F and m, such that letting $\check{F}(x',t)=(\psi_t*F^t)(x')$, one has*

$$\left(\int_{\mathbb{R}^2}\left|\int_0^\infty(\psi_t*\rho)(x')\cdot(\psi_t*F^t)(x')m(t)\frac{dt}{t}\right|^2dx'\right)^{\frac{1}{2}} \le C\|\rho\|_{\mathrm{BMO}}\|N\check{F}\|_{L^2(\mathbb{R}^2)}.$$

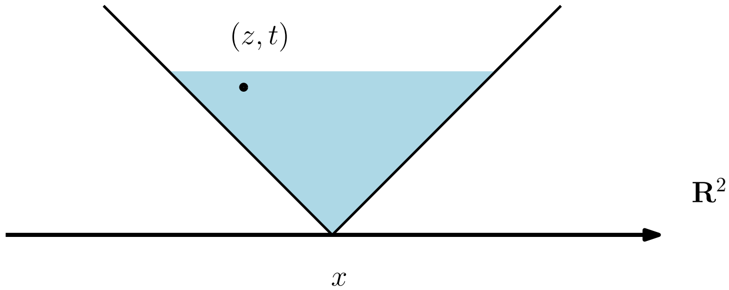

Figure 1: the non-tangential maximal function

Remark 2.4 *Via the same proof, one can indeed obtain the same conclusion for a convolution kernel verifying*

$$\int_{\mathbb{R}^2} \psi(x')dx' = 0,$$

and such that its Fourier transform satisfies

$$\begin{cases} |\partial_\xi^\alpha \hat{\psi}(\xi)| \le C(\alpha) e^{-C(\alpha)^{-1}|\xi|} & \text{for } |\xi| \ge 1; \\ |\xi|^\alpha |\partial_\xi^\alpha \hat{\psi}(\xi)| \le C(\alpha)|\xi| & \text{for } |\xi| \le 1. \end{cases} \tag{18}$$

We will prove in the appendix that $\|N\check{F}\|_{L^2(\mathbb{R}^2)}$ *is finite for the kernel in Proposition 2.3. This property can be indeed verified under the previous more general assumptions.*

The difference of Proposition 2.3 compared to the original version is that we allow a dependence in the t-variable in the second convolved function. The argument follows the original one, and in the appendix we describe how the proof in [8] adapts to this situation as well.

The result applies in particular to the (vectorial) case of the horizontal gradient of Poisson's kernel

$$\psi(x') = \nabla_h P_1(x'),$$

where P_1 is as in (13) with $s = 1$.

2.2 Useful Kernel Estimates in the Upper Half-Space

We are interested next in deriving some asymptotic estimates involving the Green's function for the bi-Laplacian with zero Dirichlet and Neumann boundary conditions in $\mathbb{R}^3_+ = \{(\eta_1, \eta_2, \eta_3) \;:\; \eta_3 > 0\}$, namely the solution $G(\xi, \eta)$ (decaying to zero as $\eta_3 \to +\infty$) of

$$\begin{cases} \Delta^2_\eta G(\xi, \cdot) = \delta_\xi & \text{in } \mathbb{R}^3_+; \\ G(\xi, \cdot) = 0 & \text{on } \partial\mathbb{R}^3_+; \\ \partial_{\eta_3} G(\xi, \cdot) = 0 & \text{on } \partial\mathbb{R}^3_+. \end{cases}$$

Recall that from Lemma 8 in [17], G has the exact expression

$$G(\xi, \eta) = \frac{1}{16\pi} |\xi - \eta| \int_1^{\frac{|\xi^* - \eta|}{|\xi - \eta|}} (t^2 - 1)\, t^{-2} dt, \tag{19}$$

where ξ^* stands for the reflection of ξ across the boundary, $\{\eta_3 = 0\}$. From an explicit computation one obtains the following formula for G

$$G(\xi, \eta) = \frac{1}{16\pi} |\xi - \eta| \left[\frac{|\xi^* - \eta|}{|\xi - \eta|} + \frac{|\xi - \eta|}{|\xi^* - \eta|} - 2 \right],$$

which after further manipulation becomes

$$G(\xi,\eta)=\frac{1}{16\pi}\frac{(|\xi^*-\eta|-|\xi-\eta|)^2}{|\xi^*-\eta|}.$$

Computing the horizontal Laplacian $\Delta_h^\eta=\frac{\partial^2}{\partial\eta_1^2}+\frac{\partial^2}{\partial\eta_2^2}$, one then finds

$$K(\xi,\eta):=\Delta_h^\eta G(\xi,\eta)=\frac{1}{8\pi}\left[\frac{1}{|\xi^*-\eta|}-\frac{1}{|\xi-\eta|}-\frac{(\xi_3-\eta_3)^2}{|\xi-\eta|^3}+\frac{\xi_3^2+\eta_3^2}{|\xi^*-\eta|^3}+6\frac{\xi_3\eta_3(\xi_3+\eta_3)^2}{|\xi^*-\eta|^5}\right]. \tag{20}$$

We observe the following three covariance properties for all $\xi,\eta\in\mathbb{R}^3_+$ and for all $v'\in\mathbb{R}^2$:

$$K(\lambda\xi,\lambda\eta)=\frac{1}{\lambda}K(\xi,\eta),\quad \lambda>0;\qquad K(\xi,\eta)=K(\eta,\xi);\qquad K(\xi+(v',0),\eta+(v',0))=K(\xi,\eta). \tag{21}$$

By these, it will be sufficient to understand the asymptotics for $|\eta|\to+\infty$ of

$$K_0(\eta)=K(\xi_0,\eta);\qquad \xi_0=(0,0,1). \tag{22}$$

We also notice that the function K_0 is even with respect to reflection around the η_3-axis, namely

$$K_0(\eta_1,\eta_2,\eta_3)=K_0(-\eta_1,-\eta_2,\eta_3). \tag{23}$$

We will now deduce some asymptotics on the function K_0.

Lemma 2.5 *For ξ_0 and K_0 as in (22), we have the upper bound*

$$|K_0(\eta)|\le\frac{C}{|\eta-\xi_0|};\qquad |\eta-\xi_0|\le 8. \tag{24}$$

Moreover, for a vector $(a,b,c)\in\mathbb{R}^3$ of unit norm, $c\ge 0$, and for $R>0$ large one has

$$K_0(\xi_0+(Ra,Rb,Rc))=\frac{3c^2\left(3a^2+3b^2-2c^2\right)}{4\pi R^3}+O\left(R^{-4}\right), \tag{25}$$

$$\nabla_\eta K_0(\xi_0+(Ra,Rb,Rc))=O(R^{-4}); \tag{26}$$

$$\nabla^{(2)}_{\eta,\eta}K_0(\xi_0+(Ra,Rb,Rc))=O(R^{-5}). \tag{27}$$

The above estimates are uniform as $R\to+\infty$ in the choice of the unit vector (a,b,c), with $c\ge 0$. Furthermore, there exists $C>0$ such that

$$K_0(\eta',\eta_3)\le\frac{C}{|\eta'|^3};\qquad \eta'\in\mathbb{R}^2, |\eta'|\ge 1,\quad \eta_3\in(0,1]. \tag{28}$$

