Find The Error Problems

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Describe the error(s) in each of the following "proofs". It is worth noting that some of the below propositions are true, while others are not. You are *not* being asked to determine whether the proposition is true. You are being asked to find the error in the "proofs", regardless of the validity of the proposition.

Proposition 0.1. *For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.*

Proof. Suppose k is any integer such that $k > 0$. If $k^2 + 2k + 1$ is composite, then $k^2 + 2k + 1 = rs$ for some integers r and s such that

$$1 < r < (k^2 + 2k + 1)$$

and

$$1 < s < (k^2 + 2k + 1).$$

Since $k^2 + 2k + 1 = rs$ where both r and s are strictly between 1 and $k^2 + 2k + 1$, then $k^2 + 2k + 1$ is not prime. Hence, $k^2 + 2k + 1$ is composite, as was to be shown. □

Proposition 0.2. *The sum of any two even integers equals $4k$ for some integer k.*

Proof. Suppose m and n are any two even integers. By definition of even, $m = 2k$ for some integer k and $n = 2k$ for some integer k. By substitution,

$$m + n = (2k) + (2k) = 4k$$

This is what was to be shown. □

Proposition 0.3. *For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.*

Proof. For $k = 2$, $k^2 + 2k + 1 = 2^2 + 2 \cdot 2 + 1 = 9$. But $9 = 3 \cdot 3$, and so 9 is composite. Hence the proposition is true. □

Proposition 0.4. *The difference between any odd integer and any even integer is odd.*

Proof. Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and by definition of even, $m = 2k$ where k is any integer. Then

$$n - m = (2k + 1) - 2k = 1.$$

But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd. □

Proposition 0.5. *The sum of any two rational numbers is a rational number.*

Proof. Suppose r and s are rational numbers. Then $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $d \neq 0$ by definition of rational. Then

$$r + s = \frac{a}{b} + \frac{c}{d}.$$

But this is a sum of two fractions, which is a fraction. So $r + s$ is a rational number since a rational number is a fraction. □

Proposition 0.6. *For any integer n, $n^2 + 5$ is not divisible by 4.*

Proof. If n is even, then $n = 2k$ for some integer k. So,

$$n^2 + 5 = (2k)^2 + 5 = 4k^2 + 5 = 4(k^2 + 1) + 1,$$

which leaves a remainder of 1 when divided by 4. Therefore $n^2 + 5$ is not divisible by 4. □

Proposition 0.7. *Let $a, b, c \in \mathbb{Z}$. Then $ac|bc$ if and only if $a|b$.*

Proof. If $ac|bc$, then

$$bc = ack \tag{1}$$

for some $k \in \mathbb{Z}$. Then, dividing both sides of equation (1) by c, we obtain

$$b = ak.$$

Since $k \in \mathbb{Z}$, we see that $a|b$, as desired. □

Note that the proof of Proposition 0.7 has two errors. One is more obvious than the other.

Proposition 0.8. *If $x \in \mathbb{R}$, then $\frac{4}{x(4-x)} \geq 1$.*

Proof. Observe that since $x - 2 \in \mathbb{R}$,

$$(x - 2)^2 \geq 0.$$

We can rearrange the above equation to get our desired result:

$$\begin{aligned}
(x-2)^2 &\geq 0 \\
x^2 - 4x + 4 &\geq 0 \\
x^2 - 4x &\geq -4 \\
x(4-x) &\leq 4 \\
4 - x &\leq \frac{4}{x} \\
1 &\leq \frac{4}{x(4-x)}
\end{aligned}$$

□

Proposition 0.9. *$\forall n \in \mathbb{Z}$, if n is odd, then $\frac{n-1}{2}$ is odd.*

Proof. Let $n \in \mathbb{Z}$ be odd. Then, $n = 2k+1$ for some $k \in \mathbb{Z}$. So,

$$\frac{n-1}{2} = \frac{2k+1-1}{2} = k,$$

which is clearly odd. □

Proposition 0.10. *Let $a, b, c \in \mathbb{Z}$. If $c|ab$, then $c|a$ and $c|b$.*

Proof. If $c|a$ then $a = ck$ for some $k \in \mathbb{Z}$, and if $c|b$, then $b = c\ell$ for some $\ell \in \mathbb{Z}$. So,

$$ab = (ck)(c\ell) = c(ck\ell).$$

Since $ck\ell \in \mathbb{Z}$, this shows that $c|ab$. □

Proposition 0.11. *Every integer is rational.*

Proof. Suppose not. Then, every integer is irrational. But then $1 = \frac{1}{1}$, which is an integer, is rational. This is a contradiction. Hence, every integer must be rational. □

Proposition 0.12. *Suppose $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$, then $2a|(b+c)$.*

Proof. If a divides b, then $b = ak$ for some $k \in \mathbb{Z}$. Since a divides c, then $c = ak$ for some $k \in \mathbb{Z}$. Therefore,

$$b + c = (ak) + (ak) = (2a)k.$$

Since $k \in \mathbb{Z}$, this shows that $2a|(b+c)$. □

Proposition 0.13. *$\forall x \in \mathbb{Z}$, $x^2 + x$ is even.*

Proof. Suppose $x \in \mathbb{Z}$ is even. Then, by definition, $x = 2k$ for some $k \in \mathbb{Z}$. Therefore,

$$\begin{aligned} x^2 + x &= (2k)^2 + (2k) \\ &= 4k^2 + 2k \\ &= 2(2k^2 + k). \end{aligned}$$

