Ovals and Width

An Introduction to Differential Geometry

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These slides are based on notes from an undergraduate course in Differential Geometry that I took at Indiana Universtiy, Bloomington. I'd like to thank Professor Bruce Solomon for an exciting introduction to the field and for the notes he provided, which I still reference today, years after graduation.

Preliminaries

Throughout this lecture, we will use c to denote the standard parametrization of the unit circle. Namely,

$$c(t) = (\cos(t), \sin(t))$$

One can prove (though, we will not do so here) that any curve with nonvanishing speed can be reparametrized to have unit speed. So, for this lecture, we will always assume that our curves has unit speed.

Ovals

Periodic and Simple Parametrizations

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Which examples from yesterday (morning or afternoon) were periodic? Which were periodic *and* simple?

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Note that a simple closed curve whose curvature is always negative can be reparametrized to have positive curvature. So, we will typically assume it always has positive curvature.

Geometric Description

Geometrically, $\kappa \neq 0$ ensures that an arc has no inflection points or “flat” points.

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Thus, $\phi = \varphi - \pi/2$, where φ is the angle with (0,1), whose derivative (according to the Proposition from yesterday) is the curvature. The same holds for the derivative of ϕ:

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$$\phi' = (\varphi - \pi/2)' = \varphi' = \kappa$$

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This "angular" reparametrization σ is called the **support parametrization** of the oval. Note that it is 2π periodic.

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$$\sigma'(\theta) = c'(\theta)(\phi^{-1})'(\theta) = \frac{c'(\theta)}{\phi'(t)} = \frac{c'(\theta)}{\kappa(\phi^{-1}(\theta))}$$

Curvature and the Support Parametrization

Proposition

Every oval has a 2π *periodic support parametrization* σ *that satisfies*

$$\sigma'(\theta) = \frac{c'(\theta)}{\kappa(\phi^{-1}(\theta))}$$

This parametrization is called the **support parametrization**.

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$$W_\theta = \sigma(\theta) \cdot c(\theta) - \sigma(\theta + \pi) \cdot c(\theta)$$

Where $\cdot$ represents the dot product of two vectors. Namely, $(x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$.

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We call h the support function of the curve.