Proof. The proof of (25) can be deduced from some cancellations after Taylor expansions of the following quantities, corresponding to each of term in (20), choosing $\eta=R(a,b,c)$, with $a^2+b^2+c^2=1$, $\xi=(0,0,1)$ and $\xi^*=(0,0,-1)$:

$$\frac{1}{(R^2a^2+R^2b^2+(Rc+1)^2)^{\frac12}}=\frac{1}{R}-\frac{c}{R^2}+\frac{2c^2-a^2-b^2}{2R^3}+O\left(\frac{1}{R^4}\right);$$

$$\frac{1}{(R^2a^2+R^2b^2+(Rc-1)^2)^{\frac12}}=\frac{1}{R}+\frac{c}{R^2}+\frac{2c^2-a^2-b^2}{2R^3}+O\left(\frac{1}{R^4}\right);$$

$$\frac{(Rc-1)^2}{\left(R^2a^2+R^2b^2+(1-Rc)^2\right)^{3/2}} = \frac{c^2}{R} + \frac{c^3-2c\left(a^2+b^2\right)}{R^2} + \frac{2c^4-11c^2\left(a^2+b^2\right)+2\left(a^2+b^2\right)^2}{2R^3} + O\left(\frac{1}{R^4}\right);$$

$$\frac{R^2c^2+1}{\left(R^2a^2+R^2b^2+(Rc+1)^2\right)^{3/2}} = \frac{c^2}{R} - \frac{3c^3}{R^2} + \frac{c^2\left(a^2+b^2\right)+2\left(a^2+b^2\right)^2+14c^4}{2R^3} + O\left(\frac{1}{R^4}\right);$$

$$\frac{Rc(Rc+1)^2}{\left(R^2a^2+R^2b^2+(Rc+1)^2\right)^{5/2}} = \frac{c^3}{R^2} + \frac{2c^2\left(a^2+b^2\right)-3c^4}{R^3} + O\left(\frac{1}{R^4}\right).$$

The proofs of the gradient and the hessian estimates can be obtained in a similar manner.

While the condition $c \geq 0$ is needed in (25) and the next two formulas to have the argument of K_0 in $\mathbb{R}^3_+$, the four estimates hold regardless of the sign of c, allowing to show (28) as well. This observation is only useful to control the kernel at (non-problematic) points whose x_3-coordinate is less than 1. ■

Using the above result, we obtain next the following properties of $K(\cdot,\cdot)$.

Proposition 2.6 *There exists $C > 0$ such that for $x_3, y_3 > 0$, $x \neq y$, the kernel K satisfies*

$$|K(x,y)| \leq \frac{C}{|x-y|}, \qquad \text{for } |x-y| \leq 8x_3. \tag{29}$$

Moreover, for any unit vector (a,b,c) with $c \geq 0$, we have, uniformly in $c \geq 0$ for $R > 2x_3$:

$$K(x, x+R(a,b,c)) = \frac{3x_3^2c^2\left(3a^2+3b^2-2c^2\right)}{4\pi R^3} + O\left(\frac{x_3^3}{R^4}\right); \tag{30}$$

$$(\nabla_\eta K)(x, x+R(a,b,c)) = O\left(\frac{x_3^2}{R^4}\right), \tag{31}$$

and

$$(\nabla^{(2)}_{\eta,\eta} K)(x, x+R(a,b,c)) = O\left(\frac{x_3^2}{R^5}\right). \tag{32}$$

Proof. We only show (30), the other two estimates being similar.

From the third and first properties in (21) we find

$$K(x, x+R(a,b,c)) = K((0,x_3),(0,x_3)+R(a,b,c)) = \frac{1}{x_3}K((0,0,1),(0,0,1)+R/x_3(a,b,c)).$$

Recalling (22), this becomes

$$K(x, x+R(a,b,c)) = \frac{1}{x_3}K_0(\xi_0 + R/x_3(a,b,c)).$$

From (25) and the observation at the end of the proof of Lemma 2.5 we then obtain

$$K(x, x+R(a,b,c)) = \frac{1}{x_3}\left(\frac{3c^2\left(3a^2+3b^2-2c^2\right)}{4\pi}\frac{x_3^3}{R^3} + O\left(\frac{x_3^4}{R^4}\right)\right),$$

as desired. ■

Remark 2.7 *We notice that, comparing with* (15), *for* $y_3 > 2x_3$

$$K(x,y) = \frac{1}{2}x_3^2 y_3 (\Delta_h P_{y_3})(x'-y') + O\left(\frac{x_3^3}{|x-y|^4}\right).$$

In fact, from a Taylor expansion of (15) *and* (30) *one finds that*

$$y_3(\Delta_h P_{y_3})(x'-y') = \frac{3y_3^2(3|x'-y'|^2 - 2y_3^2)}{2\pi(y_3^2+|x'-y'|^2)^{\frac{7}{2}}} = \frac{3(y_3-x_3)^2(3|x'-y'|^2-2(y_3-x_3)^2)}{2\pi((y_3-x_3)^2+|x'-y'|^2)^{\frac{7}{2}}} + O\left(\frac{x_3}{|x-y|^4}\right).$$

This observation will be crucial for the estimates in the next section.

Remark 2.8 *In two dimensions the counterpart of formula* (19) *becomes*

$$G(\xi,\eta) = \frac{1}{8\pi}|\xi-\eta|^2 \int_1^{\frac{|\xi^*-\eta|}{|\xi-\eta|}} (t^2-1)\,t^{-1}dt,$$

which gives, after a Taylor expansion, $G(\xi,\eta) \simeq \frac{1}{8\pi}(|\xi^*-\eta|-|\xi-\eta|)^2$ *for* $|\eta|$ *large. Taking the horizontal Laplacian, with calculations similar as above one gets (with the same notation)*

$$K(x,y) = \frac{1}{2}x_2^2 y_2 (\Delta_h P_{y_2})(x'-y') + O\left(\frac{x_2^3}{|x-y|^3}\right).$$

Therefore, the kernel K *has inverse quadratic behaviour and would lead to analogous divergence issues, which might be treated though with the same strategy.*

3 Estimates on approximate solutions to (9)

We now consider two cut-off functions in the vertical variable $\chi_{\{0\le x_3\le H/2\}}, \chi_{\{H/2\le x_3\le H\}}$ and split θ as $\theta = \theta_1 + \theta_2$, with θ_1, θ_2 periodically extended in x' and defined as

(33) $$\theta_1 = \theta\chi_{\{0\le x_3\le H/2\}}; \qquad \theta_2 = \theta\chi_{\{H/2\le x_3\le H\}}.$$

For K as in (20), we will prove in this section a key estimate involving the function (solving (12))

(34) $$\tilde{w}_1(x) := \int_{\mathbb{R}^3_+} K(x,y)\,\theta_1(y)dy$$

near the bottom of the strip $\{0 \le x_3 \le H\}$. We begin with a preliminary lemma, for which we recall Remark 2.7.

