Workshop Review 3

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1 Major Theorems

For each of the theorems described below, you should know the precise statement and how to apply it. You should also know the precise definitions of the terms in the below theorems. For those theorems marked with *, you should also know at least the key points of the proofs, because those techniques are useful in many other contexts.

- Equivalent definitions of being continuous at a point.
- Sums, products, and quotients of continuous functions are continuous.*
- Composition of continuous functions is continuous.*
- A uniformly continuous function is continuous, but not conversely.*
- Continuous functions on a compact set are uniformly continuous.*
- Heine-Borel
- Equivalent definitions of closedness.*
- Equivalent definitions of compactness.
- The image of a bounded set under a uniformly continuous function is bounded.*
- The image of a compact set under a continuous function is compact.*
- Every continuous function on a compact set obtains a maximum and a minimum.*
- Bolzano's Theorem
- Intermediate Value Theorem*

2 Starter Problems

The problems below are basic questions, which help you assess whether you understand the basic definitions and theorems.

1. Define $f : \mathbb{R} \to \mathbb{R}$ as below:

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Prove that f is continuous at 0.

2. Prove that $(0,1)$ is open.
3. Prove that $[0,1]$ is closed.
4. Prove that $[0,1)$ is neither open nor closed.
5. Starting at 9am on Monday, a hiker walked from the trailhead up a mountain and reached the summit at exactly 3pm. The hiker camped for the night and then hiked back down the same trail, again starting at 9am. On this second walk, the hiker walked very slowly at first, but walked faster on other parts of the trail and returned to the starting point in exactly six hours. Prove that there is some point on the trail that the hiker passed at exactly the same time on the two days.
6. Let $S = (0,1)$.
   (a) Prove that S is not compact, using the bounded and closed definition.
   (b) Prove that S is not compact, using the sequence definition.
   (c) Prove that S is not compact, using open covers.
7. Let E be compact and nonempty. Prove that $\sup E$ and $\inf E$ exist and are elements of E
8. Is it true that the image of a bounded set under a continuous function is bounded? Prove or give a counterexample.

3 Goldilocks Problems

The problems below are a bit harder than the ones in the section above. These are closer in difficulty to the homework problems and the problems you are likely to see on the exam.

1. Suppose $f : [0,1] \to [0,1]$ is continuous. Prove that there exists some $x_0 \in [0,1]$ such that $f(x_0) = x_0$.
2. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Prove that f is continuous. You may use the fact that $\lim_{x\to 0} e^x = 1$.
3. Let $S = [0,2] \cap \mathbb{Q}$.
   (a) Prove that S is not compact, using the bounded and closed definition.
   (b) Prove that S is not compact, using the sequence definition.
   (c) Prove that S is not compact, using open covers.
4. Suppose the functions $f, g : \mathbb{R} \to \mathbb{R}$ are continuous and that $f(r) = g(r)$ for every $r \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for every $x \in \mathbb{R}$. Hint: the rationals are dense in $\mathbb{R}$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $\mathcal{O} \subseteq \mathbb{R}$ be an open set. Prove that $f^{-1}(\mathcal{O})$ is open.
6. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $C \subseteq \mathbb{R}$ be compact. Prove that $f(C)$ is compact, using open covers.
7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Prove that the zero set of f, $f^{-1}(\{0\})$ is closed.

4 Challenge Problems

The problems in this section are the hardest on this worksheet, and require the most creativity. They are more difficult that the types of problems you are likely to see on the exam. Hence, they are a great way to deepen your understanding, but only if you are already comfortable with the problems earlier in this packet.

1. Suppose $f : E \to \mathbb{R}$ is continuous where $E \subseteq \mathbb{R}$ is closed. Prove that there exists a $g : \mathbb{R} \to \mathbb{R}$ which is continuous and for all $x \in E$, $g(x) = f(x)$. Such a function is called a *continuous extension* of f. Give a counterexample to show that the statement is false if the word "closed" is omitted.

2. Suppose $f : \mathbb{Q} \to \mathbb{R}$ is uniformly continuous. Prove that f has a continuous extension to all of $\mathbb{R}$.

3. For a nonempty set $S \subseteq \mathbb{R}$ and $x_0 \in \mathbb{R}$, we define the *distance*, $\rho_S(x_0)$ from x_0 to S as follows:

$$\rho_S(x_0) = \inf\{|s - x_0| : s \in S\}.$$

   (a) Suppose S is closed. Prove that $\rho_S(x_0) = 0$ if and only if $x_0 \in S$.

   (b) Is the above statement still true if we remove the assumption that S is closed? Prove or give a counterexample.

   (c) We can similarly define the distance between two sets S, T as follows:

$$\rho(S, T) = \inf\{|s - t| : s \in S, t \in T\}.$$

   Suppose S is closed and T is compact. Prove that there exist $s_0 \in S$ and $t_0 \in T$ such that $|s_0 - t_0| = \rho(S, T)$.