Workshop Review 2

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1 Major Theorems

For each of the theorems below, you should know the precise statement and how to apply it. You should also know the precise definitions of the terms in the below theorems. For those theorems marked with *, you should also know at least the key points of the proofs, because those techniques are useful in many other contexts.

- f converges to L at x_0 if and only if for every sequence $\{x_n\}$ converging to x_0, $f(x_n)$ also converges to L. (In particular, f has a limit at x_0 if and only if for every sequence $\{x_n\}$ converging to x_0, the sequence $f(x_n)$ converges to the same value)
- If f has a limit at x_0, then it is bounded near x_0.*
- The limit of a sum of functions is the sum of the limits, provided they exist.*
- The limit of a product of functions is the product of the limits, provided they exist.*

2 Straightforward Problems

The problems below are basic questions, which help you assess whether you understand the basic definitions and theorems.

1. Prove that $\lim\limits_{x\to 0}(x^2+4)=4$.
2. Prove that $\lim\limits_{x\to 4}\sqrt{x}=2$.
3. Prove that $\lim\limits_{x\to 3}\frac{1}{x}=\frac{1}{3}$.
4. Prove or give a counterexample: If $\lim_{x\to x_0} f(x)=L$ and x_0 is in the domain of f, then $f(x_0)=L$

3 Goldilocks Problems

The problems below are a bit harder than the ones in the section above. These are closer in difficulty to the homework problems and the problems you are likely to see on the exam.

1. Suppose $f: D\to\mathbb{R}$ with x_0 an accumulation point of D. Assume L_1 and L_2 are limits of f at x_0. Prove that $L_1=L_2$.

2. Let $D \subset \mathbb{R}$ and x_0 be an accumulation point of D. Suppose f, g, and h are all real-valued functions with domain D. Suppose further that

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in D$, and that

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L.$$

Prove that

$$\lim_{x \to x_0} g(x) = L.$$

3. Let $D_1, D_2 \subset \mathbb{R}$, and suppose $x_1 \in D_1$ is an accumulation point of D_1. Let $f : D_1 \to D_2$ be such that

$$\lim_{x \to x_1} f(x) = p \in D_2.$$

Suppose that p is an accumulation point of D_2 and that $g : D_2 \to \mathbb{R}$ is such that

$$\lim_{x \to p} g(x) = q.$$

Is it true that

$$\lim_{x \to x_1} g(f(x)) = q?$$

Prove or give a counterexample.

4. Prove that $\lim_{x \to 10} \lfloor x \rfloor \neq 10$

5. Prove or give a counterexample: Suppose $f, g : D \to \mathbb{R}$ and x_0 is an accumulation point of D. If $\lim\limits_{x \to x_0} f(x) = 0$, then $\lim\limits_{x \to x_0} f(x)g(x) = 0$.

4 Challenge Problems

The problems in this section are the hardest on this worksheet, and require the most creativity. They are more difficult that the types of problems you are likely to see on the exam. Hence, they are a great way to deepen your understanding, but only if you are already comfortable with the problems earlier in this packet.

1. Compute and prove the limit or prove that it does not exit: $\lim\limits_{x \to 0} (-1)^{\lfloor x \rfloor}$.

2. Compute and prove the limit or prove that it does not exit: $\lim\limits_{x \to 0} \sqrt[3]{x}(-1)^{\lfloor x \rfloor}$.

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisifies

$$\lim_{h \to 0} \big(f(x_0 + h) - f(x_0 - h) \big) = 0.$$

Is it true that $\lim_{x \to x_0} f(x) = f(x_0)$?

4. Suppose $f : (a, b) \to \mathbb{R}$. We say that f is **convex** if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, and $0 < \lambda < 1$.

(a) Prove that linear functions are convex.

(b) Prove that if f is convex, then $\forall x_0 \in (a, b)$,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

(c) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ such that f is incerasing and convex and g is convex. Prove that $f \circ g$ is convex.

(d) Suppose $f : (a, b) \to \mathbb{R}$ satisfies

$$f\left(\frac{x + y}{2}\right) \le \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.