Lemma 3.1 *There exists a fixed constant* $C > 0$ *such that, writing* $\tilde{w}_1(x) = I(x) + II(x)$ *with*

$$I(x) = \int_{\{0<y_3\le 4x_3\}} K(x,y)\,\theta_1(y)dy; \qquad II(x) = \int_{\{4x_3<y_3\le H/2\}} K(x,y)\,\theta_1(y)dy,$$

one has the estimates

(35) $$|I| \le Cx_3^2; \quad \left| II - \frac{x_3^2}{2}\int_{\{4x_3<y_3\le H/2\}} y_3((\Delta_h P_{y_3}) * \theta_1^{y_3})(x')dy_3 \right| \le Cx_3^2; \qquad x_3 \in (0,1].$$

In the latter formula, the convolution is taken with respect to the horizontal variables, and $\theta_1^{x_3} : \mathbb{R}^2 \to \mathbb{R}$ *denotes the restriction of* θ_1 *to the horizontal plane at height* x_3*, namely*

$$\theta_1^{x_3}(x') := \theta_1(x',x_3).$$

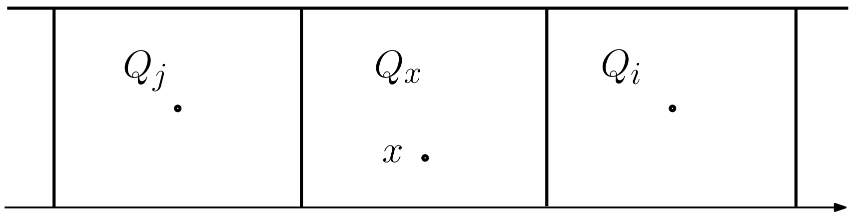

Figure 2: the cubes Q_x and $(Q_i)_i$

PROOF. To prove the first inequality in (35), we consider the cube Q_x centered at $(x', 2x_3)$ and of size $4x_3$, so its lower face lies on $\{x_3 = 0\}$, see Figure 2. We next partition the strip $\{0 < y_3 < 4x_3\}$ into Q_x and a sequence of cubes defined as follows. For $z_i \in \mathbb{Z}^2 \setminus \{(0,0)\}$, let Q_i denote the cube of size $4x_3$ with center at the point $p_i := (x' + 4x_3 z_i, 2x_3)$.

We then notice that, by (29) and the uniform bound on θ (maximum principle for T)

$$\left|\int_{Q_x} K(x,y)\theta_1(y)dy\right| \le C\left|\int_{Q_x} \frac{1}{|x-y|}dy\right| \le C\int_0^{8x_3} \frac{r^2}{r}dr \le Cx_3^2.$$

On the other hand, concerning Q_i, since the decay of K from (30) is cubic towards infinity, we have that

$$\left|\int_{Q_i} K(x,y)\theta_1(y)dy\right| \le C|Q_i| \sup_{y\in Q_i} |K(x,y)| \le Cx_3^3 \frac{x_3^2}{|x-p_i|^3}.$$

Therefore, by the choice of p_i and Q_i we have that

$$\left|\int_{\{0<y_3<4x_3\}\setminus Q_x} K(x,y)\theta_1(y)dy\right| \le Cx_3^5 \sum_i \frac{1}{|x-p_i|^3} \le Cx_3^5 \int_{x_3}^{\infty} \frac{r}{r^3}dr \le Cx_3^2.$$

The last three formulas prove that $|I| \le Cx_3^2$, as desired.

The estimate of II simply follows from Remark 2.7, that yields (still by the fact that $\theta \in L^\infty$)

$$\begin{aligned}\left|II - \frac{x_3^2}{2}\int_{\{4x_3<y_3\le H/2\}} y_3((\Delta_h P_{y_3}) * \theta_1^{y_3})(x')dy_3\right| &\le C\int_{\{4x_3<y_3\le H/2\}} \frac{x_3^3}{|x-y|^4}dy \\ &\le Cx_3^3 \int_{3x_3}^{\infty} \frac{s^2}{s^4}ds \le Cx_3^2,\end{aligned}$$

concluding the proof. ■

We next estimate the principal part of II appearing in (35), integrated versus θ in $x' \in [-HL/2, HL/2]$ and in $x_3 \in [0,1]$. In the next formula and in the proof below, we will sometimes write the differentials right after the corresponding integral signs: even though this is unconventional, we hope it will facilitate tracking the various domains of integration.

Proposition 3.2 *Let $\tilde{w}_1$ be as in* (34). *Then there exists a constant $C_1 > 0$ independent of H (large) such that for any $x_3 \in [0,1]$ one has the upper bound*

$$\left|\frac{1}{(HL)^2}\int_{[-HL/2,HL/2]^2} dx'\theta_1^{x_3}(x')\, x_3^2 \int_{\{4x_3<y_3\le H/2\}} y_3\, dy_3((\Delta_h P_{y_3}) * \theta_1^{y_3})(x')\right| \le C_1.$$

PROOF. We first localize $\theta_1^{y_3} : \mathbb{R}^2 \to \mathbb{R}$ onto the box $[-2HL, 2HL]^2$ and its complement in the (x_1, x_2)-coordinates. We write

$$\theta_1^{y_3} = g_1^{y_3} + g_2^{y_3}, \qquad \text{with} \quad g_1^{y_3} = \theta_1^{y_3}\chi_{[-2HL,2HL]^2}, \quad g_2^{y_3} = \theta_1^{y_3}\chi_{\mathbb{R}^2\setminus[-2HL,2HL]^2}.$$

We note next that for $x' \in [-HL/2, HL/2]^2$, by (15) and the fact that $\|\theta\|_{L^\infty} \le 1$, one has

$$|y_3((\Delta_h P_{y_3}) * g_2^{y_3})(x')| \le C \int_{\mathbb{R}^2 \setminus [-2HL, 2HL]^2} \frac{|\theta_1^{y_3}(w)| dw}{(y_3^2 + |x' - w|^2)^{\frac{3}{2}}} \le C \int_{\{|w| \ge HL\}} \frac{dw}{|w|^3} \le \frac{C}{H}.$$

This implies that

$$\left| \int_{[-HL/2, HL/2]^2} \theta_1^{x_3}(x') \int_{\{4x_3 \le y_3 \le H/2\}} x_3^2 y_3 ((\Delta_h P_{y_3}) * g_2^{y_3})(x') dy_3 dx' \right|$$
$$\le \int_{[-HL/2, HL/2]^2} \int_0^H dy_3 \frac{C}{H} dx' \le CH^2.$$

Therefore, we will now focus on $g_1^{y_3}$. By the semi-group property of the Poisson kernel we have that $P_{y_3} = P_{y_3/2} * P_{y_3/2}$, therefore using the symmetry and associativity of convolutions together with the fact that for $f \in L^\infty(\mathbb{R}^2)$

$$\frac{\partial}{\partial x_i'}(P_s * f) = \left(\frac{\partial}{\partial x_i'} P_s\right) * f; \qquad i = 1, 2, s > 0,$$

we find

$$\int_{[-HL/2, HL/2]^2} dx' \theta_1^{x_3}(x') \int_{\{4x_3 \le y_3 \le H/2\}} x_3^2 y_3 ((\Delta_h P_{y_3}) * g_1^{y_3})(x') dy_3$$
$$= \int_{\mathbb{R}^2} dx' \int_{\{4x_3 \le y_3 \le H/2\}} (P_{y_3/2} * (\theta_1^{x_3} \chi_{[-HL/2, HL/2]^2}))(x') x_3^2 y_3 \Delta_h (P_{y_3/2} * g_1^{y_3})(x') dy_3.$$

Integrating by parts in the horizontal variables and setting

$$\tilde{\theta}_1^{x_3} = \theta_1^{x_3}(x') \chi_{[-HL/2, HL/2]^2}(x'),$$

we find that the latter integral expression is equal to (minus) $A + B$, where

$$A = \int_{\mathbb{R}^2 \setminus [-8HL, 8HL]^2} \int_{\{4x_3 \le y_3 \le H/2\}} x_3^2 y_3 ((\nabla_h P_{y_3/2}) * \tilde{\theta}_1^{x_3})(x') \cdot ((\nabla_h P_{y_3/2}) * g_1^{y_3})(x') dy_3 dx';$$

$$B = \int_{[-8HL, 8HL]^2} \int_{\{4x_3 \le y_3 \le H/2\}} x_3^2 y_3 ((\nabla_h P_{y_3/2}) * \tilde{\theta}_1^{x_3})(x') \cdot ((\nabla_h P_{y_3/2}) * g_1^{y_3})(x') dy_3 dx'.$$

