Curves

An Introduction to Differential Geometry

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What is a Curve?

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How would you define a **curve**?

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For simplicity, we will restrict our attention to curves in 2-dimensions.

What is a Curve?

Definition

An **arc** in the plane is the image of a function $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ whose domain is an interval (a, b). We call α the **parametrization** of the curve.

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Example

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Example

$\alpha(t) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$

More Examples

Draw the curves given by the following parametrizations.

1. $\alpha(t) = (0, 2t)$, $0 \leq t \leq 1$
2. $\alpha(t) = (0, t)$, $0 \leq t \leq 2$
3. $\alpha(t) = (t, -t)$, $-1 \leq t \leq 1$
4. $\alpha(t) = (3\cos(t), 3\sin(t))$, $0 \leq t \leq 2\pi$
5. $\alpha(t) = (t\cos(t), t\sin(t))$, $0 \leq t \leq 4\pi$.
6. $\alpha(t) = (\cos(t), 2\sin(t))$, $0 \leq t \leq 2\pi$

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The **velocity vector** of a curve α at t_0 is

$$\alpha'(t_0) = (\alpha_1'(t_0), \alpha_2'(t_0))$$

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$$\alpha'(t_0) = (\alpha_1'(t_0), \alpha_2'(t_0))$$

and the **speed** at t_0 is $\left|\alpha'(t_0)\right| = \sqrt{(\alpha_1'(t_0))^2 + (\alpha_2'(t_0))^2}$.

Examples

Compute the velocity vectors and speeds of the examples from before, copied below.

1. $\alpha(t) = (0, 2t)$, $0 \leq t \leq 1$
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Definition

The **curvature** of the curve α at the point t_0 is

$$\kappa(t_0) = \frac{\alpha_1'(t_0)\alpha_2''(t_0) - \alpha_2'(t_0)\alpha_1''(t_0)}{\left|\alpha'(t_0)\right|^3}$$

Examples

Compute the curvature of the curves we've been looking at, which are copied below.

1. $\alpha(t) = (0, 2t)$, $0 \leq t \leq 1$
2. $\alpha(t) = (0, t)$, $0 \leq t \leq 2$
3. $\alpha(t) = (t, -t)$, $-1 \leq t \leq 1$
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Curvature and Angles

Definition

We say a curve α has **unit speed** if $\left|\alpha'(t)\right| = 1$.

Proposition

Suppose $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ *is a planar arc with unit speed. Define* $\varphi(t)$ *as the angle between* $\alpha'(t)$ *and* $(1,0)$. *Then,*

$$\kappa(t) = \varphi'(t)$$

Examples

Compute $\varphi(t)$ for each of the curves that we've been looking at. Confirm for each one that $\varphi'(t) = \kappa(t)$.

1. $\alpha(t) = (0, 2t)$, $0 \leq t \leq 1$
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