Workshop Review 1

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1 Major Theorems

For each of the theorems below, you should know the precise statement and how to apply it. You should also know the precise definitions of the terms in the below theorems. For those theorems marked with *, you should also know at least the key points of the proofs, because those techniques are useful in many other contexts.

- Every convergent sequence is bounded.*
- Every convergent sequence is Cauchy.*
- Balzano-Weierstass
- Every bounded sequence has at least one limit point.
- Every real number is an accumulation point of $\mathbb{Q}$ (ie the rational numbers is dense in $\mathbb{R}$)
- Every Cauchy sequence of real numbers converges to some real number.
- The limit of the sum of two convergent sequences is the sum of the limits.*
- The limit of the product of two convergent sequences is the product of the limits.*
- A sequence converges if and only if each of its subsequences converges.
- Bounded monotone sequences converge.

2 Straightforward Problems

The problems below are basic questions, which help you assess whether you understand the basic definitions and theorems.

1. Prove that $1 + \frac{1}{n}$ converges to 1.
2. Prove or give a counterexample: Every bounded sequence converges.
3. Prove that $\frac{\sin(n)}{n}$ converges to 0.
4. Let p be any postive number. Prove that $\frac{1}{n^p}$ converges to 0.
5. Let $0 < c < 1$. Prove that $\sqrt[n]{c}$ converges.

3 Trickier Problems

The problems below are a bit harder than the ones in the section above. These are closer in difficulty to the homework problems and the problems you are likely to see on the exam.

1. Prove that if $c > 1$, then $\sqrt[n]{c}$ converges.
2. Show that a_n converges to A if and only if $a_n - A$ converges to 0.
3. Suppose a_n converges to A. Define a new sequence $\{b_n\}_{n=1}^{\infty}$ by

$$b_n = \frac{a_n + a_{n+1}}{2}.$$

   Prove that b_n also converges to A.

4. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences. Prove directly that $\{a_n + b_n\}_{n=1}^{\infty}$ is Cauchy.
5. Prove that the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

   converges.

6. Prove that the sequence $\{s_n\}$ defined by

$$s_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} \cdots + \frac{1}{n^2+n}$$

   converges to 0.

7. Let $c \in \mathbb{R}$ with $|c| < 1$.
   (a) Prove that the sequence c^n converges to 0.
   (b) Let a_1 be any real number, and then define the sequence $\{a_n\}$ recursively by

$$a_{n+1} = ca_n.$$

   Prove that a_n converges to 0.
   (c) Suppose $\{b_n\}$ is a bounded sequence, and define a sequence $\{s_n\}$ by

$$s_n = c^n b_n.$$

   Does s_n converge? If so, to what limit?

8. Suppose $|a_n|$ converges to 0. Must a_n also converge to 0? Prove or give a counterexample.

4 Challenge Problems

The problems in this section are the hardest on this worksheet, and require the most creativity. They are more difficult that the types of problems you are likely to see on the exam. Hence, they are a great way to deepen your understanding, but only if you are already comfortable with the problems earlier in this packet.

1. Let $c > 0$, and prove that $\sqrt[n]{c}$ converges to 1.
2. Prove that $\sqrt[n]{n}$ converges to 1.
3. Suppose $\{a_n\}$ is a strictly increasing sequence which converges to some $A \in \mathbb{R}$. Suppose $\{b_n\}$ is a sequence such that for all $n \in \mathbb{N}$,

   $$a_n < b_n < a_{n+1}.$$

   Prove that b_n also converges to A.
4. Does every sequence have at most countably many subsequences? Does there exist a subsequence with uncountable many subsequences?
5. Suppose $\{a_n\}$ is a bounded sequence, and let M be such that $|a_n| \leq M$ for all n. Suppose a_n converges to a. Must $|a| \leq M$? Prove or give a counterexample.