We claim that A is of order $O(H^2)$: in fact, we notice that from the expression of P_s in (13) and an elementary inequality

$$|\nabla_h P_{y_3/2}(w)| \le \frac{C}{(y_3^2 + |w|^2)^{\frac{3}{2}}}; \qquad |y_3 \nabla_h P_{y_3/2}(w)| \le \frac{C y_3}{(y_3^2 + |w|^2)^{\frac{3}{2}}}.$$

Hence, for $x' \in \mathbb{R}^2 \setminus [-8HL, 8HL]^2$ we have that, by the uniform bound on θ

$$\left| ((\nabla_h P_{y_3/2}) * \tilde{\theta}_1^{x_3})(x') \right| \le \int_{\mathbb{R}^2} \frac{|\tilde{\theta}_1^{x_3}| \, dw}{(y_3^2 + |x' - w|^2)^{\frac{3}{2}}} \le C \frac{H^2}{|x'|^3}.$$

On the other hand, still for $x' \in \mathbb{R}^2 \setminus [-8HL, 8HL]^2$ we obtain

$$\left| y_3 ((\nabla_h P_{y_3/2}) * g_1^{y_3})(x') \right| \le \int_{\mathbb{R}^2} \frac{|g_1^{y_3}| \, y_3 \, dw}{(y_3^2 + |x' - w|^2)^{\frac{3}{2}}} \le C \frac{H^2}{|x'|^2}.$$

Hence we find that

$$|A| \le C \int_{\{|x'| \ge 8HL\}} x_3^2 \int_0^{H/2} dy_3 \frac{H^4}{|x'|^5} dx' \le CH^2. \tag{36}$$

We next estimate the term B: using the Cauchy-Schwartz inequality in x' we get that

$$\begin{aligned} |B| &\leq x_3^2 \left(\int_{[-8HL,8HL]^2} dx' \right)^{\frac{1}{2}} \\ &\times \left[\int_{[-8HL,8HL]^2} \left| \int_0^\infty y_3((\nabla_h P_{y_3/2}) * \tilde{\theta}_1^{x_3})(x') \cdot y_3((\nabla_h P_{y_3/2}) * g_1^{y_3})(x') \chi_S(y_3) \frac{dy_3}{y_3} \right|^2 dx' \right]^{\frac{1}{2}}, \end{aligned}$$

where we defined the set S as

$$S = \left\{ y_3 \mid 4x_3 \leq y_3 \leq \frac{H}{2} \right\}.$$

Setting now

$$t = y_3; \qquad \psi_t = t\nabla_h P_{t/2}; \qquad m(t) = \chi_S(t),$$

we can write that

$$|B| \leq CH \left[\int_{\mathbb{R}^2} \left| \int_0^\infty (\psi_t * \tilde{\theta}_1^{x_3})(x') \cdot (\psi_t * g_1^t)(x') \, m(t) \frac{dt}{t} \right|^2 dx' \right]^{\frac{1}{2}} x_3^2.$$

Noting that (in vectorial sense) $\int_{\mathbb{R}^2} \psi_t(x')dx' = 0$, we have by Theorem 2.1 that

$$|(\psi_t * \tilde{\theta}_1^{x_3})(x')|^2 \frac{dx' dt}{t}$$

is a Carleson measure, see Definition 5.2 and Theorem 2.1, since $\tilde{\theta}_1^{x_3}(\cdot)$ is in L^∞ with x_3 fixed. Moreover, we have clearly

$$\|m(t)\|_\infty \leq 1.$$

Recalling the notion of non-tangential maximal function from (17) and choosing

$$F(z,t) = g_1^t(z); \qquad \check{F}(z,t) = (\psi_t * g_1^t)(z),$$

applying Proposition 2.3 and (11) we find that

$$|B| \leq CH \|N\check{F}\|_{L^2(\mathbb{R}^2)} \|\theta_1\|_{L^\infty}. \tag{37}$$

To estimate the above L^2-norm we first notice that

$$|\check{F}(z,t)| = |(\psi_t * g_1^t)(z)| \leq C \int_{\mathbb{R}^2} \frac{|g_1^t(w)| t}{(t^2 + |z-w|^2)^{\frac{3}{2}}} dw \leq C \int_{\mathbb{R}^2} \frac{t}{(t^2 + |w|^2)^{\frac{3}{2}}} dw \leq C. \tag{38}$$

We now specialize to the case in which $x' \in \mathbb{R}^2 \setminus [-10HL, 10HL]^2$: notice that for $|z - x'| \leq t$ we have

$$t^2 + |x' - w|^2 \leq 2(t^2 + |z - w|^2),$$

which implies

$$P_t(z-w) \leq CP_t(x'-w); \qquad \text{for} \quad |z - x'| \leq t.$$

Using this inequality we get

$$|\check{F}(z,t)| \leq C \int_{\mathbb{R}^2} \frac{|g_1^t(w)| t}{(t^2 + |x'-w|^2)^{\frac{3}{2}}} dw; \qquad |z - x'| \leq t, \ x' \in \mathbb{R}^2 \setminus [-10HL, 10HL]^2.$$

Since we only integrate for $w \in [-2H, 2H]^2$ in the support of g_1^t, we then obtain

$$|\check{F}(z,t)| \leq \frac{C}{|x'|^2} \int_{\mathbb{R}^2} |g_1^t(w)| dw \leq C \frac{H^2}{|x'|^2}, \qquad |z - x'| \leq t, \ x' \in \mathbb{R}^2 \setminus [-10HL, 10HL]^2.$$

Recalling (17), combining (38) and the latter estimate we find that

$$\|N\check{F}\|_{L^2(\mathbb{R}^2)} \leq CH.$$

From the last inequality, (36) and (37) we then get the desired conclusion. ■

From Lemma 3.1 and Proposition 3.2 we derive the following consequence.

Proposition 3.3 *Let θ_1 be as in (33) and $\tilde{w}_1$ as in (34). Then there exists a fixed constant $C > 0$ independent of H (large) such that*

$$\left| \frac{1}{(HL)^2} \int_{[-HL/2,HL/2]^2} \theta_1(x', x_3)\tilde{w}_1(x', x_3)dx' \right| \leq C; \qquad x_3 \in (0, 1].$$

We next consider the above function $\tilde{w}_1$ for x_3 close to H, where we have better regularity properties due to its bi-harmonicity. The following estimates will play a crucial role in Section 4, particularly in ensuring the boundary conditions are met.

Proposition 3.4 *There exist fixed positive constants H_0 and C_0 such that for $H \geq H_0$ one has*

$$(39) \quad |\tilde{w}_1(x)| \leq C_0 H^2; \quad |\nabla \tilde{w}_1(x)| \leq C_0\, H; \quad |\nabla^{(2)} \tilde{w}_1(x)| \leq C_0, \qquad x = (x', x_3), \;\; x_3 \in [3/4\, H, 5/4\, H].$$

Proof. We proceed similarly to the proof of the first estimate in (35). Assuming without loss of generality that $x' = 0$, we let $\tilde{Q}_i$ denote the box of sides $HL/2, HL/2$ centered at $\hat{z}_i := (HL/2\, \tilde{z}_i, H/4)$, with $\tilde{z}_i \in \mathbb{Z}^2$, see Figure 3.