Since $2k^2 + k \in \mathbb{Z}$, this shows that $x^2 + x$ is even. □

Proposition 0.14. *$\exists x \in \mathbb{Q}$ such that $\forall k \in \mathbb{Z}$, $|x - k| > \frac{1}{4}$.*

Proof. Let $x = \frac{1}{2} \in \mathbb{Q}$ and $k = 2 \in \mathbb{Z}$. Then

$$|x - k| = \left|\frac{1}{2} - 2\right| = \frac{3}{2} > \frac{1}{4}$$

□

Proposition 0.15. *If k is any odd integer and m is any even integer, then $m^2 - k^2$ is odd.*

Proof. Since k is even, $k = 2a$ for some $a \in \mathbb{Z}$. Since m is odd, $m = 2a + 1$ for some $a \in \mathbb{Z}$. Then,

$$\begin{aligned} m^2 - k^2 &= (2a + 1)^2 - (2a)^2 \\ &= 4a^2 + 4a + 1 - 4a^2 \\ &= 4a + 1 \\ &= 2(2a) + 1. \end{aligned}$$

Since $2a \in \mathbb{Z}$, by definition $m^2 - k^2$ is odd. □

Proposition 0.16. *An integer x is odd if and only if $x = 2k - 1$ for some $k \in \mathbb{Z}$.*

Proof. If $x = 2k - 1$ for some $k \in \mathbb{Z}$, we can rewrite this as

$$\begin{aligned} x &= 2k - 1 \\ &= 2(k - 1) + 2 - 1 \\ &= 2(k - 1) + 1. \end{aligned}$$

Since $(k - 1) \in \mathbb{Z}$, this shows that x is odd. □

Proposition 0.17. *Let $r \in \mathbb{Q}$ and $s \in \mathbb{R}$. If $rs \in \mathbb{Q}$, then $s \in \mathbb{Q}$.*

Proof. Since $r \in \mathbb{Q}$, $\exists a, b \in \mathbb{Z}$ with $b \neq 0$ such that

$$r = \frac{a}{b}.$$

Then, if $s \in \mathbb{Q}$, $\exists c, d \in \mathbb{Z}$ with $d \neq 0$ such that

$$s = \frac{c}{d}.$$

So,

$$rs = \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}.$$

Since $ac, bd \in \mathbb{Z}$ with $bd \neq 0$, this shows that $rs \in \mathbb{Q}$. □

Proposition 0.18. *If a and b are both even integers, then $a^2 + b^2$ is divisible by 8.*

Proof. Since a and b are even integers, $a = 2x$ for some $x \in \mathbb{Z}$ and $b = 2x$ for some $x \in \mathbb{Z}$. So,

$$\begin{aligned} a^2 + b^2 &= (2x)^2 + (2x)^2 \\ &= 4x^2 + 4x^2 \\ &= 8x^2 \end{aligned}$$

Since $x^2 \in \mathbb{Z}$, this shows that $a^2 + b^2$ is divisible by 8. □

Proposition 0.19. *If a and b are even integers, then $a^4 + b^4 + 32$ is divisible by 8.*

Proof. Suppose a is an even integer, which we can take to be 2, and suppose b is an even integer, which we can take to be 4. Then

$$a^4 + b^4 + 32 = 2^4 + 4^2 + 32 = 16 + 256 + 32 = 304.$$

Since 304 is divisible by 8 (namely, 304 = 8*38), we are done. □

Proposition 0.20. *Take $n \in \mathbb{N}$. Then $n^3 + 1$ is composite.*

Proof. Suppose $n^3 + 1$ is prime. For notational simplicity, call it p. Then, since p is prime, it has no divisors. But p is an integer, and 1 divides every integer, so p has a divisor. Therefore, we have shown that p has a divisor and that p has no divisors, which is a contradiction. Therefore, $n^3 + 1$ must have been composite. □

Proposition 0.21. *Let $a, b \in \mathbb{Z}$. If $a^3 | b^2$, then $a | b$.*

Proof. Since $a^3|b^2$, there exists a $k \in \mathbb{Z}$ such that

$$b^2 = ka^3.$$

Taking the square root of both sides, we obtain

$$b = \sqrt{ka^3} = a\sqrt{ka}.$$

This shows that a times a number equals b, which proves that $a|b$. □

Proposition 0.22. *$\forall n \in \mathbb{N}$,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. Base Case: Observe that

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

So, the result holds for $n = 1$.

Inductive Step: Suppose the result holds for all $n \in \mathbb{N}$, then

$$\begin{aligned} 1 + 2 + \cdots + (n+1) &= (1 + 2 + \cdots + n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2} \end{aligned}$$

□

Proposition 0.23. *For all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.*

Proof. Suppose A and B are sets and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ and hence $A^c \cup B^c \subseteq (A \cup B)^c$. □

Proposition 0.24. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x + 3$. Then f is injective.*

Proof. Suppose $x, y \in \mathbb{R}$ such that $x = y$. Then we can multiply both sides by 2 and then add 3 to both sides to get that $2x + 3 = 2y + 3$. But this equation says that $f(x) = f(y)$. Therefore, f is injective. □

Proposition 0.25. *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then f is injective.*

Proof. Suppose $x, y \in \mathbb{R}$ are such that $f(x) = f(y)$, ie $x^2 = y^2$. Taking the square root of both sides, we see that $x = y$. Hence, f is injective. □