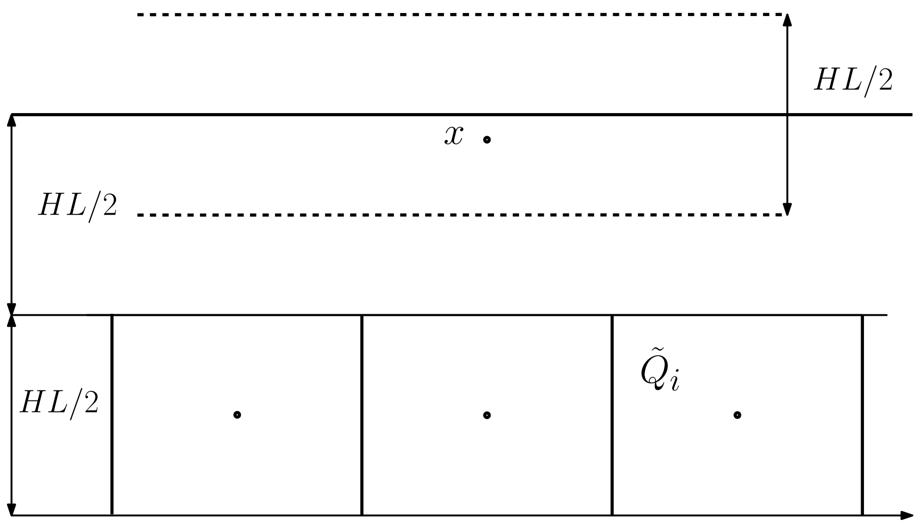

Figure 3: the boxes $(\tilde{Q}_i)_i$

Using (30) and the second property in (21) we have that, for $x' = 0$ and $x_3 \in [3/4\, H, 5/4\, H]$,

$$(40) \qquad \left| \int_{\tilde{Q}_i} K(x,y)\theta_1(y)dy \right| = \left| \int_{\tilde{Q}_i} K(y,x)\theta_1(y)dy \right| \leq C \left| \int_{\tilde{Q}_i} y_3^2 dy \right| \frac{1}{|x - \hat{z}_i|^3} \leq CH^5 \frac{1}{|x - \hat{z}_i|^3}.$$

Recalling the above definition of $\tilde{z}_i$ and $\hat{z}_i$ and the fact that $x_3 \in [3/4\, H, 5/4\, H]$, we have that

$$(41) \qquad |x - \hat{z}_i| \geq C^{-1} H (1 + |\tilde{z}_i|^2)^{\frac{1}{2}}.$$

The latter two formulas imply

$$(42)\qquad |\tilde{w}_1(x)| \le CH^2 \sum_{\tilde{z}_i \in \mathbb{Z}^2} (1+|\tilde{z}_i|^2)^{-\frac{3}{2}} \le CH^2 \int_0^\infty \frac{s}{(1+s^2)^{\frac{3}{2}}} ds \le CH^2.$$

Concerning the gradient estimate, recalling (21), the fact that $K(x,y) = K(y,x)$ gives

$$(43)\qquad \nabla_\xi K(x,y) = \nabla_\eta K(y,x),$$

and similarly

$$(44)\qquad \nabla_{\xi,\xi} K(x,y) = \nabla_{\eta,\eta} K(y,x).$$

From these and (31)-(32) we deduce that

$$|\nabla_\xi K(x,y)| \le C \frac{y_3^2}{|x-\hat{z}_i|^4}; \qquad |\nabla^{(2)}_{\xi,\xi} K(x,y)| \le C \frac{y_3^2}{|x-\hat{z}_i|^5}; \qquad y \in \tilde{Q}_i.$$

Analogously to (40), these imply that

$$\left| \int_{\tilde{Q}_i} \nabla_\xi K(x,y) \tilde{\theta}_1(y) dy \right| \le CH^5 \frac{1}{|x-\hat{z}_i|^4}; \qquad \left| \int_{\tilde{Q}_i} \nabla^{(2)}_{\xi,\xi} K(x,y) \tilde{\theta}_1(y) dy \right| \le CH^5 \frac{1}{|x-\hat{z}_i|^5}.$$

Since $x_3 \ge 3/4H$ and $y_3 \le H/2$, we are avoiding the singularity of K and we can write that

$$(\nabla \tilde{w}_1)(x) = \int_{\{0 \le y_3 \le H/2\}} \nabla_\xi K(x,y)\, \theta_1(y) dy; \qquad (\nabla^{(2)} \tilde{w}_1)(x) = \int_{\{0 \le y_3 \le H/2\}} \nabla^{(2)}_{\xi,\xi} K(x,y)\, \theta_1(y) dy.$$

By the latter formulas and (41), reasoning as for (42) we then find

$$|(\nabla \tilde{w}_1)(x)| \le CH \sum_{\tilde{z}_i \in \mathbb{Z}^2} (1+|\tilde{z}_i|^2)^{-2} \le CH \int_0^\infty \frac{s}{(1+s^2)^2} ds \le CH;$$

$$|(\nabla^{(2)} \tilde{w}_1)(x)| \le C \sum_{\tilde{z}_i \in \mathbb{Z}^2} (1+|\tilde{z}_i|^2)^{-\frac{5}{2}} \le C \int_0^\infty \frac{s}{(1+s^2)^{\frac{5}{2}}} ds \le C.$$

This concludes the proof. ■

Remark 3.5 *One can get estimates on higher-order derivatives of $\tilde{w}_1$ for $x_3 \in [3/4\,H, 5/4\,H]$ as follows. Consider the scaled function*

$$\check{w}_1(x) := \frac{1}{H^2} \hat{w}_1(Hx); \qquad x = (x', x_3), \quad x' \in \mathbb{R}^2, x_3 \in [3/4, 5/4].$$

Then this is bi-harmonic, horizontally L-periodic and by the previous proposition also uniformly bounded, together with its first- and second-order derivatives. Standard regularity theory implies that

$$|\nabla^{(j)} \check{w}_1| \le C_j \qquad \textit{for} \quad x_3 \in [7/8, 9/8]; \quad |j| \ge 3,$$

which gives in turn

$$(45)\qquad |\nabla^{(j)} \hat{w}_1| \le C_j H^{2-j} \qquad \textit{for} \quad x_3 \in [7/8\,H, 9/8\,H]; \quad |j| \ge 3.$$

4 Corrections to fit boundary data and conclusion

We need next to adjust the above function $\tilde{w}_1$, obtained convoluting K with θ_1, see (33), to achieve the correct boundary data on $\{x_3 = H\}$. In order to do it, we would therefore need to consider $\tilde{w}_1 - \hat{w}_1$, where $\hat{w}_1$ is an (LH)-periodic function in x_1, x_2 that satisfies

$$\begin{cases} \Delta^2 \hat{w}_1 = 0 & \text{on } \mathbb{R}^2 \times [0, H]; \\ \hat{w}_1 = \tilde{w}_1 & \text{on } \{x_3 = H\}; \\ \partial_{x_3} \hat{w}_1 = \partial_{x_3} \tilde{w}_1 & \text{on } \{x_3 = H\}. \end{cases} \tag{46}$$

Recall also that, by Proposition 3.4,

$$|\tilde{w}_1| \leq C\,H^2; \qquad |\partial_{x_3} \tilde{w}_1| \leq C\,H, \qquad \text{on } \{x_3 = H\}.$$

Throughout this section we will assume for simplicity that $L = 1$, the general case requiring only obvious modifications. We can solve (46) via Fourier decomposition in the horizontal variables: for $\mathbf{k} \in \mathbb{Z}^2$ and $x' \in \mathbb{R}^2$, we write

$$\hat{w}_1(x', x_3) = \sum_{\mathbf{k} \in \mathbb{Z}^2} a_{\mathbf{k}}(x_3) e^{2\pi i \, \frac{\mathbf{k}}{H} \cdot x'}. \tag{47}$$

The function $a_{\mathbf{k}}$ satisfies the fourth-order ODE

$$\begin{cases} a_{\mathbf{k}}'''' - 2 \left| \frac{\mathbf{k}}{H} \right|^2 a_{\mathbf{k}}'' + \left| \frac{\mathbf{k}}{H} \right|^4 a_{\mathbf{k}} = 0 & \text{in } [0, H]; \\ a_{\mathbf{k}} = a_{\mathbf{k}}' = 0 & \text{in } \{x_3 = 0\}; \\ a_{\mathbf{k}} = b_{\mathbf{k}} & \text{on } \{x_3 = H\}; \\ a_{\mathbf{k}}' = c_{\mathbf{k}} & \text{on } \{x_3 = H\}, \end{cases}$$

where $b_{\mathbf{k}}$, $c_{\mathbf{k}}$ are the Fourier components of $\tilde{w}_1$ and $\partial_{x_3} \tilde{w}_1$ respectively on $\{x_3 = H\}$, defined by

$$\tilde{w}_1(x', H) = \sum_{\mathbf{k} \in \mathbb{Z}^2} b_{\mathbf{k}} e^{2\pi i \, \frac{\mathbf{k}}{H} \cdot x'}; \qquad (\partial_{x_3} \tilde{w}_1)(x', H) = \sum_{\mathbf{k} \in \mathbb{Z}^2} c_{\mathbf{k}} e^{2\pi i \, \frac{\mathbf{k}}{H} \cdot x'}. \tag{48}$$

From an explicit computation, see also Section C in [26] for a related one, it follows that for $\mathbf{k} = \mathbf{0}$

$$a_{\mathbf{0}} = A_{\mathbf{0}} x_3^2 + B_{\mathbf{0}} x_3^3; \qquad A_{\mathbf{0}} = \frac{2 b_{\mathbf{0}}}{H^2} - \frac{c_{\mathbf{0}}}{2H}, \qquad B_{\mathbf{0}} = \frac{c_{\mathbf{0}}}{2H^2} - \frac{b_{\mathbf{0}}}{H^3}, \tag{49}$$

while for $\mathbf{k} \neq \mathbf{0}$

$$a_{\mathbf{k}}(x_3) = x_3 (B_{\mathbf{k}} \sinh(|\mathbf{k}|/H\, x_3) - A_{\mathbf{k}} |\mathbf{k}|/H \cosh(|\mathbf{k}|/H\, x_3)) + A_{\mathbf{k}} \sinh(|\mathbf{k}|/H\, x_3), \tag{50}$$

where

$$A_{\mathbf{k}} = \frac{2((b_{\mathbf{k}} - c_{\mathbf{k}} H) \sinh(|\mathbf{k}|) + b_{\mathbf{k}} |\mathbf{k}| \cosh(|\mathbf{k}|))}{\cosh(2|\mathbf{k}|) - 2|\mathbf{k}|^2 - 1};$$

$$B_{\mathbf{k}} = 2 \frac{\sinh(|\mathbf{k}|) \left(b_{\mathbf{k}} |\mathbf{k}|^2 / H + c_{\mathbf{k}} \right) - c_{\mathbf{k}} |\mathbf{k}| \cosh(|\mathbf{k}|)}{\cosh(2|\mathbf{k}|) - 2|\mathbf{k}|^2 - 1}.$$

Due to Remark 3.5, the Fourier components of $\tilde{w}_1$ and $\partial_{x_3} \tilde{w}_1$ for $x_3 = H$ decay fast in $\mathbf{k}$: more precisely, for any integer ℓ there exists $C_\ell > 0$, independent of H large, such that

$$\sum_{\mathbf{k} \in \mathbb{Z}^2} \frac{|b_{\mathbf{k}}/H^2|^2 + |c_{\mathbf{k}}/H|^2}{1 + |\mathbf{k}|^\ell} \leq C_\ell. \tag{51}$$

This will imply uniform convergence of the series (in $\mathbf{k}$) of $\hat{w}_1$ for $x_3 \in [0, 1]$ and of that of $\tilde{w}_1 - \hat{w}_1$ for $x_3 \in [H - 1, H]$. We indeed prove the following result.

Proposition 4.1 *Let $\hat{w}_1$ be as in (46). There exist fixed constants H_0 and C_0 such that for all $H \geq H_0$*

$$|\hat{w}_1| \leq C_0 \quad \text{in } \{0 \leq x_3 \leq 1\}; \qquad |\tilde{w}_1 - \hat{w}_1| \leq C_0 \quad \text{in } \{H-1 \leq x_3 \leq H\}.$$

PROOF. We first estimate the zero-mode $a_{\mathbf{0}}$ in (49). For $x_3 \in [0,1]$, $a_{\mathbf{0}}(x_3)$ is clearly bounded since $A_{\mathbf{0}}$ and $B_{\mathbf{0}}$ are, due to Proposition 3.4.

Let us now pass to higher-order modes. We first notice that, since the denominators in $A_{\mathbf{k}}$, $B_{\mathbf{k}}$ are bounded below by $C^{-1}e^{2|\mathbf{k}|}$, then

$$|A_{\mathbf{k}}| \leq C(|b_{\mathbf{k}}| + H|c_{\mathbf{k}}|)|\mathbf{k}|e^{-|\mathbf{k}|}; \qquad |B_{\mathbf{k}}| \leq C(|b_{\mathbf{k}}|/H + |c_{\mathbf{k}}|)|\mathbf{k}|^2 e^{-|\mathbf{k}|}. \tag{52}$$

With a change of variables, we can write

$$a_{\mathbf{k}}(x_3) = \left(A_{\mathbf{k}} + \frac{H}{|\mathbf{k}|}B_{\mathbf{k}}\, z\right)\sinh z - zA_{\mathbf{k}}\cosh z; \qquad z = \frac{|\mathbf{k}|}{H}x_3. \tag{53}$$

This function vanishes at $z=0$ and has also vanishing first-order derivative at $z=0$, while its second-order derivative is given by

$$2\frac{H}{|\mathbf{k}|}B_{\mathbf{k}}\cosh z + \frac{H}{|\mathbf{k}|}B_{\mathbf{k}} z\sinh z - A_{\mathbf{k}}(\sinh z + z\cosh z).$$

By a Taylor expansion in integral form, one finds that

$$|a_{\mathbf{k}}(x_3)| \leq Cz^2 e^z \left(|A_{\mathbf{k}}| + \frac{H}{|\mathbf{k}|}|B_{\mathbf{k}}|\right); \qquad z = \frac{|\mathbf{k}|}{H}x_3,$$

which implies

$$|a_{\mathbf{k}}(x_3)| \leq C\left(\frac{|\mathbf{k}|}{H}\right)^2 e^{\frac{|\mathbf{k}|}{H}}\left(|A_{\mathbf{k}}| + \frac{H}{|\mathbf{k}|}|B_{\mathbf{k}}|\right); \qquad x_3 \in [0,1].$$

From (52), it follows that

$$|a_{\mathbf{k}}(x_3)| \leq C\left(\frac{|\mathbf{k}|}{H}\right)^2 e^{\frac{|\mathbf{k}|}{H}}(|b_{\mathbf{k}}| + H|c_{\mathbf{k}}|)|\mathbf{k}|e^{-|\mathbf{k}|} \leq C|\mathbf{k}|^3 e^{-\frac{1}{2}|\mathbf{k}|}\left(\frac{|b_{\mathbf{k}}|}{H^2} + \frac{|c_{\mathbf{k}}|}{H}\right); \qquad x_3 \in [0,1],$$

so by (51) the series in (47) converges arbitrarily fast for $x_3 \in [0,1]$. This proves the first statement in the proposition.

We turn next to the second assertion. It will be sufficient to control the second derivative in x_3 of $\hat{w}_1$. Using (53), we have that

$$\begin{aligned}
\frac{d^2}{dx_3^2}a_{\mathbf{k}}(x_3) &= \left(\frac{dz}{dx_3}\right)^2 \frac{d^2}{dz^2}a_{\mathbf{k}}(x_3) \\
&= \left(\frac{|\mathbf{k}|}{H}\right)^2\left[2\frac{H}{|\mathbf{k}|}B_{\mathbf{k}}\cosh z - A_{\mathbf{k}}\sinh z + z\left(\frac{H}{|\mathbf{k}|}B_{\mathbf{k}}\sinh z - A_{\mathbf{k}}\cosh z\right)\right].
\end{aligned}$$

To evaluate this quantity for $x_3 \in [H-1, H]$, we write

$$\begin{aligned}
&2\frac{H}{|\mathbf{k}|}B_{\mathbf{k}}\cosh z - A_{\mathbf{k}}\sinh z + z\left(\frac{H}{|\mathbf{k}|}B_{\mathbf{k}}\sinh z - A_{\mathbf{k}}\cosh z\right) \\
= &\left[\left(2\frac{H}{|\mathbf{k}|}B_{\mathbf{k}} - A_{\mathbf{k}}\right) + z\left(\frac{H}{|\mathbf{k}|}B_{\mathbf{k}} - A_{\mathbf{k}}\right)\right]\sinh z + \left(2\frac{H}{|\mathbf{k}|}B_{\mathbf{k}} - A_{\mathbf{k}}\right)e^{-z} - z\left(\frac{H}{|\mathbf{k}|}B_{\mathbf{k}} - A_{\mathbf{k}}\right)e^{-z}.
\end{aligned}$$

For $x_3 \in [H-1, H]$, by (52) the latter expression can be bounded by

$$Cze^z\left(\frac{H}{|\mathbf{k}|}|B_{\mathbf{k}}| + |A_{\mathbf{k}}|\right) \leq C|\mathbf{k}|e^{|\mathbf{k}|}(|b_{\mathbf{k}}| + H|c_{\mathbf{k}}|)|\mathbf{k}|e^{-|\mathbf{k}|} \leq C|\mathbf{k}|^2(|b_{\mathbf{k}}| + H|c_{\mathbf{k}}|).$$

The above formula for $\frac{d^2}{dx_3^2}a_{\mathbf{k}}(x_3)$ then implies

$$\left|\frac{d^2}{dx_3^2}a_{\mathbf{k}}(x_3)\right| \leq C\frac{|\mathbf{k}|^4}{H^4}\frac{d^2}{dx_3^2}a_{\mathbf{k}}(x_3); \qquad x_3 \in [H-1, H].$$

By (51) this function is uniformly bounded, giving the conclusion by the last estimate in (39) and by the fact that $\tilde{w}_1 - \hat{w}_1$ vanishes and has zero normal derivative on the plane $\{x_3 = H\}$. ■

We can now prove our main result.

Proof of Theorem 1.2. Let θ_1, θ_2 be as in (33), and let w_i, $i = 1, 2$, be the solution of

$$\begin{cases} \Delta^2 w_i = \Delta_h \theta_i & \text{in } \{0 \leq x_3 \leq H\}; \\ w_i = 0 & \text{on } \{x_3 = 0\} \cup \{x_3 = H\}; \\ \partial_{x_3} w_i = 0 & \text{on } \{x_3 = 0\} \cup \{x_3 = H\}. \end{cases}$$

If $\tilde{w}_1$ is as in (34) and $\hat{w}_1$ as in (46), then by uniqueness $w_1 = \tilde{w}_1 - \hat{w}_1$. Proposition 3.3 (recalling that $\theta = \theta_1$ for $x_3 \leq H/2$) and the first estimate in Proposition 4.1 imply that $\langle \theta\, w_1 \rangle$ is uniformly bounded for $x_3 \in (0, 1]$.

On the other hand, we could apply the second estimate in Proposition 4.1 to the function $w_2(x', H - x_3)$ and again use the L^∞-bound on θ to show that also $\langle \theta\, w_2 \rangle$ is uniformly bounded for $x_3 \in (0, 1]$. The conclusion follows then from formula (10). ■

5 Appendix

In this appendix we collect some remarks to help the reader understand the proof of Proposition 2.3 and to point out the elementary observations that allow us to adapt the original one for Theorem 33 in [8]. In fact this modification is already suggested by the proof of that result. We collect first some facts from the book [27].

Lemma 5.1 ([27], page 62) *Let P_t denote the Poisson kernel, see* (13)*. Then its Fourier transform has the expression*

$$\hat{P}_t(\xi) = e^{-2\pi t|\xi|}, \qquad \xi \in \mathbb{R}^2.$$

In particular, $\hat{P}_t$ is exponentially decaying at infinity and is in the Schwartz space at infinity.

It is useful to recall the following definition.

Definition 5.2 *Let μ be a measure on $\mathbb{R}^3_+$. Let us consider the* Carleson box

$$T(x^0, h) := \left\{(x, h) \in \mathbb{R}^2 \times \mathbb{R} \mid |x - x^0| < h, 0 \leq t < h\right\}.$$

We say that μ is a Carleson measure *if there is an absolute constant C_1 such that for all $h > 0$*

$$\mu(T(x^0, h)) \leq C_1 h^2.$$

Recalling the definition of non-tangential maximal function in (17), we have the following result.

Lemma 5.3 ([27], page 236) *Let $F(x, t)$ be any measurable function on $\mathbb{R}^3_+$, and let μ be a Carleson measure. Then, for $0 < p < +\infty$*

$$\int_{\mathbb{R}^3_+} |F(x,t)|^p d\mu \leq C(p)\, C_1 \int_{\mathbb{R}^2} |NF|^p dx,$$

where C_1 is the constant in Definition 5.2 and and NF is the non-tangential maximal function of F.

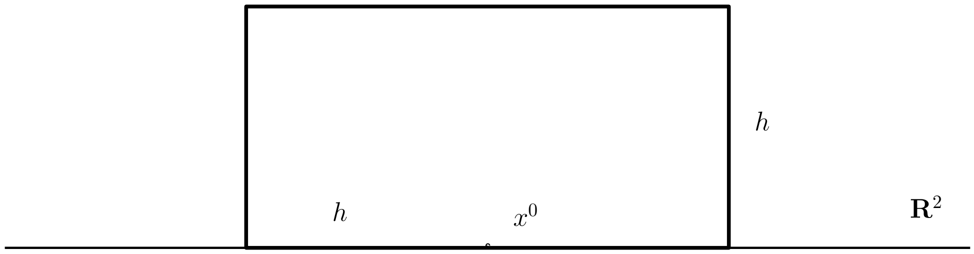

Figure 4: A Carleson box

The case $p = 2$ is employed in [8]. We will call *bump functions* approximate identities of the type

$$\varphi_t(x) = \frac{1}{t^n}\varphi(x/t).$$

Even though Proposition 2.3 is stated for vectorial kernels, it will be enough to consider scalar ones. We will employ as bump functions $t\nabla_h P_t$, where $\nabla_h P_t$ stands for the horizontal derivatives of the Poisson kernel P_t on $\mathbb{R}^2$. The Fourier transform of this function when $t = 1$ is

$$\xi_j \hat{P}(\xi) = \xi_j e^{-2\pi|\xi|},$$

see e.g. [27, page 61]. We have the following estimates, for all multi-indices α

$$\begin{cases} |\partial_\xi^\alpha \hat{P}(\xi)| \leq C(\alpha) e^{-C(\alpha)^{-1}|\xi|} & \text{for } |\xi| \geq 1; \\ |\xi|^\alpha |\partial_\xi^\alpha \hat{P}(\xi)| \leq C(\alpha)|\xi| & \text{for } |\xi| \leq 1. \end{cases} \tag{54}$$

Thus the above inequalities imply the decay assumption in Theorem 33 of [8], reported in Remark 2.4.

The proof of Theorem 33 in [8] has a key step, namely Lemma 5 on page 151. Here one establishes the theorem under the provisional assumption that the bump functions have compactly supported Fourier transform, with the origin not lying in the support of the Fourier transform of any of the bump functions, and also the origin not lying in the algebraic sum of the supports of the Fourier transform of the bump functions.

In the general case where assumptions (54) hold true, the arguments on pages 152-153 in [8] reduce matters to Lemma 5 there. We now make some remarks on the proof of Lemma 5, which is already implicit in the proof displayed in [8], but that we now highlight. We collect real variable facts used in Lemma 5 which are standard.

Lemma 5.4 *For two functions $\varphi, \psi \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ one has*

(i) $\widehat{\varphi * \psi}(\xi) = \hat{\varphi}(\xi)\hat{\psi}(\xi)$*;*

(ii) $\widehat{\varphi\psi}(\xi) = (\hat{\varphi} * \hat{\psi})(\xi)$*;*

(iii) *the support of $\hat{\varphi} * \hat{\psi}$ is contained in the algebraic sum of* supp $\hat{\varphi}$ *and* supp $\hat{\psi}$.

In [8], one considers integrals of the type

$$\int_0^\infty (f * \varphi_t)(x)(b * \psi_t)(x) m(t) \frac{dt}{t},$$

namely

$$\int_0^\infty (F(\cdot, t) * \varphi_t)(x)(b * \psi_t)(x) \frac{dt}{t},$$

with

$$F(x',t) = f(x')m(t). \tag{55}$$

In our case, for Proposition 2.3, we need to estimate integrals of the type

$$\int_0^\infty (F(\cdot,t) * \varphi_t)(x)(b * \psi_t)(x)\frac{dt}{t},$$

with $b \in L^\infty(\mathbb{R}^2)$ (or even BMO), $\varphi_t = \psi_t$ and a more general dependence of F on t, compared to (55).

The latter integral is a paraproduct, see [8], Appendix I. To estimate the $L^2(\mathbb{R}^2)$-norm of the above expression, in [8] one proceeds via duality and consider, for h of class $L^2(\mathbb{R}^2)$, the operator

$$h \quad \longmapsto \quad \int_{\mathbb{R}^2} h(x)\left[\int_0^\infty (F(\cdot,t) * \varphi_t)(x)(b * \psi_t)(x)m(t)\frac{dt}{t}\right] dx. \tag{56}$$

The main idea of Lemma 5 in [8] (see page 151 there), under the above-mentioned assumptions on the supports of $\hat{\varphi}$ and $\hat{\psi}$, is that one can write, for $\delta > 0$

$$[(F(\cdot,t) * \varphi_t)(\cdot)(b * \psi_{\delta t})(\cdot)]m(t) \qquad \text{as} \qquad \psi_{3,t} * [(F(\cdot,t) * \varphi_t)(\cdot)(b * \psi_{\delta t})(\cdot)]m(t).$$

Here $\psi_{3,t}(x) = \frac{1}{t^2}\psi_3(x/t)$ is a bump function for which $\hat{\psi}_3$ is compactly supported, smooth, such that $\hat{\psi}_3 \equiv 1$ on supp $\hat{\varphi}(\cdot)$ + supp $\hat{\psi}(\delta\,\cdot)$ and with supp $\hat{\psi}_3$ contained either in a ball or in a proper annulus around the origin. When applying then L^2-duality in (56), one can then pass the convolution with $\psi_{3,t}$ to h and use Littlewood-Paley estimates.

To explain these steps in more detail, consider the Fourier transform in x of

$$(F(\cdot,t) * \varphi_t)(x)(b * \psi_{\delta t})(x)m(t). \tag{57}$$

By Lemma 5.4, this is given by

$$[\hat{F}(\xi,t)\widehat{\varphi_t}(\xi)] * [\hat{b}(\xi)\widehat{\psi_t}(\xi)]m(t). \tag{58}$$

The point is now that the support of $\hat{F}(\xi,t)\widehat{\varphi_t}(\xi)$ lies in the support of $\widehat{\varphi_t}$ and the support of $\hat{b}(\xi)\widehat{\psi_t}(\xi)$ lies in the support of $\widehat{\psi_t}$, and for this to hold it is irrelevant that $F(x,t)$ depends on t or that there is the extra factor $m(t)$. Thus the support of (58) lies in the algebraic sum of the supports of $\widehat{\varphi_t}$ and $\widehat{\psi_t}$.

By the above discussion, the dual pairing of h and the expression in (57) becomes

$$\int_{\mathbb{R}^2} h(x)\int_0^\infty \psi_{3,t} * [(F(\cdot,t) * \varphi_t)(b * \psi_{\delta t})](x)m(t)\frac{dt}{t}dx,$$

and after switching convolutions it can be written as

$$\int_{\mathbb{R}^2}\int_0^\infty (\psi_{3,t} * h)(x)[(F(\cdot,t) * \varphi_t)(b * \psi_{\delta t})](x)m(t)\frac{dt}{t}dx.$$

Using the fact that $m \in L^\infty$ and the Cauchy-Schwarz inequality, this quantity can be bounded by

$$\left(\int_{\mathbb{R}^2}\int_0^\infty |(\psi_{3,t} * h)(x)|^2\frac{dt}{t}dx\right)^{\frac{1}{2}} \times \left(\int_{\mathbb{R}^2}\int_0^\infty |(F(\cdot,t) * \varphi_t)(x)|^2|(b * \psi_{\delta t})(x)|^2\frac{dt}{t}dx\right)^{\frac{1}{2}}. \tag{59}$$

We apply Lemma 1 in [8] to the first term in the latter formula to conclude that it is bounded by a fixed constant times $\|h\|_{L^2(\mathbb{R}^2)}$. To the second term we apply Lemma 5.3. Recall that we set

$$(F(\cdot,t) * \varphi_t)(x) = \check{F}(x,t):$$

from Theorem 2.1 we see that $|(b * \psi_{\delta t})(x)|^2 \frac{dt}{t} dx$ is a Carleson measure since b is in L^∞, so the second term in (59), using Lemma 5.3, can be bounded by

$$\left(\int_{\mathbb{R}^2} (N\check{F})^2(x) dx \right)^{\frac{1}{2}},$$

as desired. Notice that the bound is uniform in $\delta > 0$ because of the scaling invariance in t of both $\frac{dt}{t}$ and the expression of $N\check{F}$. For general kernels as in Proposition 2.3, see the comments before Lemma 5.4, we follow the arguments on pages 152-153 in [8].

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